Mechanics of Materials

Chapter 7

Deflections of Beams and Shafts

Tishk International University
Civil Engineering Department
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Mechanics of Materials
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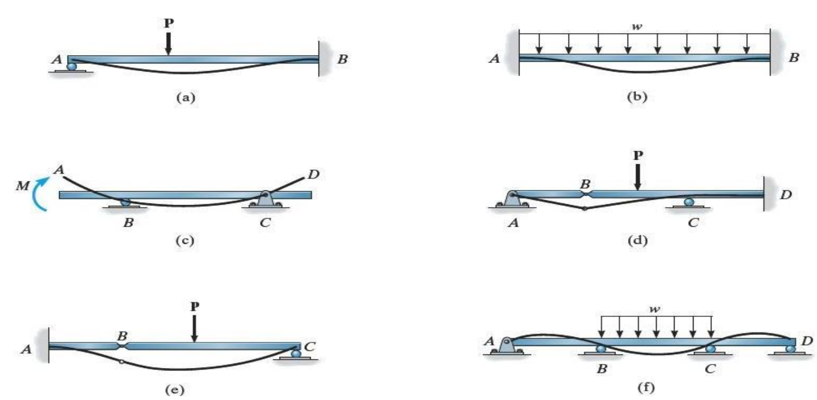
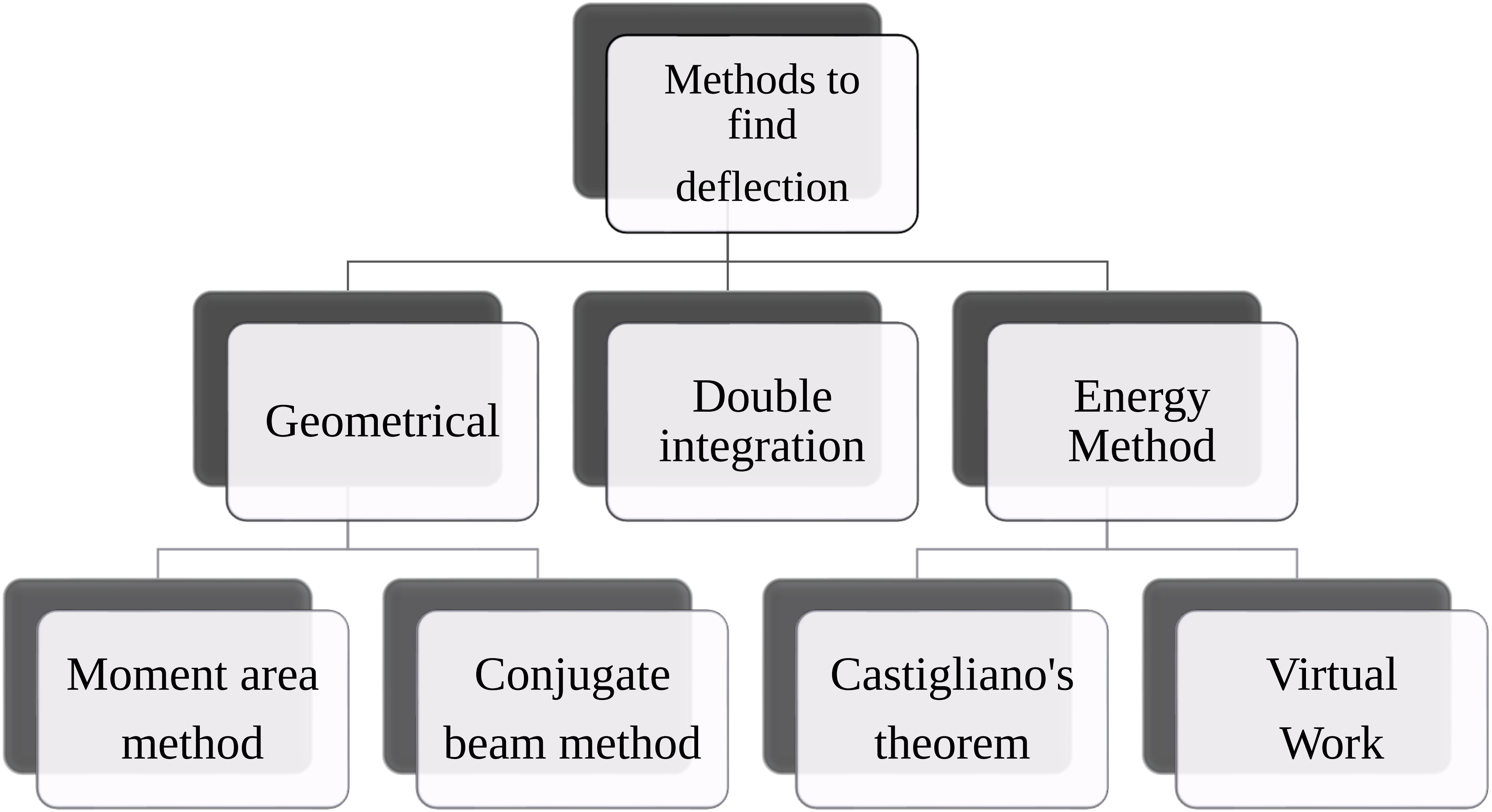


CHAPTER OUTLINE

DEFLECTIONS

- Calculation of deflections is an important part of structural analysis
- Excessive beam deflection can be seen as a mode of failure.
  - Extensive glass breakage in tall buildings can be attributed to excessive deflections
  - Large deflections in buildings are unsightly (and unnerving) and can cause cracks in ceilings and walls.
  - Deflections are limited to prevent undesirable vibrations

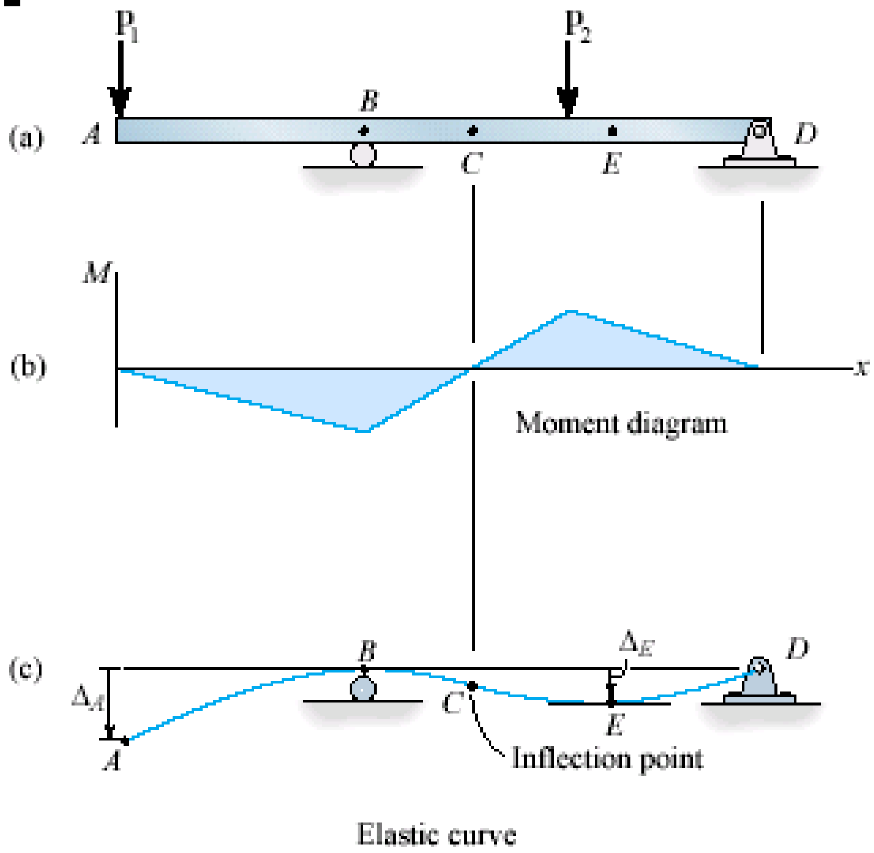
METHODS TO FIND DEFLECTION



The Elastic Curve

Beam Deflection

- Bending changes the initially straight longitudinal axis of the beam into a curve that is called the **Deflection Curve** or **Elastic Curve**

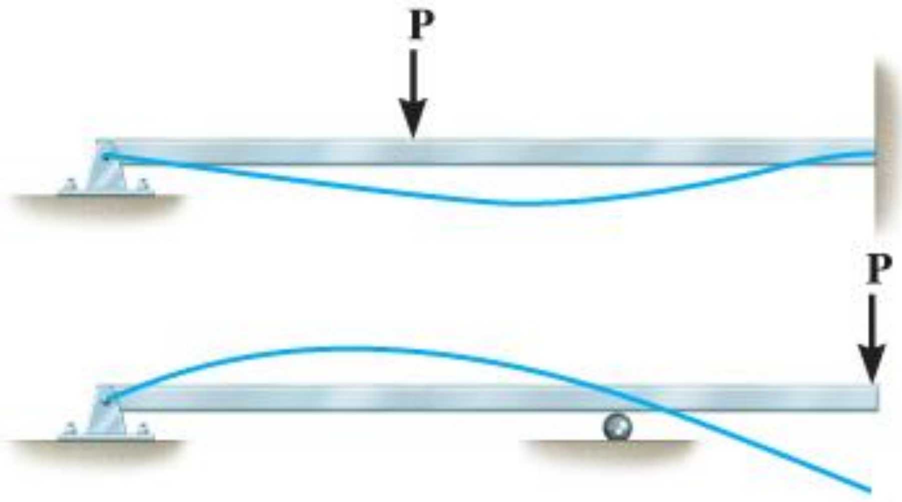


Beam Deflection

- To determine the deflection curve:
  - Draw shear and moment diagram for the beam
  - Directly under the moment diagram draw a line for the beam and label all supports
  - At the supports displacement is zero
  - Where the moment is negative, the deflection curve is concave downward.
  - Where the moment is positive the deflection curve is concave upward
  - Where the two curve meet is the Inflection Point

7.1 THE ELASTIC CURVE

- It is useful to sketch the deflected shape of the loaded beam, to "visualize" computed results and partially check the results.
- The deflection diagram of the longitudinal axis that passes through the centroid of each x-sectional area of the beam is called the elastic curve.



7.1 THE ELASTIC CURVE

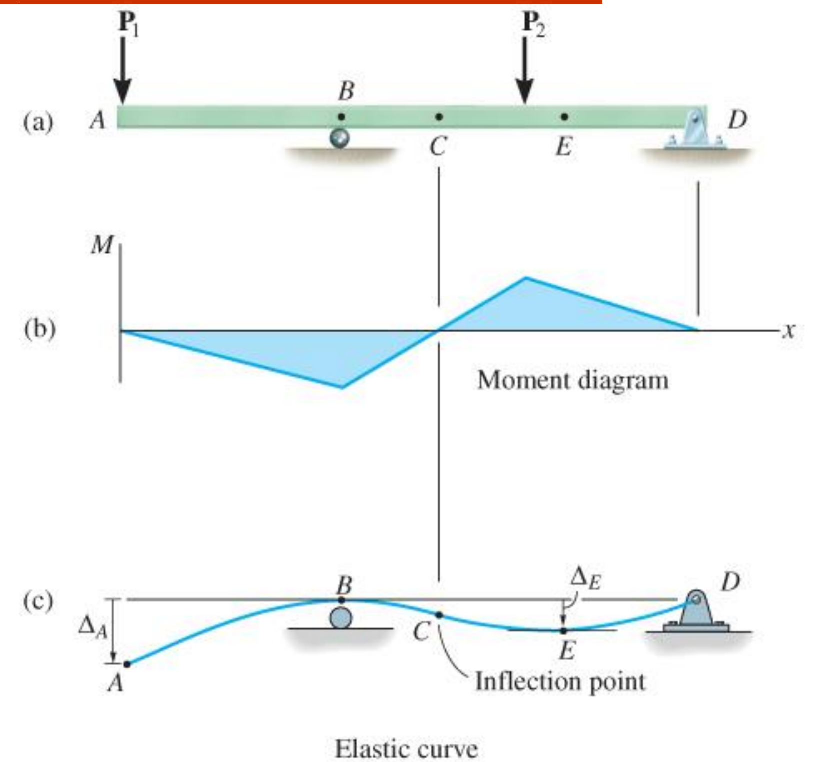
- Draw the moment diagram for the beam first before creating the elastic curve.
- Use beam convention as shown and established in chapter 6.

Positive internal moment concave upwards
(a)

Negative internal moment concave downwards
(b)

7.1 THE ELASTIC CURVE

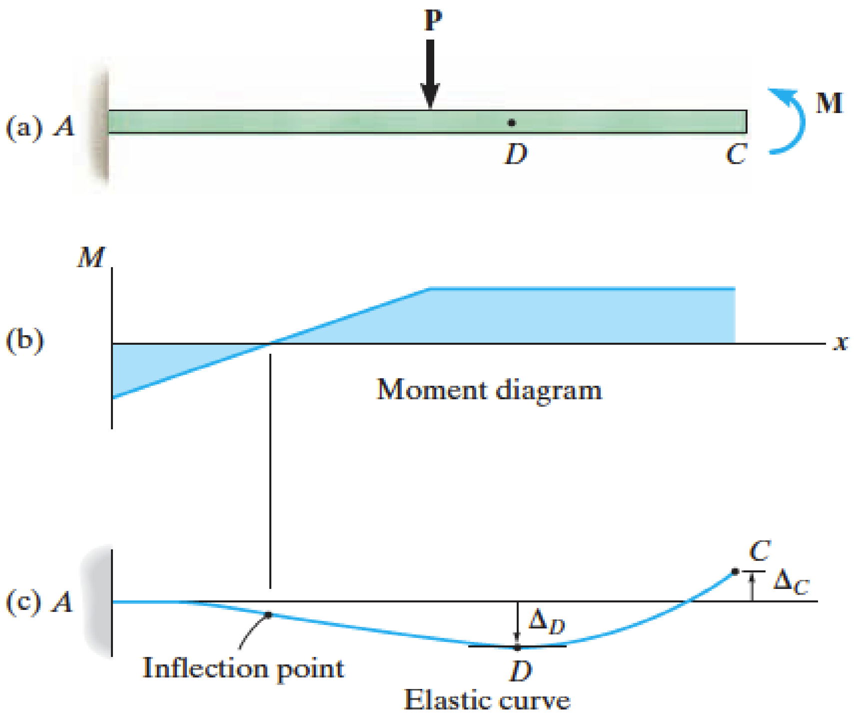
- For example, due to roller and pin supports at B and D, displacements at B and D is zero.
- For region of -ve moment AC, elastic curve concave downwards.
- Within region of +ve moment CD, elastic curve concave upwards.
- At pt C, there is an inflection pt where curve changes from concave up to concave down (zero moment).



It should also be noted that the displacements ΔA and ΔE are especially critical. At point E the slope of the elastic curve is zero, and there the beam's deflection may be a maximum. Whether ΔE is actually greater than ΔA depends on the relative magnitudes of P1 and P2 and the location of the roller at B.

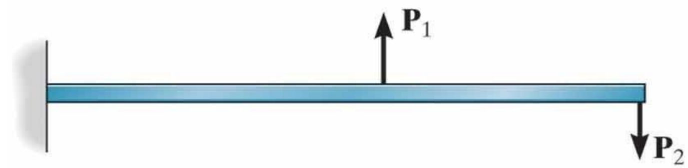
7.1 THE ELASTIC CURVE

Following these same principles, note how the elastic curve was constructed. Here the beam is cantilevered from a fixed support at A and therefore the elastic curve must have both zero displacement and zero slope at this point. Also, the largest displacement will occur either at D, where the slope is zero, or at C.

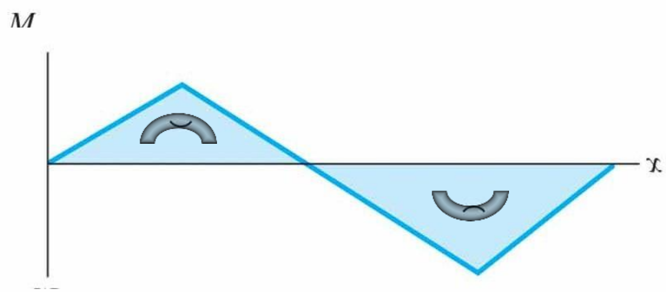


Deflected Shape

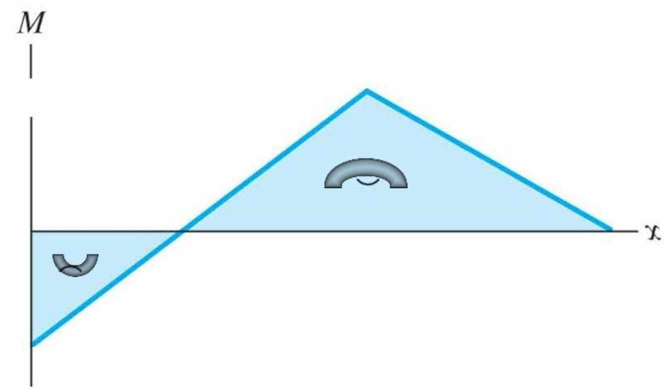
beam



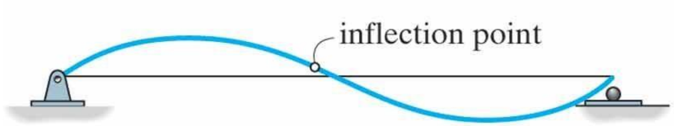
beam



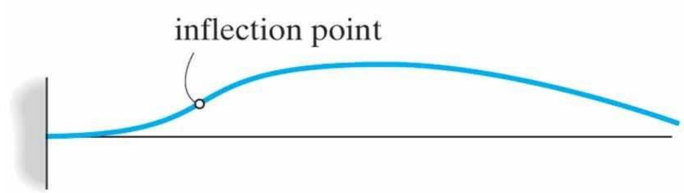
moment diagram



moment diagram



deflection curve

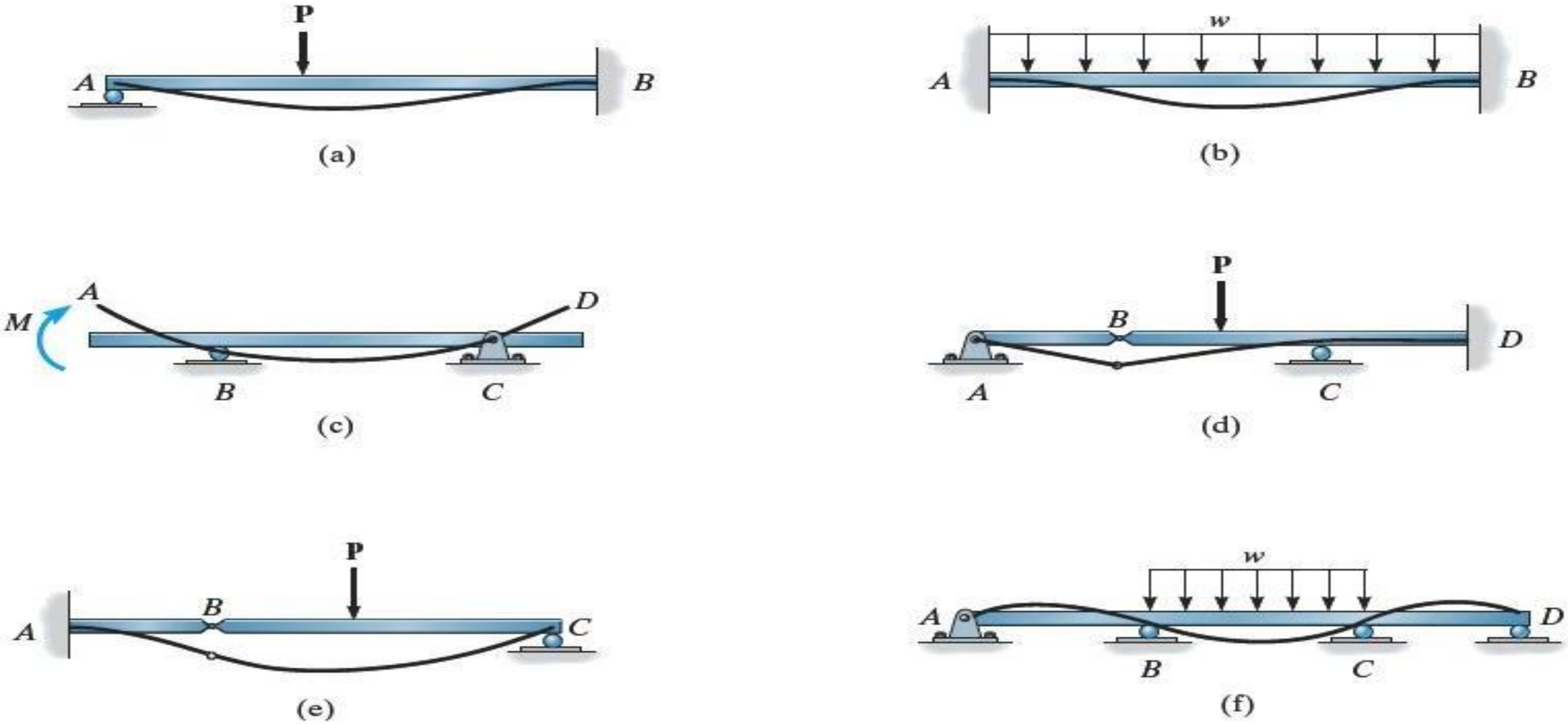


deflection curve

7.1 THE ELASTIC CURVE

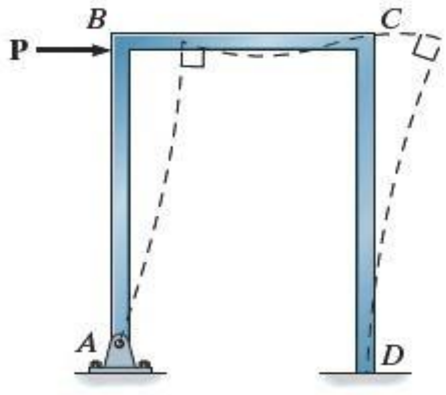
Example A

Draw the deflected shape for each of the beams shown

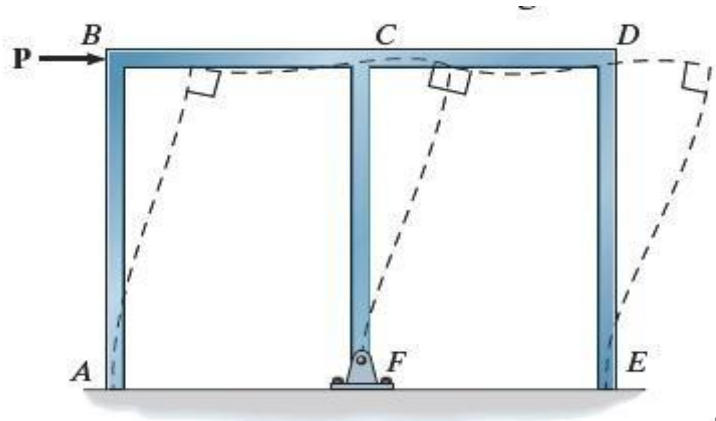


Example B

Draw the deflected shape for each of the frames shown



(a)



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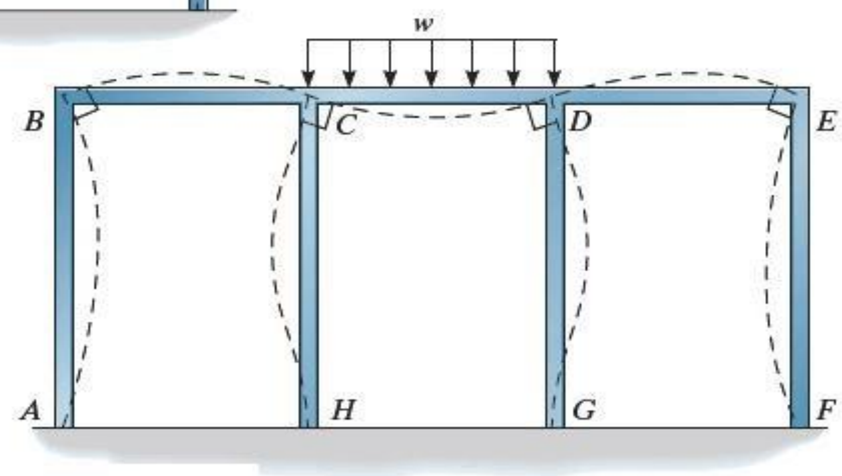
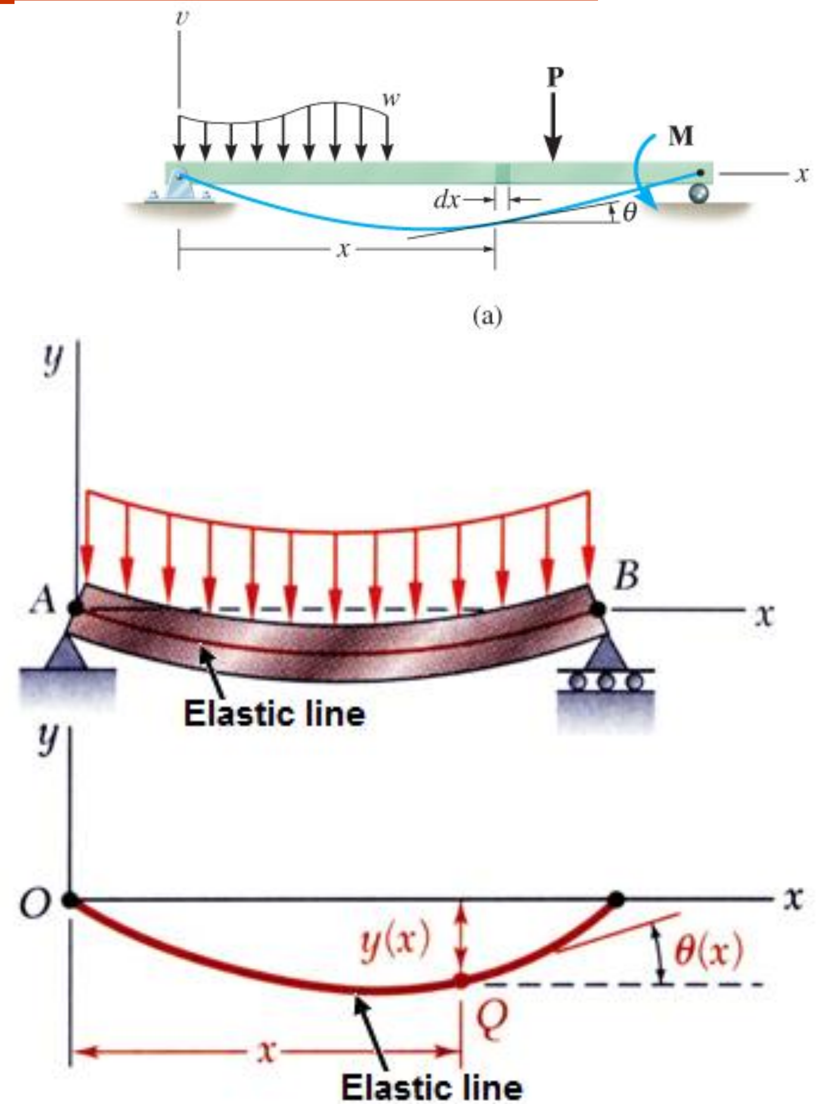


(c)

7.1 THE ELASTIC CURVE

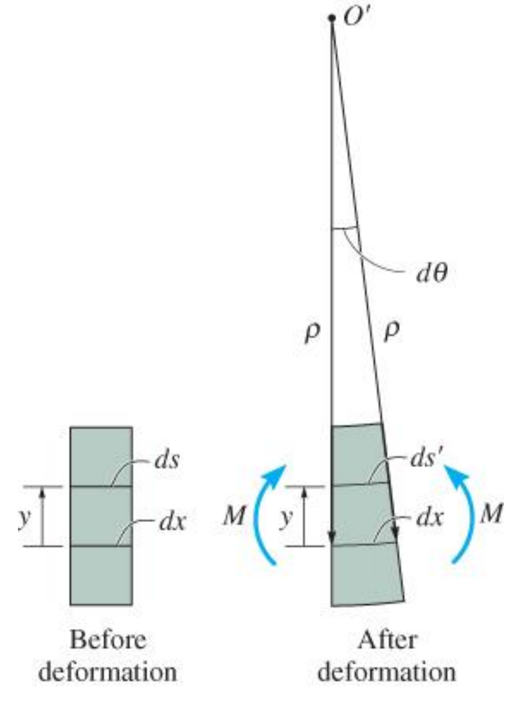
- *x* axis extends +ve to the right, along longitudinal axis of beam.
- A differential element of undeformed width *dx* is located.
- *y* axis extends +ve upwards from *x* axis. It measures the displacement of the centroid on x-sectional area of element.
- A "localized" *y* coordinate is specified for the position of a fiber in the element.
- It is measured +ve upward from the neutral axis.



7.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Limit analysis to the case of initially straight beam elastically deformed by loads applied perpendicular to beam's *x* axis and lying in the *x*-*v* plane of symmetry for beam's x-sectional area.
- Internal moment **M** deforms element such that angle between x-sections is $d\theta$.
- Arc *dx* is a part of the elastic curve that intersects the neutral axis for each x-section.
- Radius of curvature for this arc defined as the distance ρ, measured from center of curvature *O'* to *dx*.



(b)

7.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Strain in arc ds, at position y from neutral axis, is

$$\varepsilon = \frac{ds' - ds}{ds}$$

But $ds = dx = \rho d\theta$ and $ds' = (\rho - y)d\theta$

$$\varepsilon = \frac{[(\rho - y)d\theta - \rho d\theta_s]}{\rho d\theta} \text{ or}$$

$$\frac{1}{\rho} = -\frac{\varepsilon}{y} \qquad (7-1)$$

7.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- If material is homogeneous and shows linear-elastic behavior, Hooke's law applies. Since flexure formula also applies, we combing the equations to get

$$\frac{1}{\rho} = \frac{M}{EI} \qquad (7\text{-}2)$$

ρ = radius of curvature at a specific pt on elastic curve (1/ρ is referred to as the curvature).

M = internal moment in beam at pt where is to be determined.

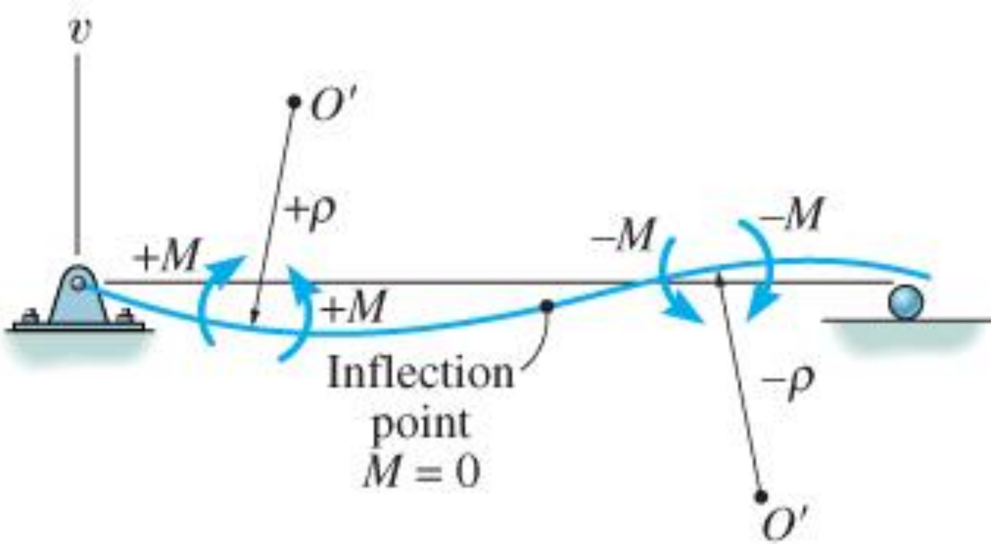
E = material's modulus of elasticity.

I = beam's moment of inertia computed about neutral axis.

7.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- EI is the flexural rigidity and is always positive.
- Sign for ρ depends on the direction of the moment.
- As shown, when M is +ve, ρ extends above the beam. When M is –ve, ρ extends below the beam.



7.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Using flexure formula, $\sigma = -My/I$, curvature is also

$$\frac{1}{\rho} = -\frac{\sigma}{Ey} \qquad (7\text{-}3)$$

- Eqns 7-2 and 7-3 valid for either small or large radii of curvature.

Slope and Displacement by Integration

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Let's represent the curvature in terms of ν and x.

$$\frac{1}{\rho}=\frac{d^2\upsilon/dx^2}{\left[1+\left(d\upsilon/dx\right)^2\right]^{3/2}}$$

- Substitute into Eqn 7-2

$$\frac{d^2\upsilon/dx^2}{\left[1+\left(d\upsilon/dx\right)^2\right]^{3/2}}=\frac{M}{EI} \qquad \textbf{(7-4)}$$

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Slope of elastic curve determined from dv/dx is very small and its square will be negligible compared with unity.
- Therefore, by approximation $1/\rho = d^2v/dx^2$, Eqn 7-4 rewritten as

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI} \qquad (7\text{-}5)$$

- Differentiate each side w.r.t. x and substitute $V = dM/dx$, we get

$$\frac{d}{dx}\left(EI\frac{d^2\upsilon}{dx^2}\right) = V(x) \qquad (7\text{-}6)$$

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Differentiating again, using $-w = dV/dx$ yields

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\upsilon}{dx^2}\right)=-w(x) \qquad (7\text{-}7)$$

- Flexural rigidity is constant along beam, thus

$$EI\frac{d^4\upsilon}{dx^4}=-w(x) \qquad (7\text{-}8)$$

$$EI\frac{d^3\upsilon}{dx^3}=V(x) \qquad (7\text{-}9)$$

$$EI\frac{d^2\upsilon}{dx^2}=M(x) \qquad (7\text{-}10)$$

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

From the Equation $EI\dfrac{d^2y}{dx^2} = M_x$

$EI\dfrac{d^4y}{dx^4} = -w$ Shear force density (Load)

$EI\dfrac{d^3y}{dx^3} = V_x$ SHear force

$EI\dfrac{d^2y}{dx^2} = M_x BendingMoment$

$\dfrac{dy}{dx} = q$ SLope

y=d=Deflection,Displacement

Flexural reigidity=EI

$$EI\frac{d^2y}{dx^2} = M_x$$

$$V_x = \int -w\,dx$$

$$M_x = \int V_x\,dx$$

$$EI\frac{d^2y}{dx^2} = M_x$$

$$q = slope = \frac{1}{EI}\int M_x\,dx$$

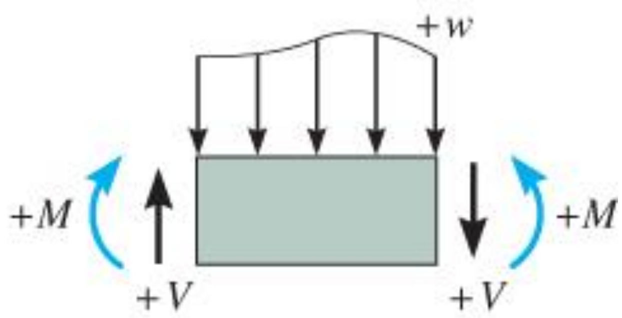
$$d = deflection = \int q\,dx$$

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Generally, it is easier to determine the internal moment M as a function of x, integrate twice, and evaluate only two integration constants.
- For convenience in writing each moment expression, the origin for each x coordinate can be selected arbitrarily.

Sign convention and coordinates

- Use the proper signs for M, V and w.



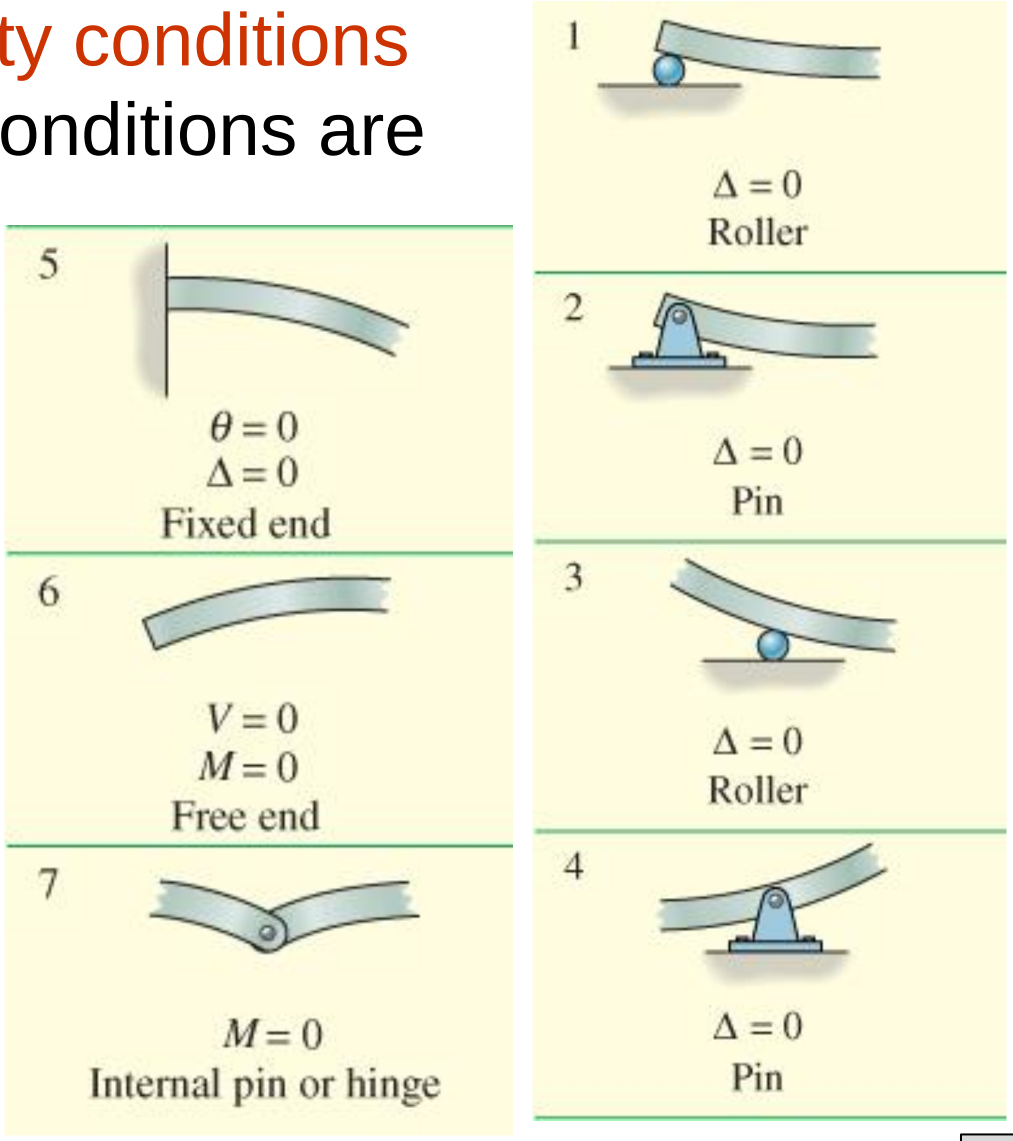
Positive sign convention

(a)

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

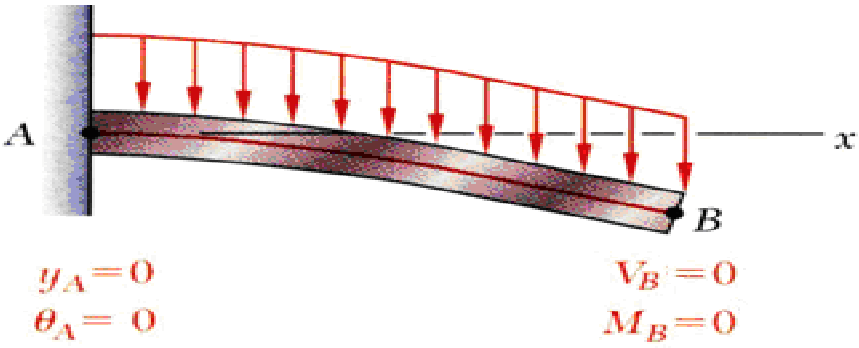
Boundary and continuity conditions

- Possible boundary conditions are shown here.



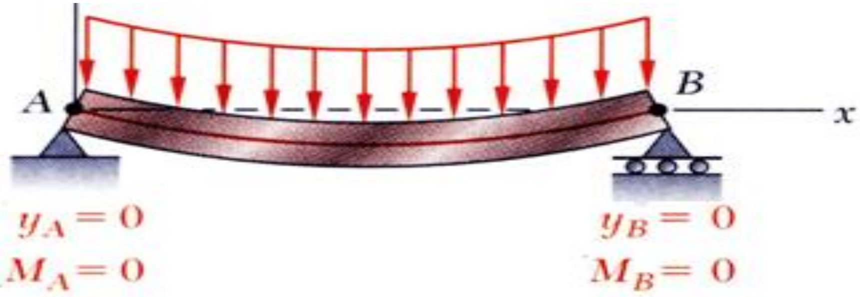
Boundary conditions

Clamped or Built in support or Fixed end :
(Point A)
Deflection =0
Slope=0
Moment is not 0



Free end:
(Point B)
Deflection is not 0
Slope is not
Moment= 0

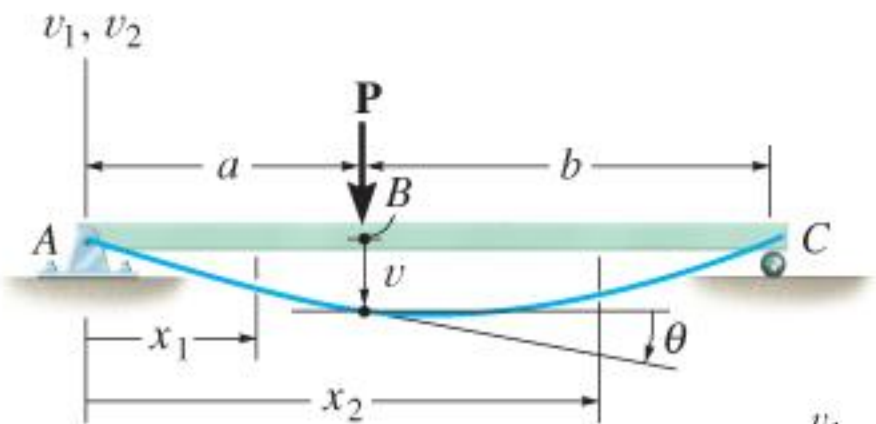
End restrained against rotation but free to deflection)
Deflection is not 0
Slope=0
Shear is 0



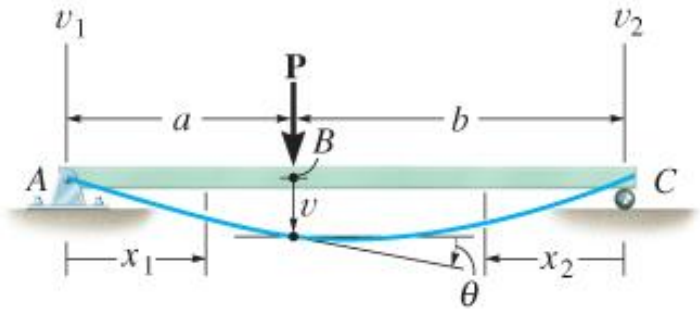
7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Boundary and continuity conditions

- If a single x coordinate cannot be used to express the eqn for beam's slope or elastic curve, then continuity conditions must be used to evaluate some of the integration constants.



(a)



(b)

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Elastic curve

- Draw an exaggerated view of the beam's elastic curve.
- Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.
- Establish the x and v coordinate axes.
- The x axis must be parallel to the undeflected beam and can have an origin at any pt along the beam, with +ve direction either to the right or to the left.

Procedure for analysis

Elastic curve

- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities.
- Choose these coordinates so that they will simplify subsequent algebraic work.

Procedure for analysis

Load or moment function

- For each region in which there is an x coordinate, express that loading w or the internal moment M as a function of x.
- In particular, always assume that M acts in the +ve direction when applying the eqn of moment equilibrium to determine $M = f(x)$.

Procedure for analysis

Slope and elastic curve

- Provided EI is constant, apply either the load eqn $EI\ d^4 v/dx^4 = -w(x)$, which requires four integrations to get $v = v(x)$, or the moment eqns $EI\ d^2 v\ /dx^2 = M(x)$, which requires only two integrations. For each integration, we include a constant of integration.
- Constants are evaluated using boundary conditions for the supports and the continuity conditions that apply to slope and displacement at pts where two functions meet.

7.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Slope and elastic curve

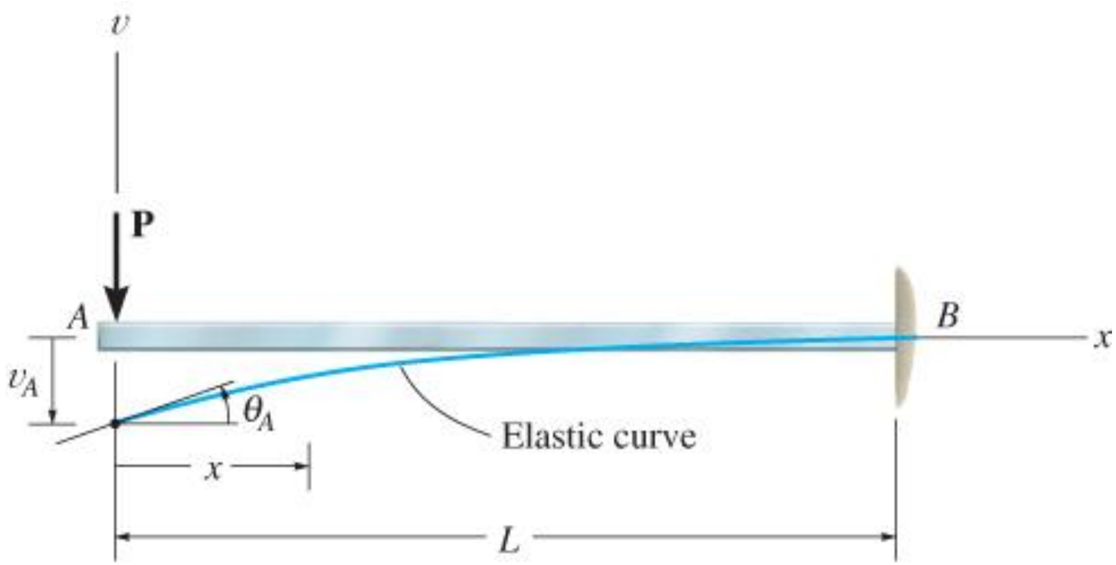
- Once constants are evaluated and substituted back into slope and deflection eqns, slope and displacement at specific pts on elastic curve can be determined.
- The numerical values obtained is checked graphically by comparing them with sketch of the elastic curve.
- Realize that +ve values for slope are counterclockwise if the x axis extends +ve to the right, and clockwise if the x axis extends +ve to the left. For both cases, +ve displacement is upwards.

Assumptions and Limitations

- Deflections caused by shearing action negligibly small compared to bending
- Deflections are small compared to the cross- sectional dimensions of the beam
- All portions of the beam are acting in the elastic range
- Beam is straight prior to the application of loads

EXAMPLE 7.1

Cantilevered beam shown is subjected to a vertical load **P** at its end. Determine the eqn of the elastic curve. EI is constant.



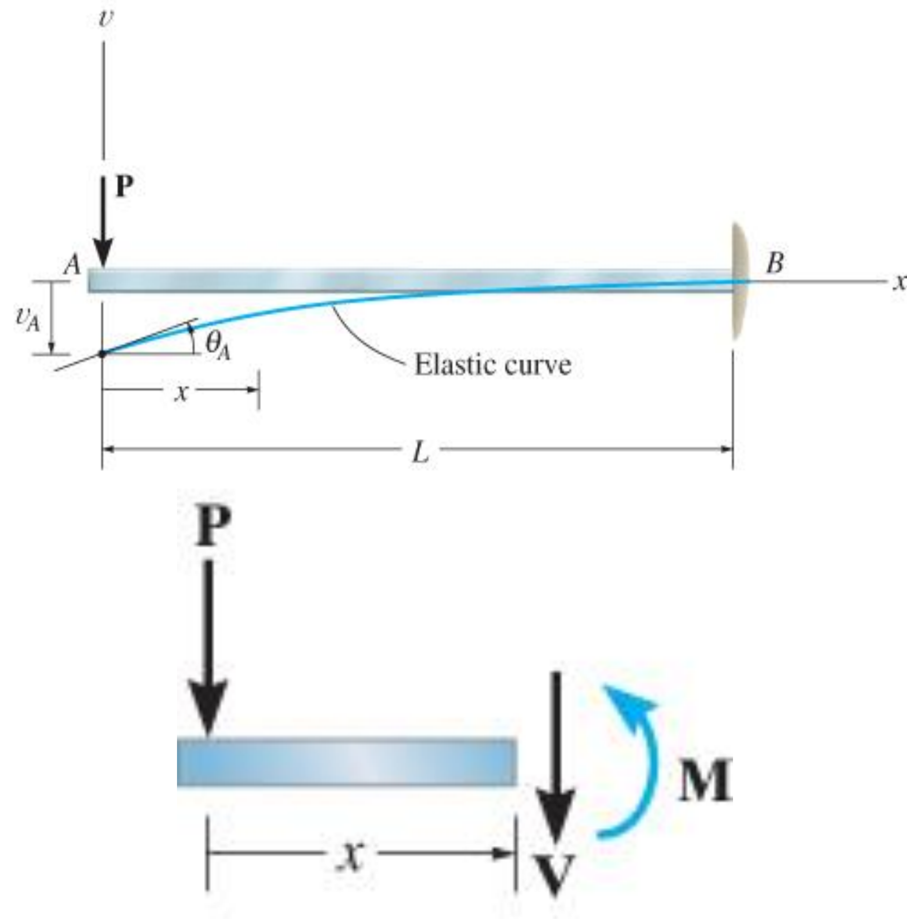
(a)

EXAMPLE 7.1 (CONT.)

Elastic curve: Load tends to deflect the beam. By inspection, the internal moment can be represented throughout the beam using a single *x* coordinate.

Moment function: From free-body diagram, with **M** acting in the +ve direction, we have

$$M = -Px$$



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EXAMPLE 7.1 (CONT.)

Slope and elastic curve:

Applying Eqn 7-10 and integrating twice yields

$$EI\frac{d^2\upsilon}{dx^2} = -Px \qquad (1)$$

$$EI\frac{d\upsilon}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EI\upsilon = -\frac{Px^3}{6} + C_1 x + C_2 \qquad (3)$$

EXAMPLE 7.1 (CONT.)

Slope and elastic curve:

Using boundary conditions $dv/dx = 0$ at $x = L$, and $v = 0$ at $x = L$, Eqn (2) and (3) becomes

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

EXAMPLE 7.1 (CONT.)

Slope and elastic curve:

Thus, $C_1 = PL^2/2$ and $C_2 = PL^3/3$. Substituting these results into Eqns (2) and (3) with $\theta = dv/dx$, we get

$$\theta = -\frac{P}{2EI}\left(L^2 - x^2\right)$$

$$\upsilon = \frac{P}{6EI}\left(-x^3 + 3L^2x - 2L^3\right)$$

Maximum slope and displacement occur at A ($x = 0$),

$$\theta_A = \frac{PL^2}{2EI} \qquad \upsilon_A = -\frac{PL^3}{3EI}$$

EXAMPLE 7.1 (CONT.)

Slope and elastic curve:

Positive result for θ_A indicates counterclockwise rotation and negative result for A indicates that ν_A is downward.

Consider beam to have a length of 5 m,

support load P = 30 kN,

made of steel having E_{st} = 200 GPa.

$I = 84.8(10^6)\ \text{mm}^4$.

EXAMPLE 7.1 (CONT.)

Slope and elastic curve:

From Eqns (4) and (5),

$$\theta_A = \frac{30\text{ kN}\left(10^3\text{ N/kN}\right)\times\left[5\text{ m}\left(10^3\text{ mm/m}\right)^2\right]^2}{2\left[200\left(10^3\right)\text{N/mm}^2\right]\left(84.8\left(10^6\right)\text{mm}^4\right)} = 0.0221\text{ rad}$$

$$\upsilon_A = -\frac{30\text{ kN}\left(10^3\text{ N/kN}\right)\times\left[5\text{ m}\left(10^3\text{ mm/m}\right)^2\right]^3}{3\left[200\left(10^3\right)\text{N/mm}^2\right]\left(84.8\left(10^6\right)\text{mm}^4\right)} = -73.7\text{ mm}$$

EXAMPLE 7.1 (CONT.)

SOLUTION 2

Using Eqn 7-8 to solve the problem. Here $w(x) = 0$ for $0 \le x \le L$, so that upon integrating once

$$EI\frac{d^4\upsilon}{dx^4} = 0$$

$$EI\frac{d^3\upsilon}{dx^3} = C'_1 = V$$

EXAMPLE 7.1 (Con.)

Solution II

Shear constant C'_1 can be evaluated at $x = 0$, since $V_A = -P$. Thus, $C'_1 = -P$. Integrating again yields the form of Eqn 7-10,

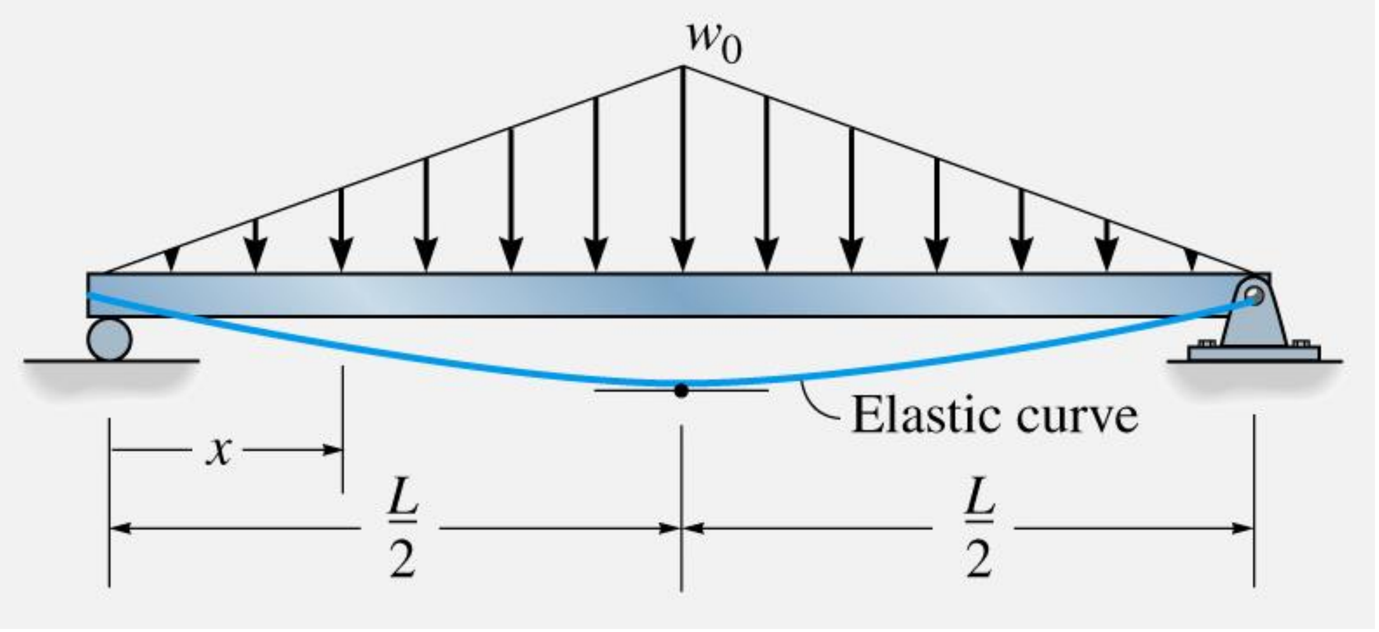
$$EI\frac{d^3\upsilon}{dx^3} = -P$$

$$EI\frac{d^2\upsilon}{dx^2} = -Px + C'_2 = M$$

Here, $M = 0$ at $x = 0$, so $C'_2 = 0$, and as a result, we obtain Eqn 1 and solution proceeds as before.

EXAMPLE 7.2

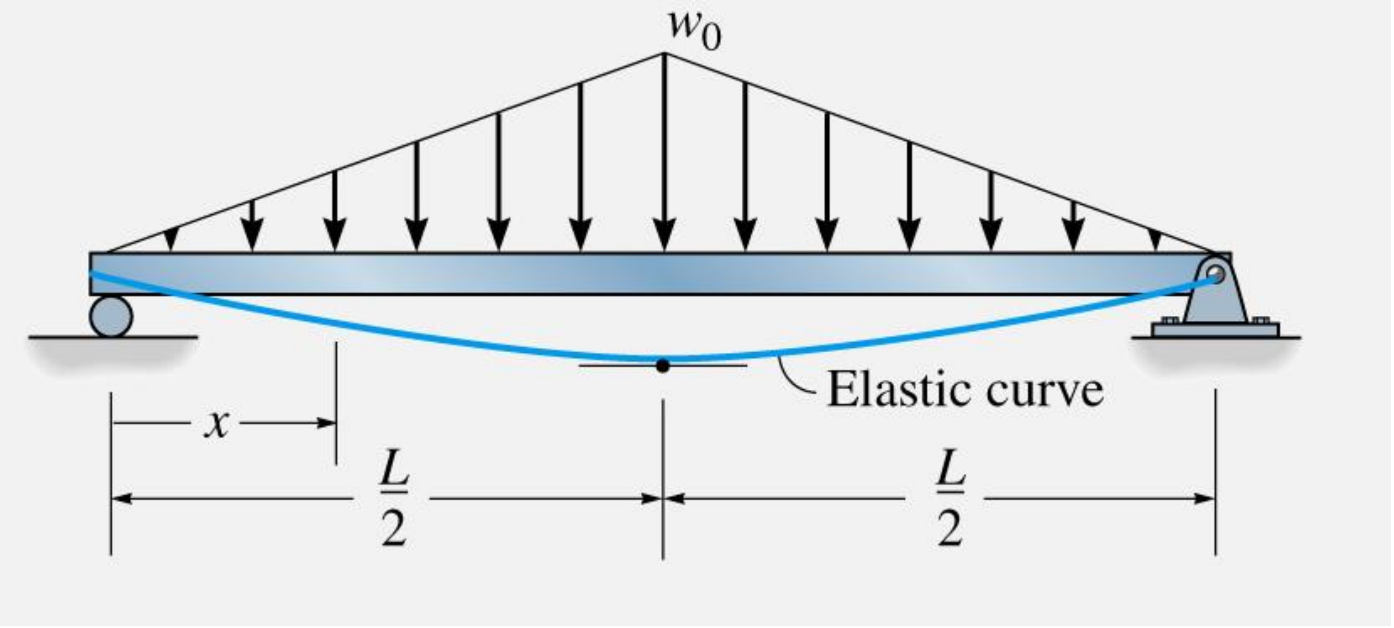
The beam shown in Figure below, supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



EXAMPLE 7.2 (Con.)

SOLUTION I

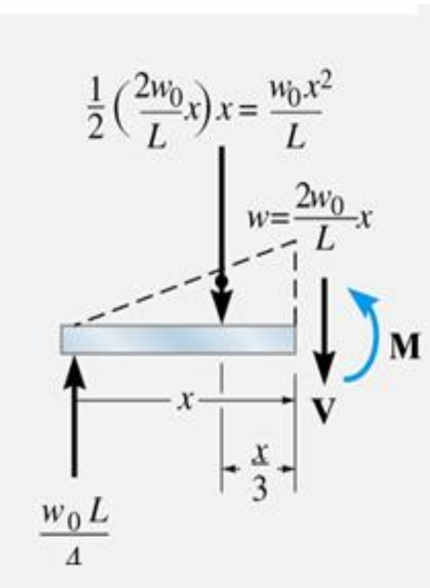
Elastic Curve. Due to symmetry, only one x coordinate is needed for the solution, in this case $0 \leq x \leq L/2$. The beam deflects as shown. The maximum deflection occurs at the center since the slope is zero at this point.



EXAMPLE 7.2 (Con.)

Moment Function. A free-body diagram of the segment on the left is shown in Figure below. The equation for the distributed loading is

$$w = \frac{2w_0}{L}x \qquad (1)$$



Hence,

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad M + \frac{w_0x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0L}{4}(x) = 0$$

$$M = -\frac{w_0x^3}{3L} + \frac{w_0L}{4}x$$

EXAMPLE 7.2 (Con.)

Slope and Elastic Curve. Using Eq. 7–10 and integrating twice, we have

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x \qquad (2)$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

EXAMPLE 7.2 (Con.)

The constants of integration are obtained by applying the boundary condition $v = 0$ at $x = 0$ and the symmetry condition that $dv/dx = 0$ at $x = L/2$. This leads to

$$C_1 = -\frac{5w_0L^3}{192} \qquad C_2 = 0$$

Hence,

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 - \frac{5w_0L^3}{192}$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 - \frac{5w_0L^3}{192}x$$

Determining the maximum deflection at $x = L/2$, we have

$$v_{\max} = -\frac{w_0L^4}{120EI}$$ ***Ans.***

EXAMPLE 7.2 (Con.)

SOLUTION II

Starting with the distributed loading, Eq. 1, and applying Eq. 7–8, we have

$$EI\frac{d^4v}{dx^4} = -\frac{2w_0}{L}x$$

$$EI\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + C_1'$$

Since $V = +w_0L/4$ at $x = 0$, then $C_1' = w_0L/4$. Integrating again yields

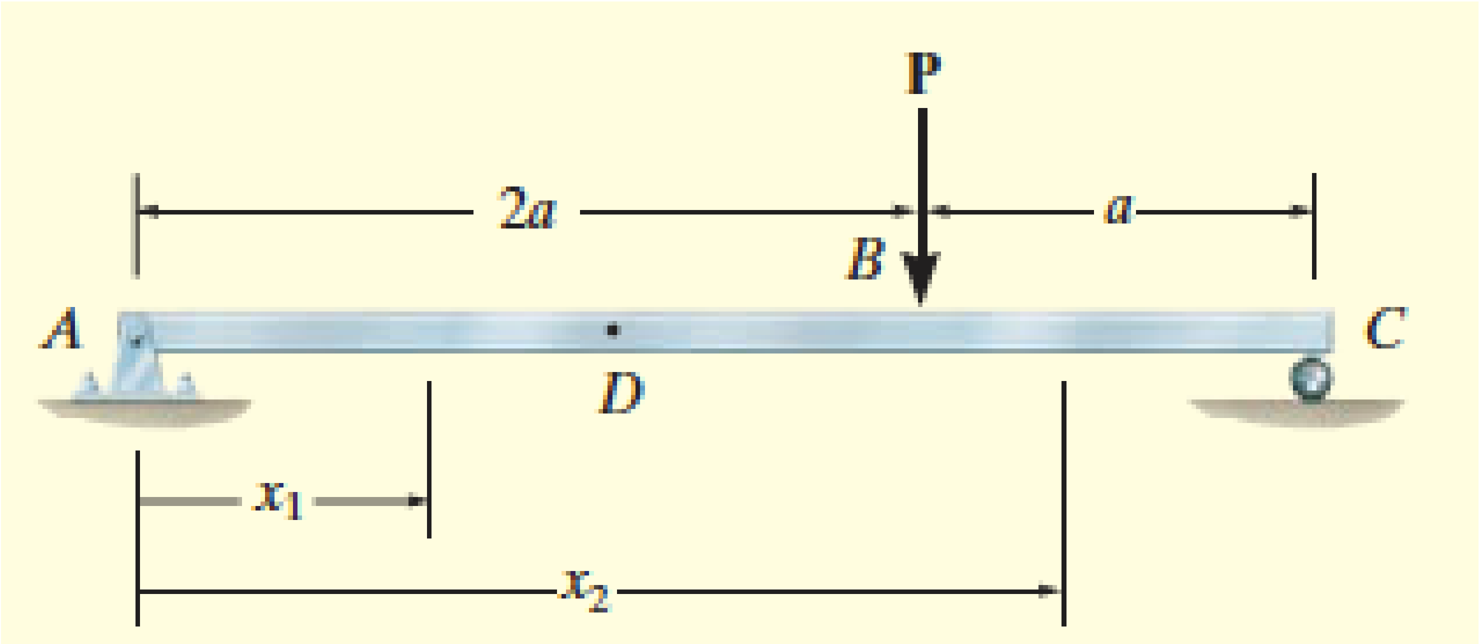
$$EI\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + \frac{w_0L}{4}$$

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x + C_2'$$

Here $M = 0$ at $x = 0$, so $C_2' = 0$. This yields Eq. 2. The solution now proceeds as before.

EXAMPLE 7.3

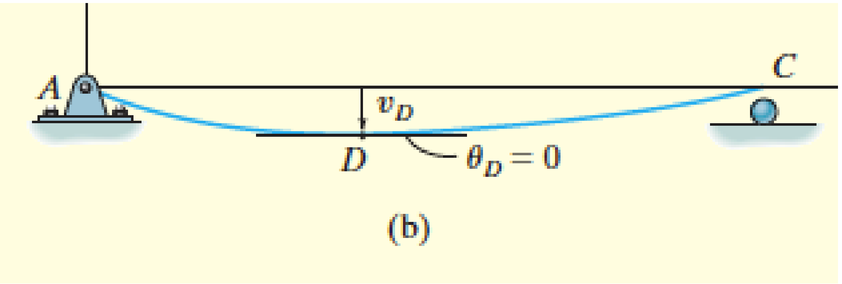
The simply supported beam shown in Fig. is subjected to the concentrated force P. Determine the maximum deflection of the beam. **EI** is constant.



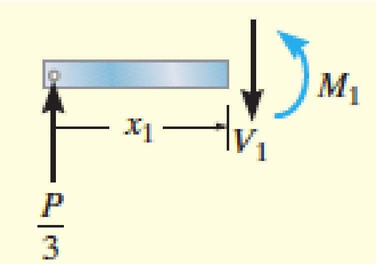
EXAMPLE 7.3 (Con.)

Solution

The beam deflects as shown in Fig. b. Two coordinates must be used, since the moment function will change at P. Here we will take x1 and x2 having the same origin at A.



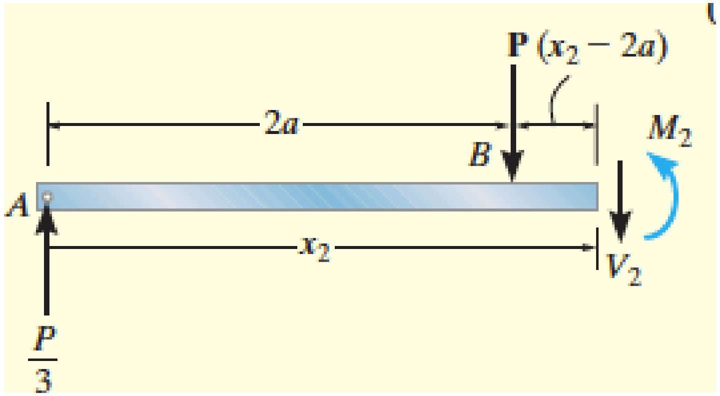
(b)



From free body diagrams

$$M_1 = \frac{P}{3}x_1,$$

$$M_2 = \frac{P}{3}x_2 - P(x_2 - 2a) = \frac{2P}{3}(3a - x_2)$$



EXAMPLE 7.3 (Con.)

Slope and Elastic Curve. Applying Eq. 7–10 for M_1, and integrating twice yields

$$EI\frac{d^2v_1}{dx_1^2} = \frac{P}{3}x_1$$

$$EI\frac{dv_1}{dx^1} = \frac{P}{6}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = \frac{P}{18}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

Likewise for M_2,

$$EI\frac{d^2v_2}{dx_2^2} = \frac{2P}{3}(3a - x_2)$$

$$EI\frac{dv_2}{dx_2} = \frac{2P}{3}\left(3ax_2 - \frac{x_2^2}{2}\right) + C_3 \qquad (3)$$

$$EIv_2 = \frac{2P}{3}\left(\frac{3}{2}ax_2^2 - \frac{x_2^3}{6}\right) + C_3x_2 + C_4 \qquad (4)$$

EXAMPLE 7.3 (Con.)

The four constants are evaluated using *two* boundary conditions, namely, $x_1 = 0$, $v_1 = 0$ and $x_2 = 3a$, $v_2 = 0$. Also, *two* continuity conditions must be applied at B, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$ and $v_1 = v_2$ at $x_1 = x_2 = 2a$. Substitution as specified results in the following four equations:

$$v_1 = 0 \text{ at } x_1 = 0; \qquad 0 = 0 + 0 + C_2$$

$$v_2 = 0 \text{ at } x_2 = 3a; \qquad 0 = \frac{2P}{3}\left(\frac{3}{2}a(3a)^2 - \frac{(3a)^3}{6}\right) + C_3(3a) + C_4$$

$$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}; \qquad \frac{P}{6}(2a)^2 + C_1 = \frac{2P}{3}\left(3a(2a) - \frac{(2a)^2}{2}\right) + C_3$$

$$v_1(2a) = v_2(2a); \frac{P}{18}(2a)^3 + C_1(2a) + C_2 = \frac{2P}{3}\left(\frac{3}{2}a(2a)^2 - \frac{(2a)^3}{6}\right) + C_3(2a) + C_4$$

EXAMPLE 7.3 (Con.)

Solving these equations we get

$$C_1 = -\frac{4}{9}Pa^2 \qquad C_2 = 0$$

$$C_3 = -\frac{22}{9}Pa^2 \qquad C_4 = \frac{4}{3}Pa^3$$

Thus Eqs. 1–4 become

$$\frac{dv_1}{dx_1} = \frac{P}{6EI}x_1^{\,2} - \frac{4}{9}\frac{Pa^2}{EI} \qquad (5)$$

$$v_1 = \frac{P}{18EI}x_1^{\,3} - \frac{4}{9}\frac{Pa^2}{EI}x_1 \qquad (6)$$

$$\frac{dv_2}{dx_2} = \frac{2Pa}{EI}x_2 - \frac{P}{3EI}x_2^{\,2} - \frac{22}{9}\frac{Pa^2}{EI} \qquad (7)$$

$$v_2 = \frac{Pa}{EI}x_2^{\,2} - \frac{P}{9EI}x_2^{\,3} - \frac{22}{9}\frac{Pa^2}{EI}x_2 + \frac{4}{3}\frac{Pa^3}{EI} \qquad (8)$$

EXAMPLE 7.3 (Con.)

By inspection of the elastic curve, Fig. *b*, the maximum deflection occurs at *D*, somewhere within region *AB*. Here the slope must be zero. From Eq. 5,

$$\frac{1}{6}x_1^2 - \frac{4}{9}a^2 = 0$$

$$x_1 = 1.633a$$

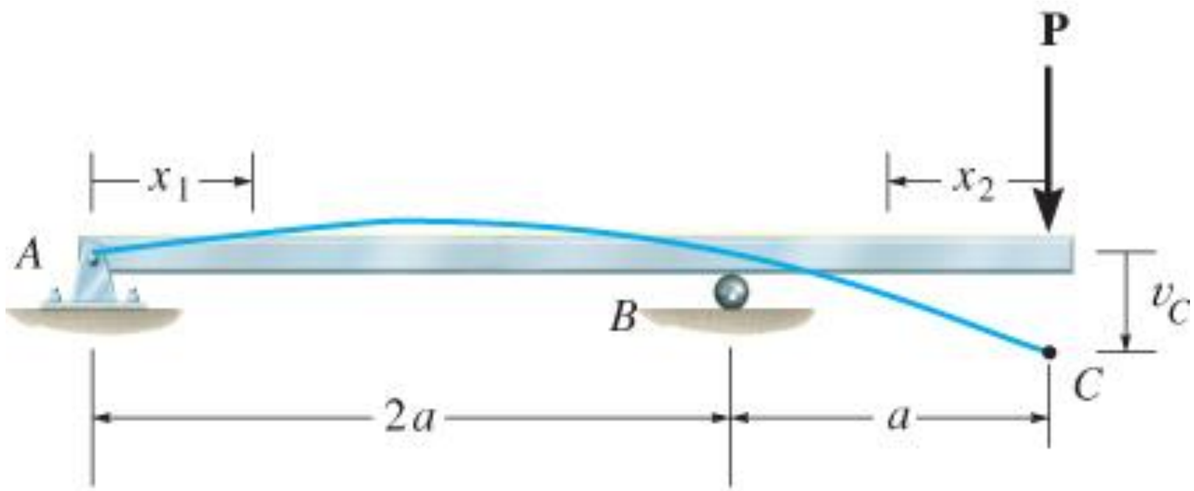
Substituting into Eq. 6,

$$v_{max} = -0.484\,\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.

EXAMPLE 7.4

Beam is subjected to load P at its end. Determine the displacement at C. EI is a constant.

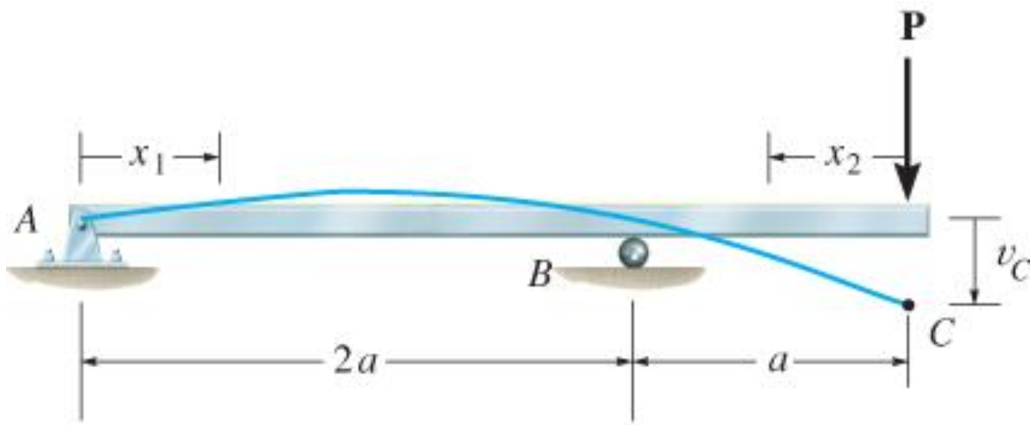


(a)

EXAMPLE 7.4 (Con.)

Elastic curve

Beam deflects into shape shown. Due to loading, two x coordinates will be considered, $0 \leq x_1 \leq 2a$ and $0 \leq x_2 \leq a$, where x_2 is directed to the left from C since internal moment is easy to formulate.

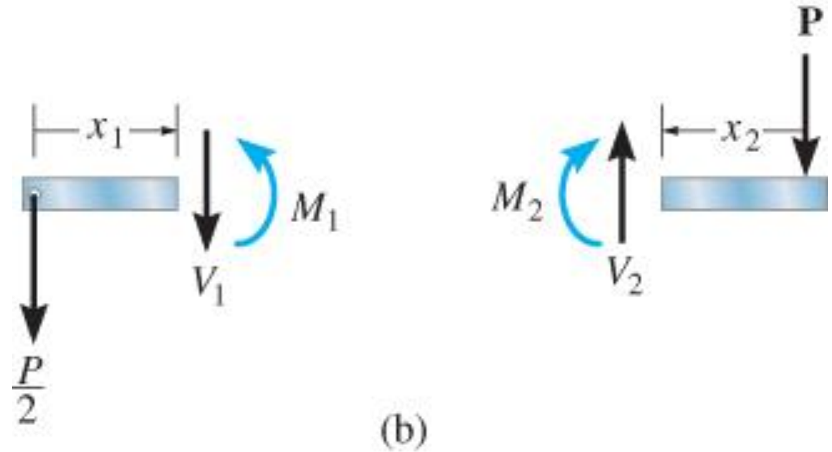


(a)

EXAMPLE 7.4 (Con.)

Moment functions

Using free-body diagrams, we have



$$M_1 = -\frac{P}{2}x_1 \qquad M_2 = -Px_2$$

Slope and Elastic curve: Applying Equation,

for $0 \le x_1 \le 2a$ $\quad EI = \frac{d^2\upsilon_1}{dx_1^2} = -\frac{P}{2}x_1$

$$EI\frac{d\upsilon_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EI\upsilon_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

EXAMPLE 7.4 (Con.)

Slope and Elastic curve:

for $0 \le x_2 \le a$ $$EI\frac{d^2\upsilon_2}{dx_2{}^2} = -Px_2$$

$$EI\frac{d\upsilon_2}{dx_2} = -\frac{P}{2}x_2{}^2 + C_3 \qquad (3)$$

$$EI\upsilon_2 = -\frac{P}{6}x_2{}^3 + C_3x_2 + C_4 \qquad (4)$$

EXAMPLE 7.4 (Con.)

Slope and Elastic curve:

The four constants of integration determined using three boundary conditions, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = a$ and a discontinuity eqn.

Here, continuity of slope at roller requires $dv_1/dx_1 = -dv_2/dx_2$ at $x_1 = 2a$ and $x_2 = a$.

$$\upsilon_1 = 0 \; at \; x_1 = 0; \qquad 0 = 0 + 0 + C2$$

$$\upsilon_1 = 0 \; at \; x_1 = 2a; \quad 0 = -\frac{P}{12}(2a)^2 + C_1(2a) + C_2$$

EXAMPLE 7.4 (Con.)

Slope and Elastic curve:

$$\upsilon_2 = 0 \ at \ x_2 = a; \qquad 0 = -\frac{P}{6}a^3 + C_3 a + C_4$$

$$\frac{d\upsilon_1(2a)}{dx_1} = -\frac{d\upsilon_2(a)}{dx_2}; \qquad -\frac{P}{4}(2a)^2 + C_1 = -\left(-\frac{P}{2}(a)^2 + C_3\right)$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{7}{6}Pa_2 \qquad C_4 = -Pa^3$$

EXAMPLE 7.4 (Con.)

Slope and Elastic curve:

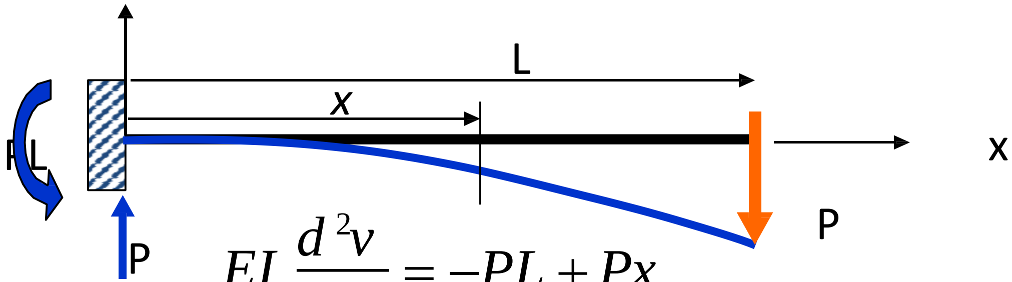
Substituting C_3 and C_4 into Eqn (4) gives

$$\upsilon_2 = -\frac{P}{6EI}x_2^{\,3} + \frac{7Pa^2}{6EI}x_2 - \frac{Pa^3}{EI}$$

Displacement at C is determined by setting $x_2 = 0$,

$$\upsilon_C = -\frac{Pa^3}{EI}$$

EXAMPLE 7.5



$$EI\frac{d^2v}{dx^2} = M$$

$$M = -PL + Px$$

$$EI\frac{d^2v}{dx^2} = -PL + Px$$

Integrating once: $EI\frac{dv}{dx} = -PLx + P\frac{x^2}{2} + c_1$

$$@\ x = 0,\ \frac{dv}{dx} = 0 \Rightarrow EI(0) = -PL(0) + P\frac{(0)^2}{2} + c_1 \Rightarrow c_1 = 0$$

Integrating twice: $EIv = -\frac{PLx^2}{2} + P\frac{x^3}{6} + c_2$

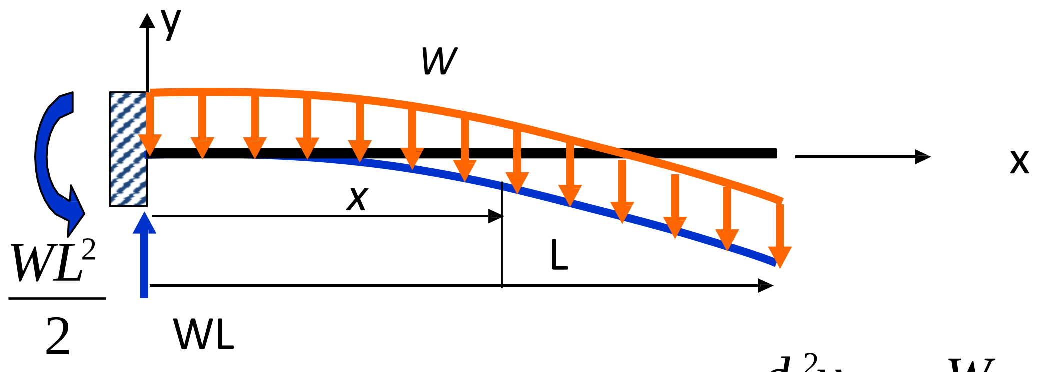
$$@\ x = 0,\ v = 0 \Rightarrow EI(0) = -\frac{PL}{2}(0)^2 + P\frac{(0)^3}{6} + c_2 \Rightarrow c_2 = 0$$

$$EIv = -\frac{PLx^2}{2} + P\frac{x^3}{6}$$

$$@\ x = L \Rightarrow v = v_{max}$$

$$EIv_{max} = -\frac{PL\,L^2}{2} + P\frac{L^3}{6} = -\frac{PL^3}{3} \Rightarrow v_{max} = -\frac{PL^3}{3EI}$$

EXAMPLE 7.6



$$M = -\frac{W}{2}(L-x)^2$$

$$EI\frac{d^2v}{dx^2} = M$$

@ x $$EI\frac{d^2v}{dx^2} = -\frac{W}{2}(L-x)^2$$

Integrating once $$EI\frac{dv}{dx} = \frac{W}{2}\frac{(L-x)^3}{3} + c_1$$

@ x = 0 $$\frac{dv}{dx} = 0 \Rightarrow EI(0) = \frac{W}{2}\frac{(L-0)^3}{3} + c_1 \Rightarrow \boxed{c_1 = -\frac{WL^3}{6}}$$

$$\therefore \boxed{EI\frac{dv}{dx} = \frac{W}{6}(L-x)^3 - \frac{WL^3}{6}}$$

EXAMPLE 7.6 (Con.)

Integrating twice $EIv = -\frac{W}{6}\frac{(L-x)^4}{4} - \frac{WL^3}{6}x + c_2$

@ x = 0 $\quad v = 0 \quad \Rightarrow \quad EI(0) = -\frac{W}{6}\frac{(L-0)^4}{4} - \frac{WL^3}{6}(0) + c_2 \Rightarrow c_2 = \frac{WL^4}{24}$

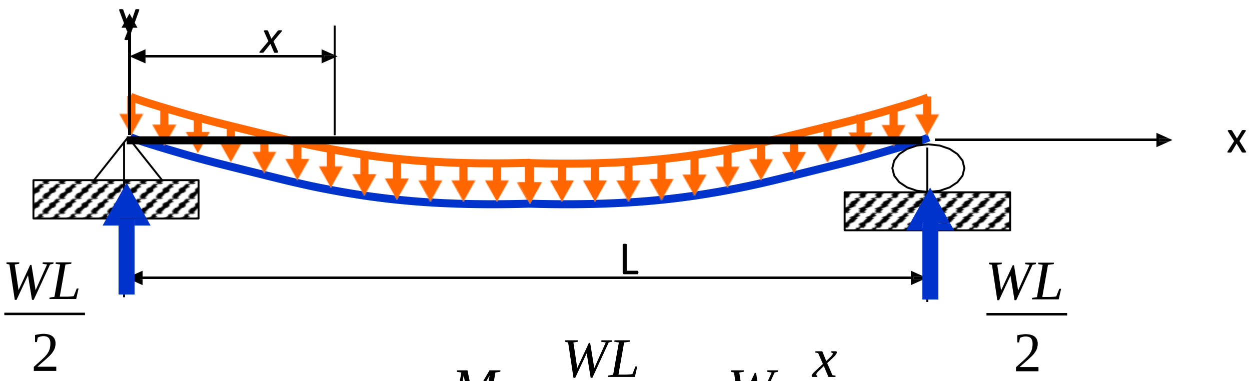
$$EIv = -\frac{W}{24}(L-x)^4 - \frac{WL^3}{6}x + \frac{WL^4}{24}$$

Max. occurs @ x = L

$$EIv_{\max} = -\frac{W L^4}{6} + \frac{WL^4}{24} = -\frac{WL^4}{8} \Rightarrow v_{\max} = -\frac{WL^4}{8EI}$$

$$\Delta_{\max} = \frac{WL^4}{8EI}$$

EXAMPLE 7.7



$$M = \frac{WL}{2}x - Wx\frac{x}{2}$$

$$EI\frac{d^2v}{dx^2} = \frac{WL}{2}x - W\frac{x^2}{2}$$

Integrating

$$EI\frac{dv}{dx} = \frac{WL}{2}\frac{x^2}{2} - \frac{W}{2}\frac{x^3}{3} + c_1$$

Since the beam is symmetric $\quad @\ x = \frac{L}{2} \quad \frac{dv}{dx} = 0$

$$@\ x = \frac{L}{2} \qquad EI(0) = \frac{WL}{2}\frac{\left(\frac{L}{2}\right)^2}{2} - \frac{W}{2}\frac{\left(\frac{L}{2}\right)^3}{3} + c_1 \Rightarrow \boxed{c_1 = -\frac{WL^3}{24}}$$

$$\therefore \quad \boxed{EI\frac{dv}{dx} = \frac{WL}{4}x^2 - \frac{W}{6}x^3 - \frac{WL^3}{24}}$$

EXAMPLE 7.7 (Con.)

Integrating $EIv = \frac{WL}{4}.\frac{x^3}{3} - \frac{W}{6}.\frac{x^4}{4} - \frac{WL^3}{24}x + c_2$

$$EI(0) = \frac{WL}{4}.\frac{(0)^3}{3} - \frac{W}{6}.\frac{(0)^4}{4} - \frac{WL^3}{24}(0) + c_2$$

@ x = 0 v = 0 $\Rightarrow EI(0) = \frac{WL}{4}.\frac{(0)^3}{3} - \frac{W}{6}.\frac{(0)^4}{4} - \frac{WL^3}{24}(0) + c_2$

$\Rightarrow$ $c_2 = 0$

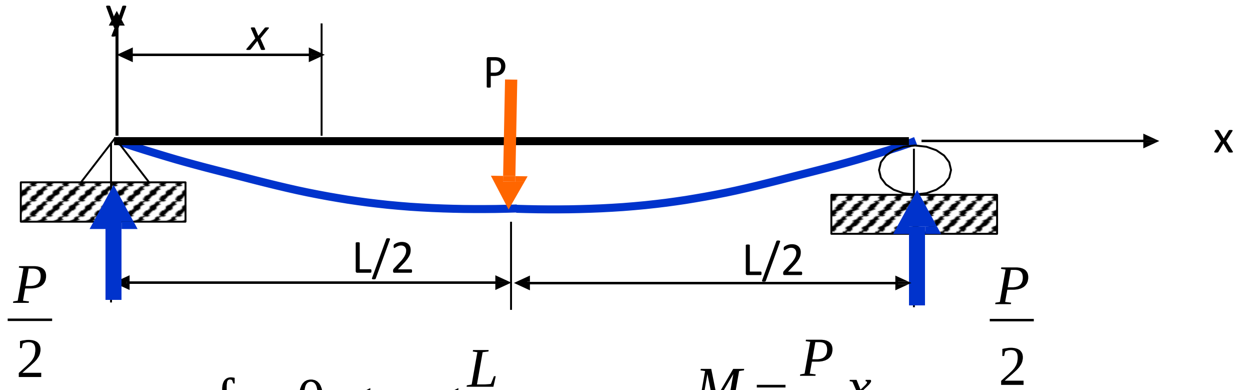
$$\therefore \quad EIv = \frac{WL}{12}x^3 - \frac{W}{24}x^4 - \frac{WL^3}{24}x$$

Max. occurs @ x = L /2

$$EIv_{max} = -\frac{5WL^4}{384}$$

$$\Delta_{max} = \frac{5WL^4}{384EI}$$

EXAMPLE 7.8



$$\text{for } 0 < x < \frac{L}{2} \qquad M = \frac{P}{2}x$$

$$EI\frac{d^2v}{dx^2} = \frac{P}{2}x \qquad \text{for } 0 < x < \frac{L}{2}$$

Integrating

$$EI\frac{dv}{dx} = \frac{P}{2}\frac{x^2}{2} + c_1$$

Since the beam is symmetric $\quad @\ x = \frac{L}{2} \quad \frac{dv}{dx} = 0$

$$@\ x = \frac{L}{2} \qquad EI(0) = \frac{P}{2}\frac{\left(\frac{L}{2}\right)^2}{2} + c_1 \Rightarrow \boxed{c_1 = -\frac{PL^2}{16}}$$

$$\boxed{\therefore \quad EI\frac{dv}{dx} = \frac{P}{4}x^2 - \frac{PL^2}{16}}$$

EXAMPLE 7.8 (Con.)

Integrating $$EIv = \frac{P}{4}\frac{x^3}{3} - \frac{PL^2}{16}x + c_2$$

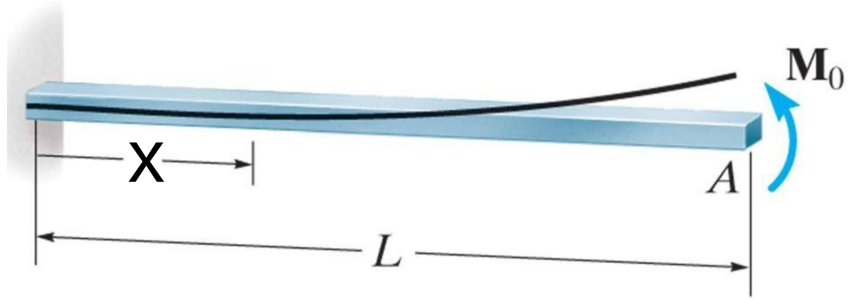
@ x = 0 v = 0 $$\Rightarrow EI(0) = \frac{P}{4}\frac{(0)^3}{3} - \frac{PL^2}{16}(0) + c_2 \quad \Rightarrow \boxed{c_2 = 0}$$

$$\boxed{\therefore EIv = \frac{P}{12}\frac{x^3}{} - \frac{PL^2}{16}x}$$

Max. occurs @ x = L /2 $$\boxed{EIv_{max} = -\frac{PL^3}{48}}$$

$$\boxed{\Delta_{max} = \frac{PL^3}{48EI}}$$

EXAMPLE 7.9



$$EI\frac{d^2v}{dx^2} = M_0 \quad (1)$$

$$EI\frac{dv}{dx} = M_0x + C_1 \quad (2)$$

$$EIv = \frac{M_0x^2}{2} + C_1x + C_2 \quad (3)$$

Using the boundary conditions $dv/dx = 0$ at $x = 0$ and $v = 0$ at $x = 0$, then $C_1 = C_2 = 0$. Substituting these results into Eqs. (2) and (3) with $\theta = dv/dx$, we get

$$\theta = \frac{M_0x}{EI}$$

$$v = \frac{M_0x^2}{2EI} \qquad \textit{Ans.}$$

Slope and Displacement by the Moment-Area Method

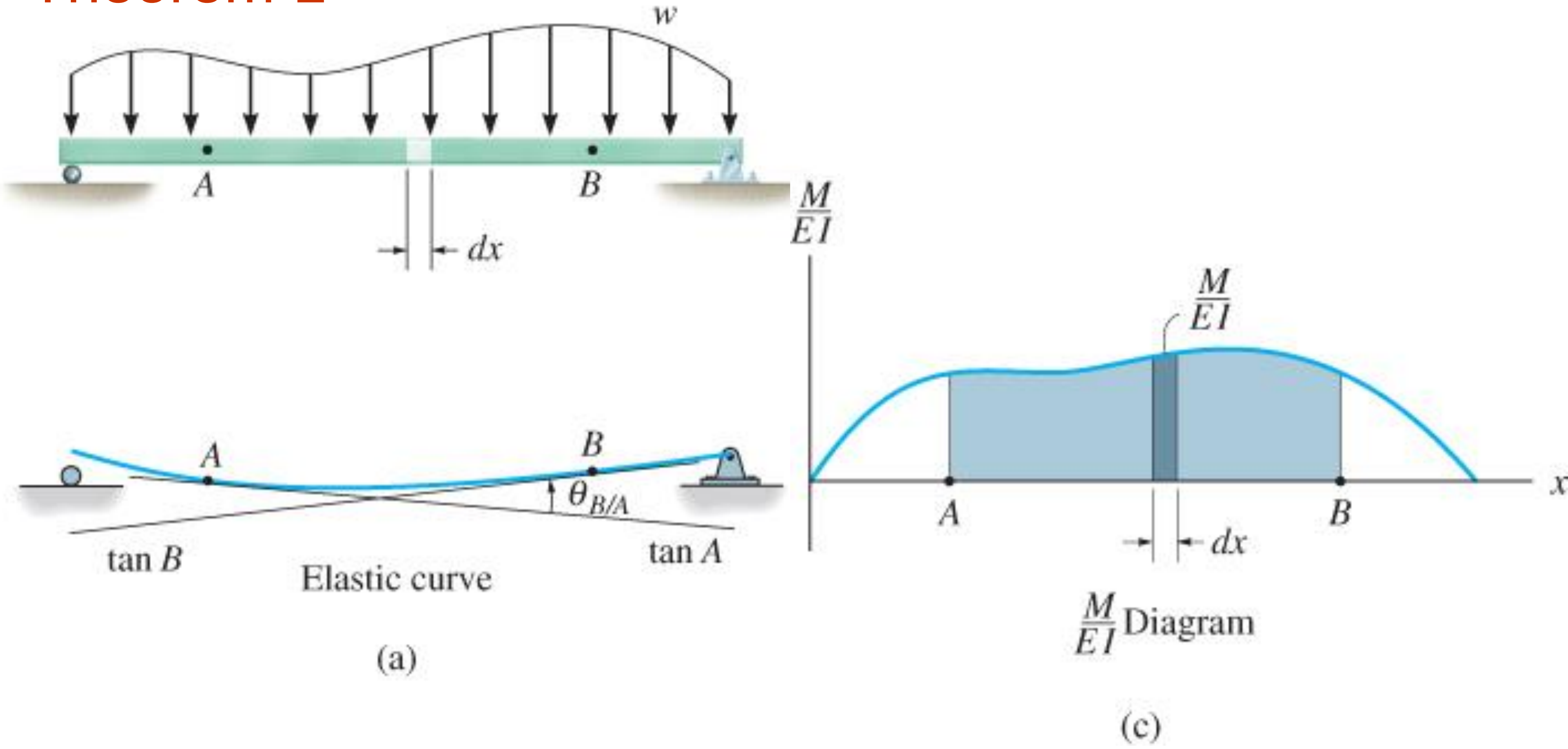
- Assumptions:
  - beam is initially straight,
  - is elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and
  - deformations are caused by bending.

Theorem 1

- The angle between the tangents at any two pts on the elastic curve equals the area under the *M/EI* diagram between these two pts.

$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx \qquad (7\text{-}19)$$

Theorem 1

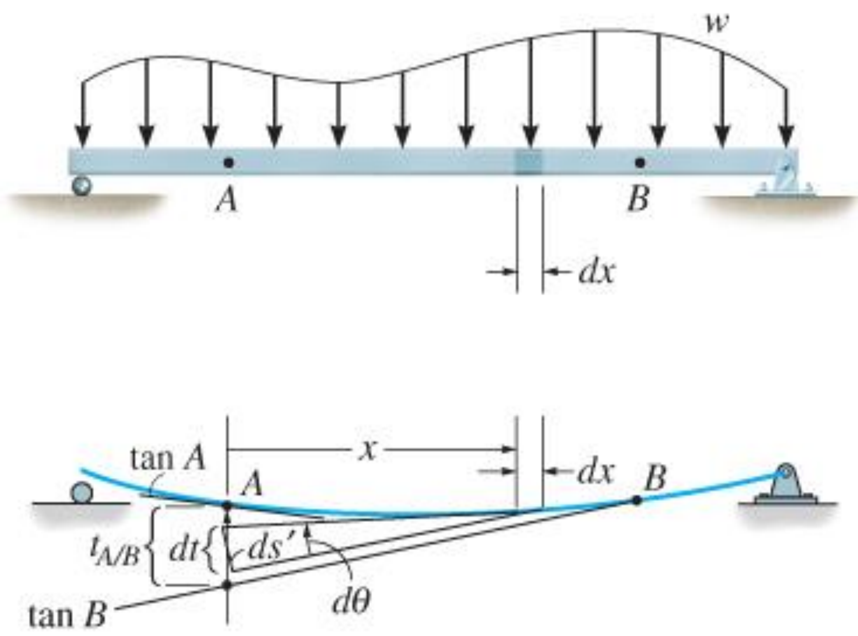


w
A
B
dx
A
B
$\theta_{B/A}$
tan B
tan A
Elastic curve
(a)
$\frac{M}{EI}$
$\frac{M}{EI}$
x
A
B
dx
$\frac{M}{EI}$ Diagram
(c)

Theorem 2

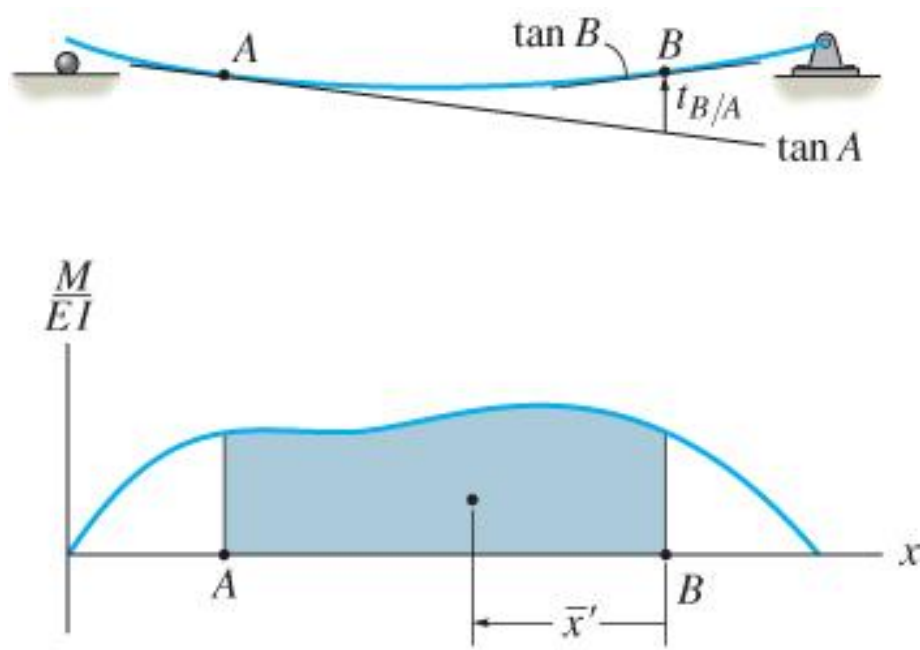
- The vertical deviation of the tangent at a pt (A) on the elastic curve w.r.t. the tangent extended from another pt (B) equals the moment of the area under the ME/I diagram between these two pts (A and B).
- This moment is computed about pt (A) where the vertical deviation ($t_{A/B}$) is to be determined.

Theorem 2



w
A
B
dx
tan A
x
A
dx
B
$t_{A/B}$
dt
ds′
dθ
tan B

(a)



A
tan B
B
$t_{B/A}$
tan A
$\frac{M}{EI}$
x
A
B
$\bar{x}'$

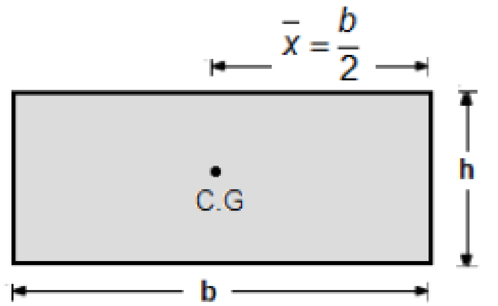
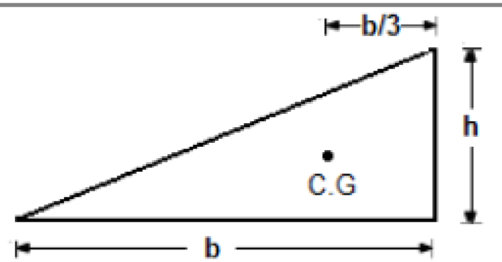
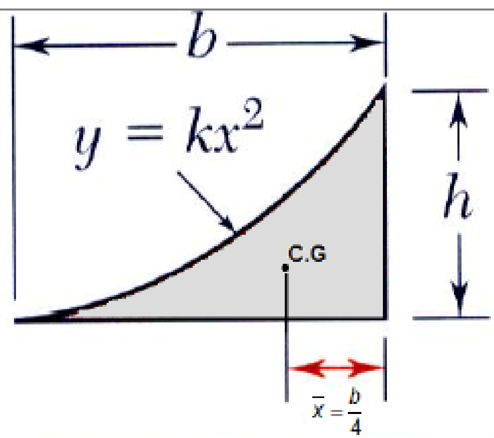
(c)

Procedure for analysis

M/EI Diagram

- Determine the support reactions and draw the beam's *M/EI* diagram.
- If the beam is loaded with concentrated forces, the *M/EI* diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to compute.
- If the loading consists of a series of distributed loads, the *M/EI* diagram will consist of parabolic or perhaps higher-order curves, and we use the table on the inside front cover to locate the area and centroid under each curve.

7.3 SLOPE & DISPLACEMENT BY THE MOMENT-AREA METHOD

Shape	BM Diagram	Area	Distance from C.G
1. Rectangle		$A = bh$	$\bar{x} = \frac{b}{2}$
2. Triangle			$\bar{x} = \frac{b}{3}$
3. Parabola			$\bar{x} = \frac{b}{4}$

Procedure for analysis

Elastic curve

- Draw an exaggerated view of the beam's elastic curve.
- Recall that pts of zero slope and zero displacement always occur at a fixed support, and zero displacement occurs at all pin and roller supports.
- If it is difficult to draw the general shape of the elastic curve, use the moment (M/EI) diagram.
- Realize that when the beam is subjected to a +ve moment, the beam bends concave up, whereas -ve moment bends the beam concave down.

Procedure for analysis

Elastic curve

- An inflection pt or change in curvature occurs when the moment if the beam (or M/EI) is zero.
- The unknown displacement and slope to be determined should be indicated on the curve.
- Since moment-area theorems apply only between two tangents, attention should be given as to which tangents should be constructed so that the angles or deviations between them will lead to the solution of the problem.
- The tangents at the supports should be considered, since the beam usually has zero displacement and/or zero slope at the supports.

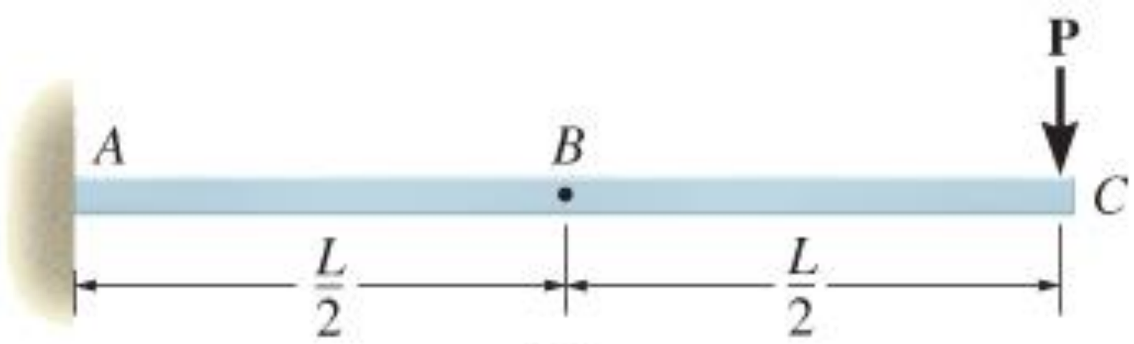
Procedure for analysis

Moment-area theorems

- Apply Theorem 1 to determine the angle between any two tangents on the elastic curve and Theorem 2 to determine the tangential deviation.
- The algebraic sign of the answer can be checked from the angle or deviation indicated on the elastic curve.
- A positive $\theta_{B/A}$ represents a counterclockwise rotation of the tangent at B w.r.t. tangent at A, and a +ve $t_{B/A}$ indicates that pt B on the elastic curve lies above the extended tangent from pt A.

EXAMPLE 7.10

Determine the slope of the beam shown at pts B and C. EI is constant.



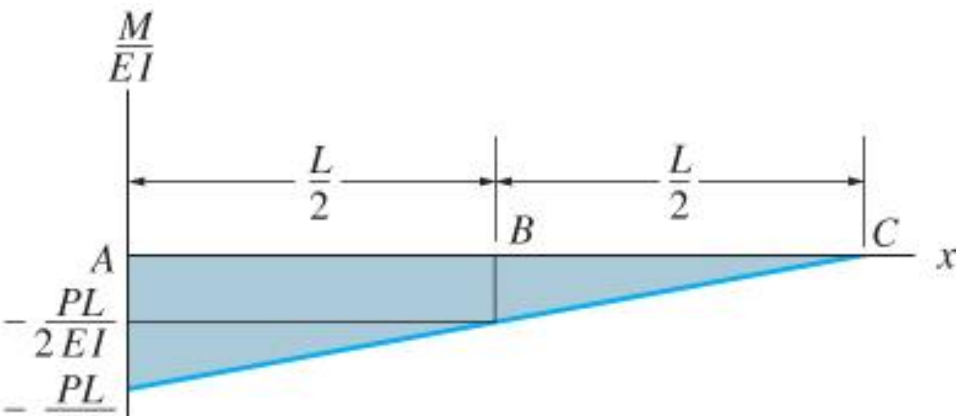
(a)

EXAMPLE 7.10 (Con.)

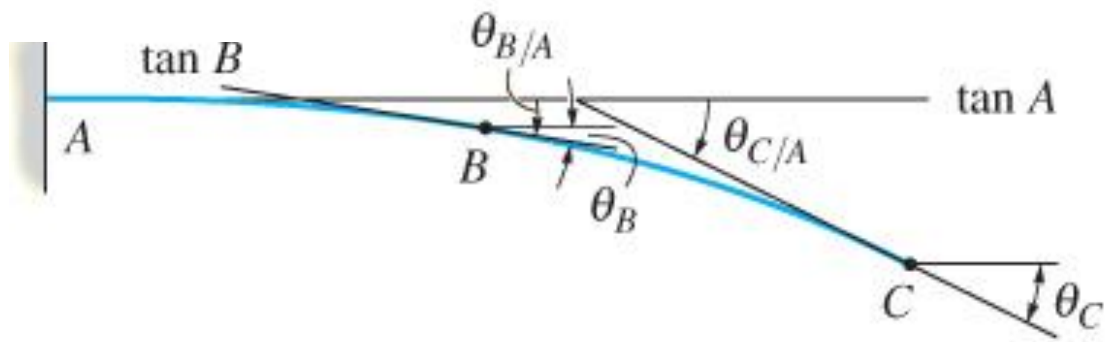
M/EI diagram: See below.

Elastic curve:

The force **P** causes the beam to deflect as shown.



(b)



(c)

EXAMPLE 7.10 (Con.)

Elastic curve:

The tangents at B and C are indicated since we are required to find B and C. Also, the tangent at the support (A) is shown. This tangent has a known zero slope. By construction, the angle between tan A and tan B, $\theta_{B/A}$, is equivalent to θ_B, or

$$\theta_B = \theta_{B/A} \quad \text{and} \quad \theta_C = \theta_{C/A}$$

EXAMPLE 7.10 (Con.)

Moment-area theorem:

Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between pts A and B, that is,

$$\theta_B = \theta_{B/A} = \left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) + \frac{1}{2}\left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)$$

$$= -\frac{3PL^2}{8EI}$$

EXAMPLE 7.10 (Con.)

Moment-area theorem:

The negative sign indicates that angle measured from tangent at A to tangent at B is clockwise. This checks, since beam slopes downward at B.

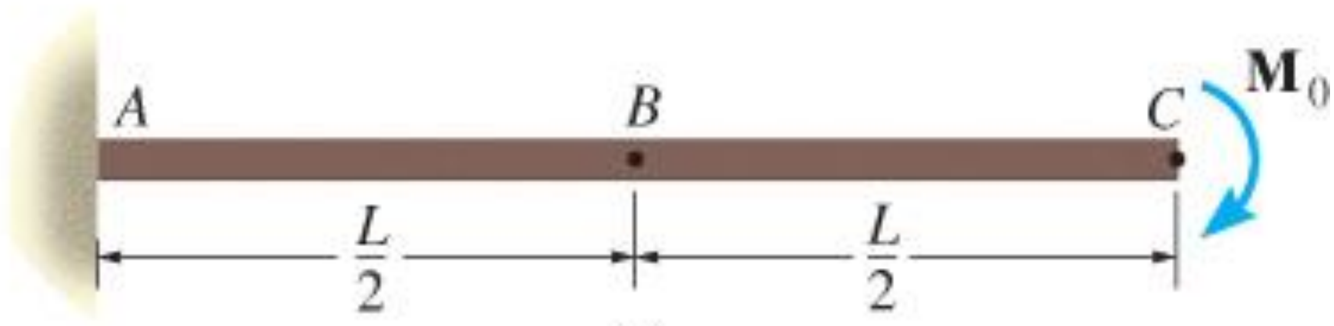
Similarly, area under the M/EI diagram between pts A and C equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{PL}{EI}\right)L$$

$$= -\frac{PL^2}{2EI}$$

EXAMPLE 7.11

Determine the displacement of pts B and C of beam shown. EI is constant.



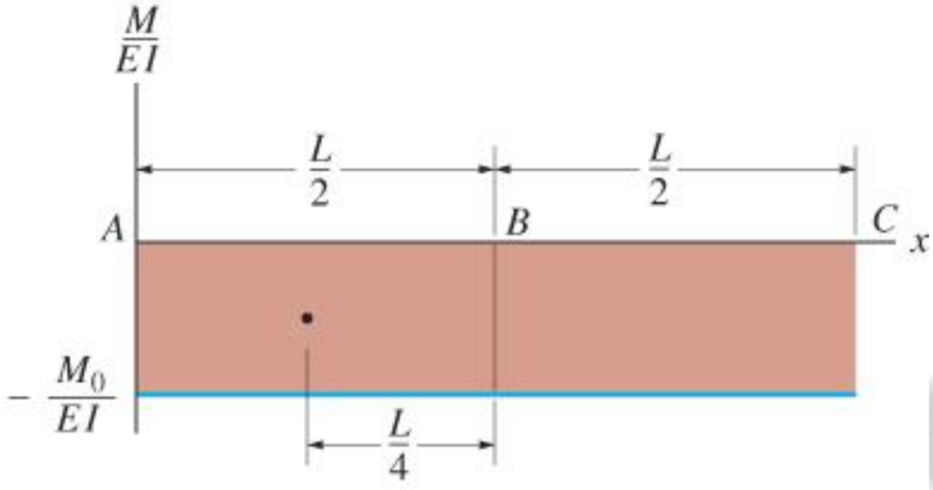
(a)

EXAMPLE 7.11 (Con.)

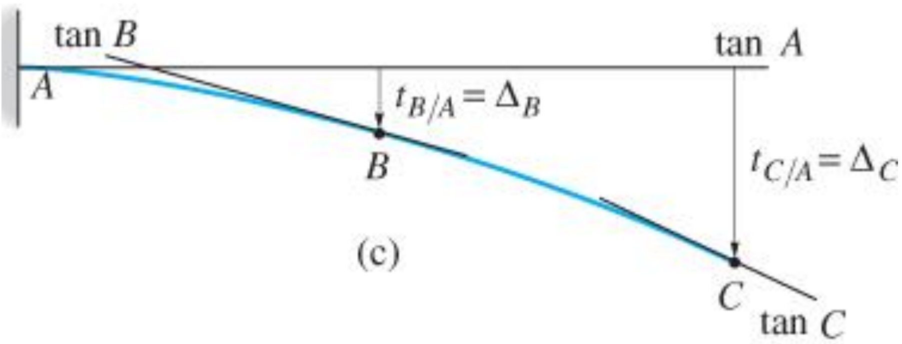
M/EI diagram: See below.

Elastic curve:

The couple moment at C cause the beam to deflect as shown.



(b)



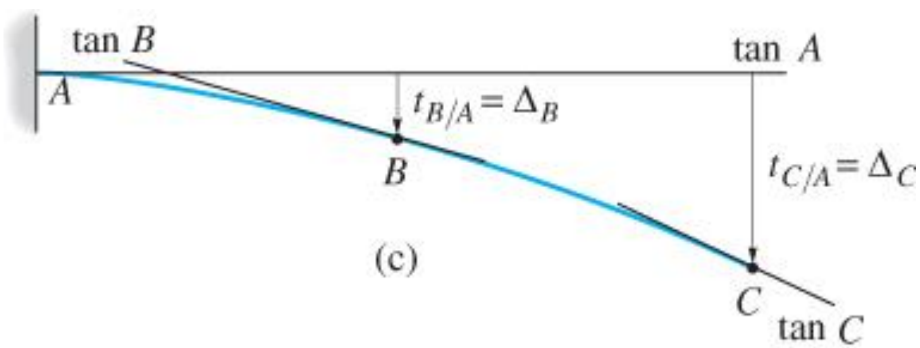
(c)

EXAMPLE 7.11 (Con.)

Elastic curve:

The required displacements can be related directly to deviations between the tangents at B and A and C and A. Specifically, Δ_B is equal to deviation of tan A from tan B,

$$\Delta_B = t_{B/A} \qquad \Delta_C = t_{C/A}$$



EXAMPLE 7.11 (Con.)

Moment-area theorem:

Applying Theorem 2, $t_{B/A}$ is equal to the moment of the shaded area under the M/EI diagram between A and B computed about pt B, since this is the pt where tangential deviation is to be determined. Hence,

$$\Delta_B = t_{B/A} = \left(\frac{L}{4}\right)\left[\left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\right] = -\frac{M_0 L^2}{8EI}$$

EXAMPLE 7.11 (Con.)

Moment-area theorem:

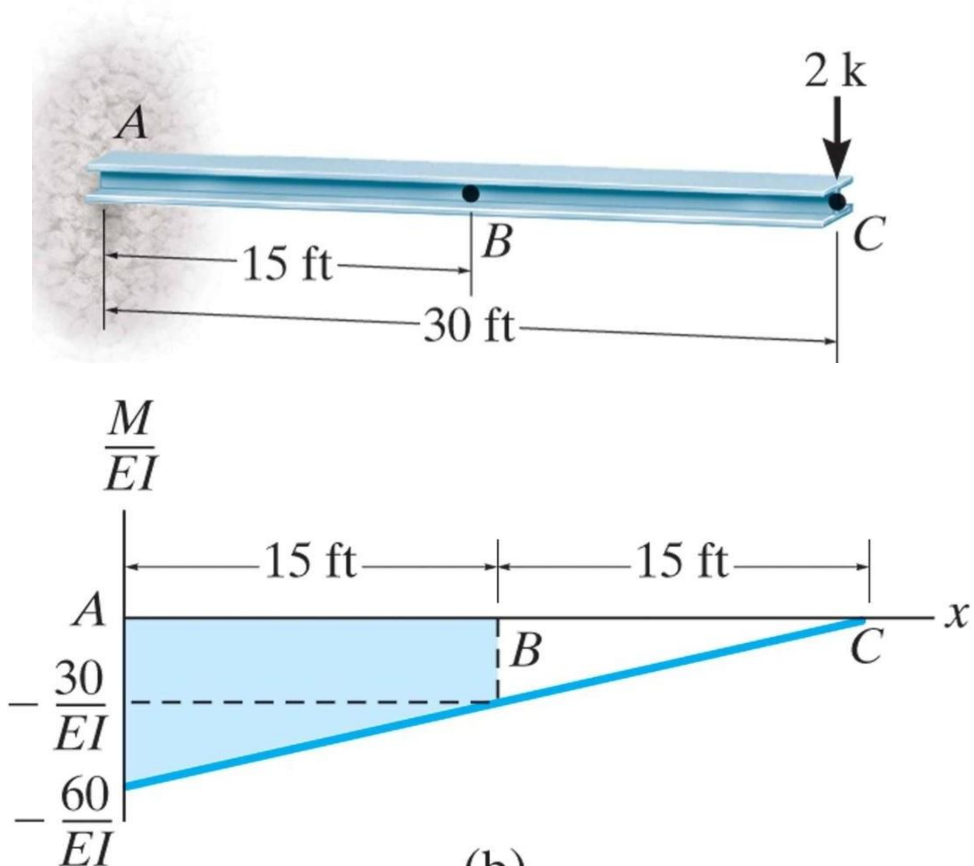
Likewise, for $t_{C/A}$ we must determine the moment of the area under the entire M/EI diagram from A to C about pt C. We have

$$\Delta_C = t_{C/A} = \left(\frac{L}{2}\right)\left[\left(-\frac{M_0}{EI}\right)(L)\right] = -\frac{M_0 L^2}{2EI}$$

Since both answers are –ve, they indicate that pts B and C lie below the tangent at A. This checks with the figure.

EXAMPLE 7.12

Determine the slope at points B and C of the beam shown in Fig. Take $E = 29(10^3)$ ksi and $I = 600 \text{ in}^4$.

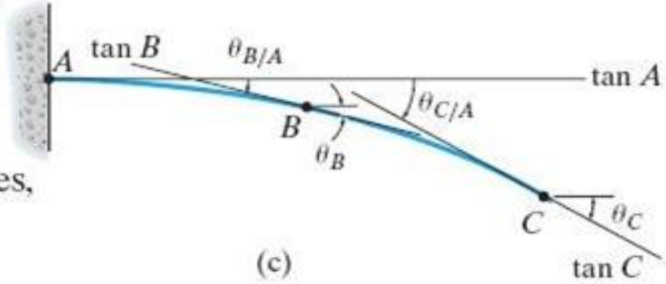


(b)

EXAMPLE 7.12 (Con.)

$$\theta_B = \theta_{B/A} = -\left(\frac{30\ \text{k}\cdot\text{ft}}{EI}\right)(15\ \text{ft}) - \frac{1}{2}\left(\frac{60\ \text{k}\cdot\text{ft}}{EI} - \frac{30\ \text{k}\cdot\text{ft}}{EI}\right)(15\ \text{ft})$$

$$= -\frac{675\ \text{k}\cdot\text{ft}^2}{EI}$$



(c)

Substituting numerical data for E and I, and converting feet to inches, we have

$$\theta_B = \frac{-675\ \text{k}\cdot\text{ft}^2(144\ \text{in}^2/1\ \text{ft}^2)}{29(10^3)\ \text{k/in}^2(600\ \text{in}^4)}$$

$$= -0.00559\ \text{rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle is measured clockwise from A,

In a similar manner, the area under the M/EI diagram between points A and C equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{60\ \text{k}\cdot\text{ft}}{EI}\right)(30\ \text{ft}) = -\frac{900\ \text{k}\cdot\text{ft}^2}{EI}$$

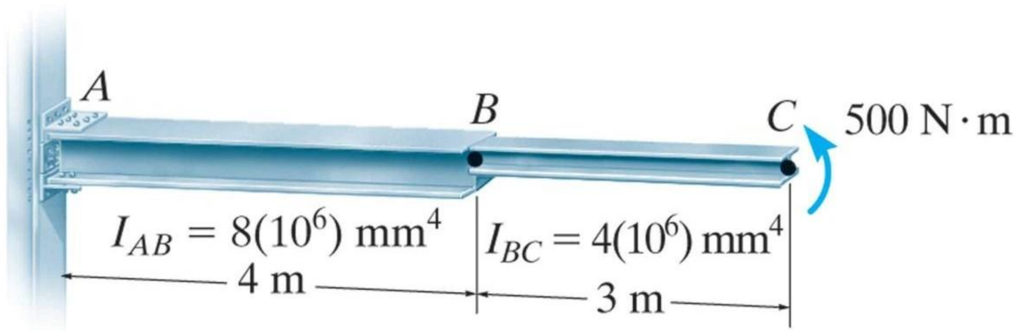
Substituting numerical values for EI, we have

$$\theta_C = \frac{-900\ \text{k}\cdot\text{ft}^2(144\ \text{in}^2/\text{ft}^2)}{29(10^3)\ \text{k/in}^2(600\ \text{in}^4)}$$

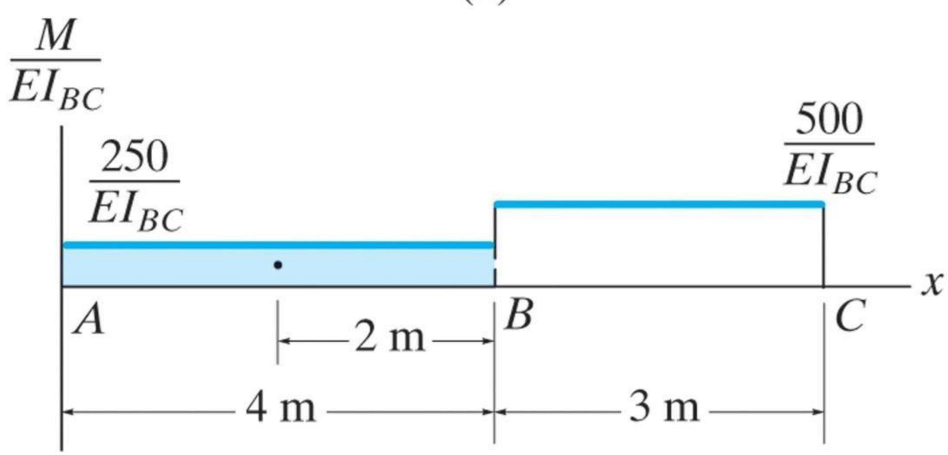
$$= -0.00745\ \text{rad} \qquad \textit{Ans.}$$

EXAMPLE 7.13

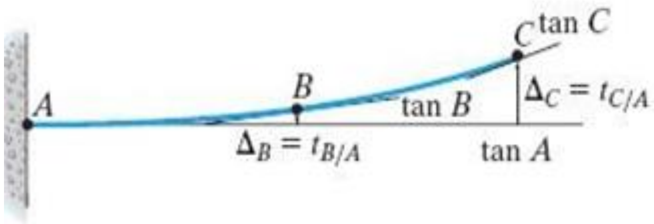
Determine the deflection at points B and C of the beam Values for the moment of inertia of each segment are indicated in the figure. Take E = 200 GPa.



(a)



EXAMPLE 7.13 (Con.)



(c)

$$\Delta_B = t_{B/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](2\ \text{m}) = \frac{2000\ \text{N}\cdot\text{m}^3}{EI_{BC}}$$

Substituting the numerical data yields

$$\Delta_B = \frac{2000\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)\ \text{mm}^4(1\ \text{m}^4/(10^3)^4\ \text{mm}^4)]}$$

$$= 0.0025\ \text{m} = 2.5\ \text{mm.} \qquad \textit{Ans.}$$

Likewise, for $t_{C/A}$ we must compute the moment of the entire M/EI_{BC} diagram from A to C about point C. We have

$$\Delta_C = t_{C/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](5\ \text{m}) + \left[\frac{500\ \text{N}\cdot\text{m}}{EI_{BC}}(3\ \text{m})\right](1.5\ \text{m})$$

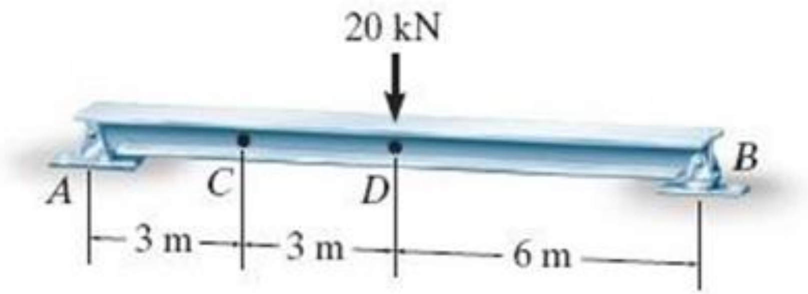
$$= \frac{7250\ \text{N}\cdot\text{m}^3}{EI_{BC}} = \frac{7250\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)(10^{-12})\ \text{m}^4]}$$

$$= 0.00906\ \text{m} = 9.06\ \text{mm} \qquad \textit{Ans.}$$

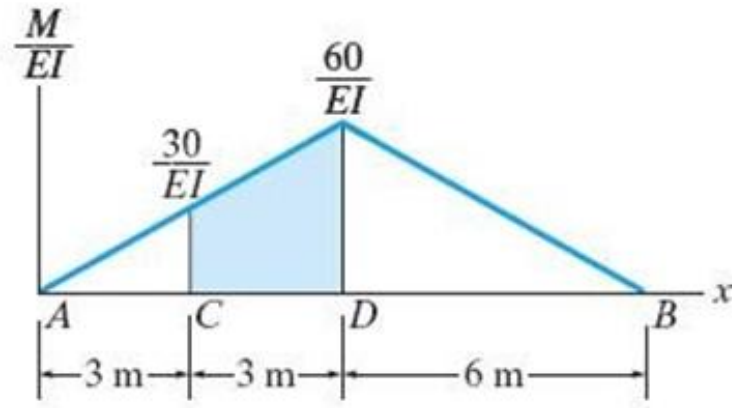
Since both answers are *positive*, they indicate that points B and C lie *above* the tangent at A.

EXAMPLE 7.14

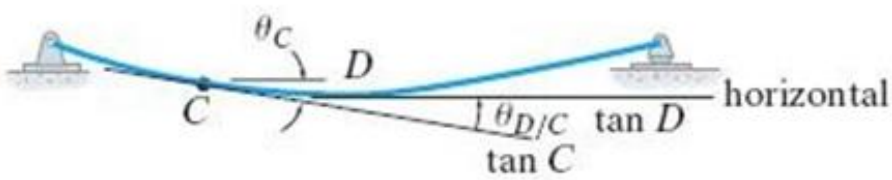
Determine the slope at point C of the beam $E = 200$ GPa, $I = 6(10^6)$ mm^4.



(a)



(b)



(c)

$$\theta_C = \theta_{D/C}$$

EXAMPLE 7.14 (Con.)

$$\theta_C = \theta_{D/C}$$

Moment-Area Theorem. Using Theorem 1, $\theta_{D/C}$ is equal to the shaded area under the *M/EI* diagram between points *C* and *D*. We have

$$\theta_C = \theta_{D/C} = 3\text{ m}\left(\frac{30\text{ kN}\cdot\text{m}}{EI}\right) + \frac{1}{2}(3\text{ m})\left(\frac{60\text{ kN}\cdot\text{m}}{EI} - \frac{30\text{ kN}\cdot\text{m}}{EI}\right)$$

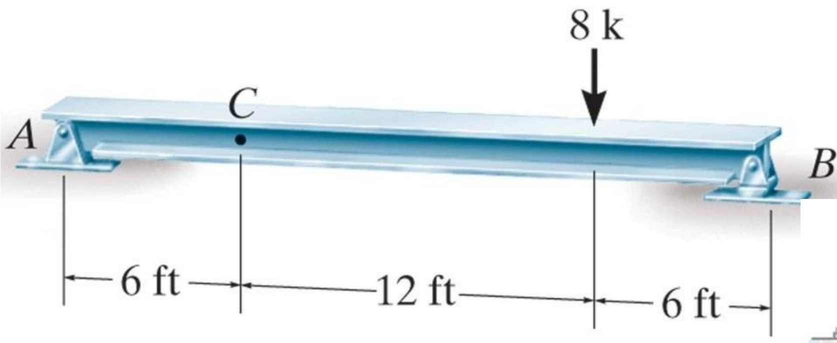
$$= \frac{135\text{ kN}\cdot\text{m}^2}{EI}$$

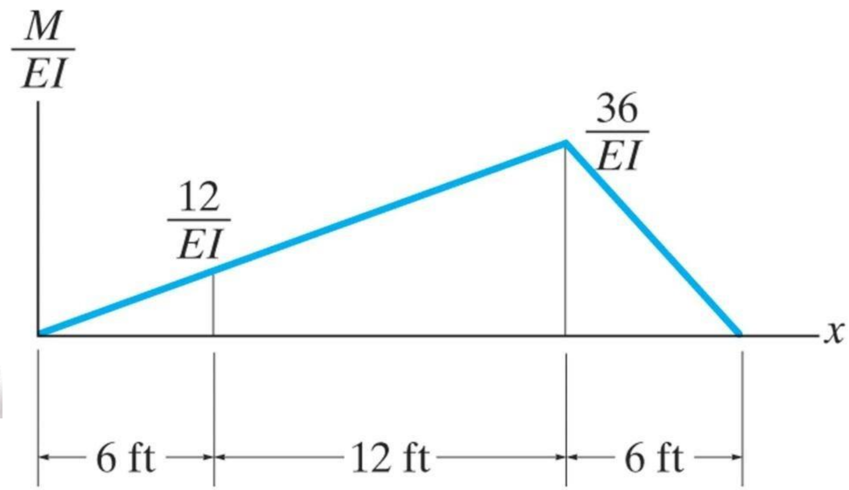
Thus,

$$\theta_C = \frac{135\text{ kN}\cdot\text{m}^2}{[200(10^6)\text{ kN/m}^2][6(10^6)(10^{-12})\text{ m}^4]} = 0.112\text{ rad} \qquad \textit{Ans.}$$

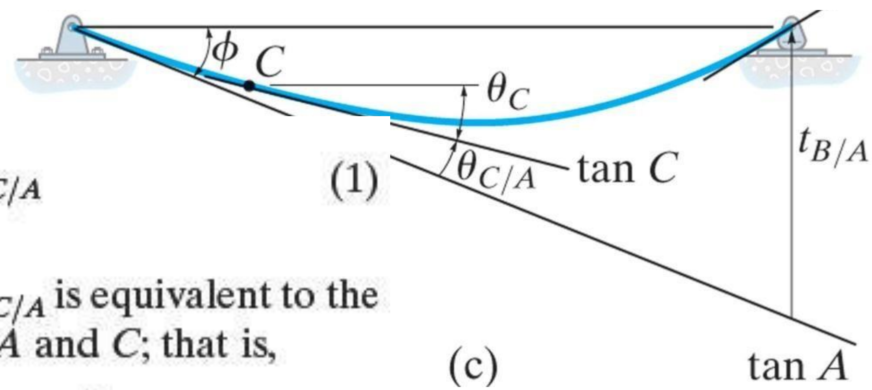
EXAMPLE 7.15

Determine the slope at point C of the beam $E = 29(10^3)$ ksi, $I = 600\text{ in}^4$.





an B

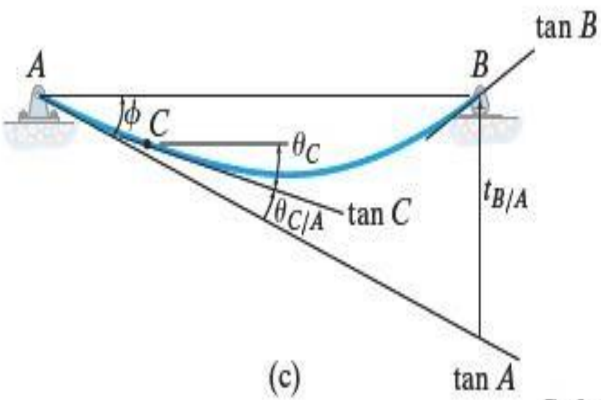


$$\theta_C = \phi - \theta_{C/A} = \frac{t_{B/A}}{24} - \theta_{C/A} \qquad (1)$$

Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(6\text{ ft})\left(\frac{12\text{ k}\cdot\text{ft}}{EI}\right) = \frac{36\text{ k}\cdot\text{ft}^2}{EI}$$

EXAMPLE 7.15 (Con.)



(c)

Fig. 8–18

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B, since this is the point where the tangential deviation is to be determined. We have

$$t_{B/A} = \left[6\text{ ft} + \frac{1}{3}(18\text{ ft})\right]\left[\frac{1}{2}(18\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right] + \frac{2}{3}(6\text{ ft})\left[\frac{1}{2}(6\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{4320\text{ k}\cdot\text{ft}^3}{EI}$$

Substituting these results into Eq. 1, we have

$$\theta_C = \frac{4320\text{ k}\cdot\text{ft}^3}{(24\text{ ft})\,EI} - \frac{36\text{ k}\cdot\text{ft}^2}{EI} = \frac{144\text{ k}\cdot\text{ft}^2}{EI}$$

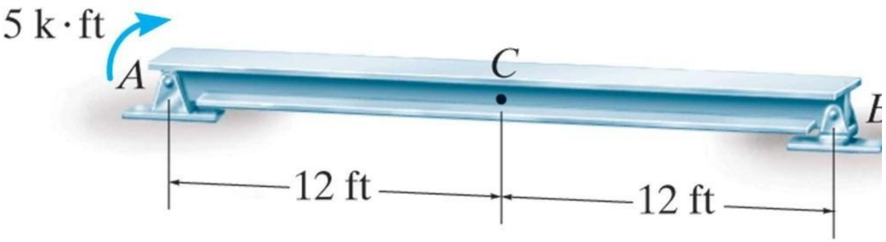
so that

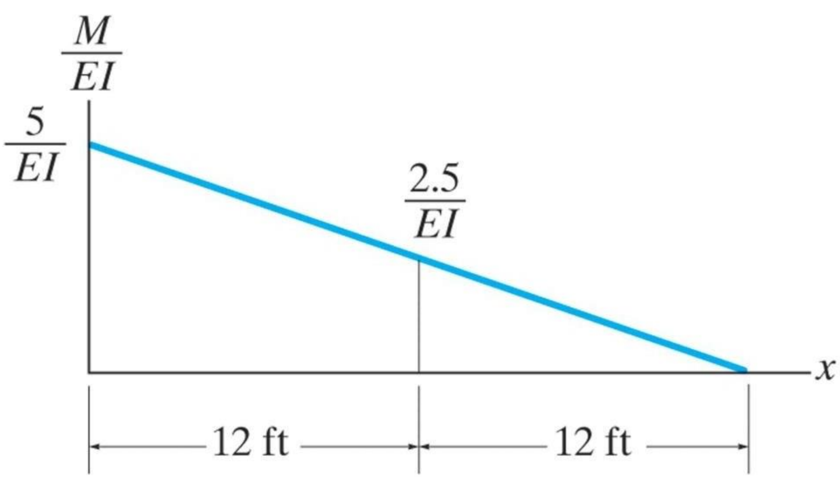
$$\theta_C = \frac{144\text{ k}\cdot\text{ft}^2}{29(10^3)\text{ k/in}^2(144\text{ in}^2/\text{ft}^2)\;600\text{ in}^4(1\text{ ft}^4/(12)^4\text{ in}^4)} = 0.00119\text{ rad}$$

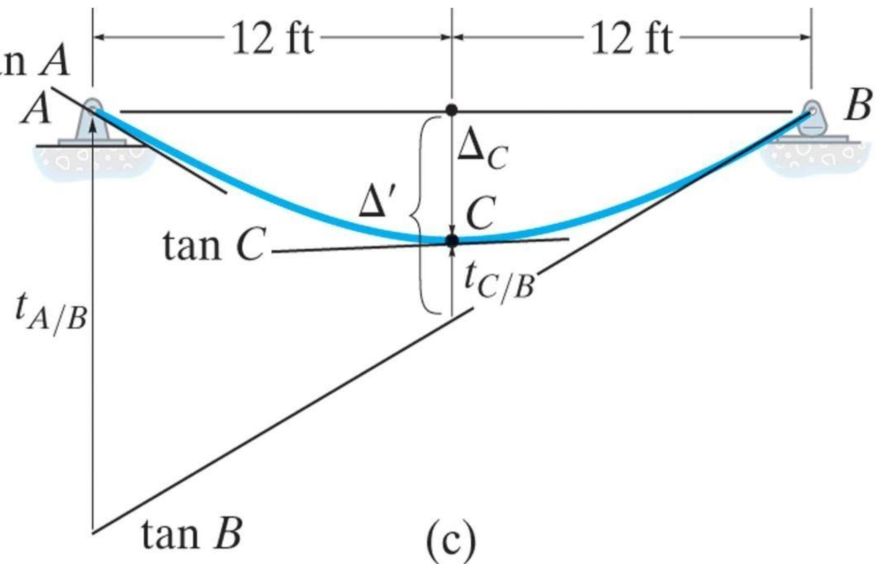
Ans.

EXAMPLE 7.16

Determine the deflection at C of the beam $E = 29(10^3)$ ksi, $I = 21$ in^4.







$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B}$$

EXAMPLE 7.16 (Con.)

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$. Here $t_{A/B}$ is the moment of the M/EI diagram between A and B about point A,

$$t_{A/B} = \left[\frac{1}{3}(24\text{ ft})\right]\left[\frac{1}{2}(24\text{ ft})\left(\frac{5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{480\text{ k}\cdot\text{ft}^3}{EI}$$

and $t_{C/B}$ is the moment of the M/EI diagram between C and B about C.

$$t_{C/B} = \left[\frac{1}{3}(12\text{ ft})\right]\left[\frac{1}{2}(12\text{ ft})\left(\frac{2.5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{60\text{ k}\cdot\text{ft}^3}{EI}$$

Substituting these results into Eq. (1) yields

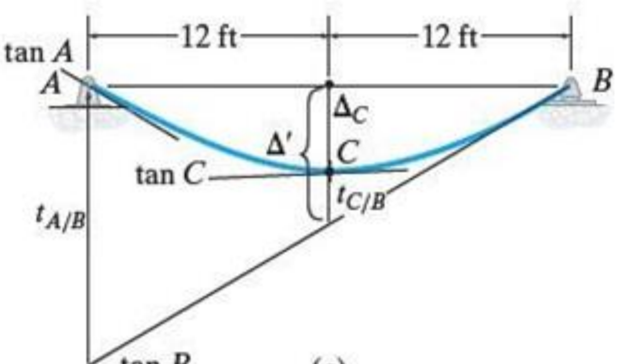
$$\Delta_C = \frac{1}{2}\left(\frac{480\text{ k}\cdot\text{ft}^3}{EI}\right) - \frac{60\text{ k}\cdot\text{ft}^3}{EI} = \frac{180\text{ k}\cdot\text{ft}^3}{EI}$$

Working in units of kips and inches, we have

$$\Delta_C = \frac{180\text{ k}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(21\text{ in}^4)}$$

$$= 0.511\text{ in.}$$

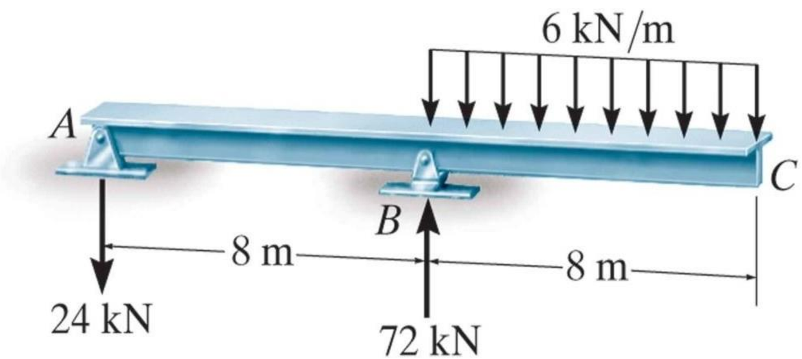
Ans.

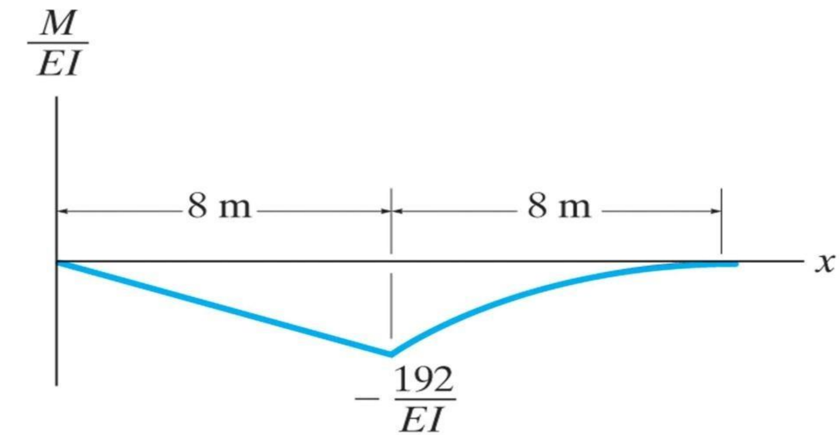


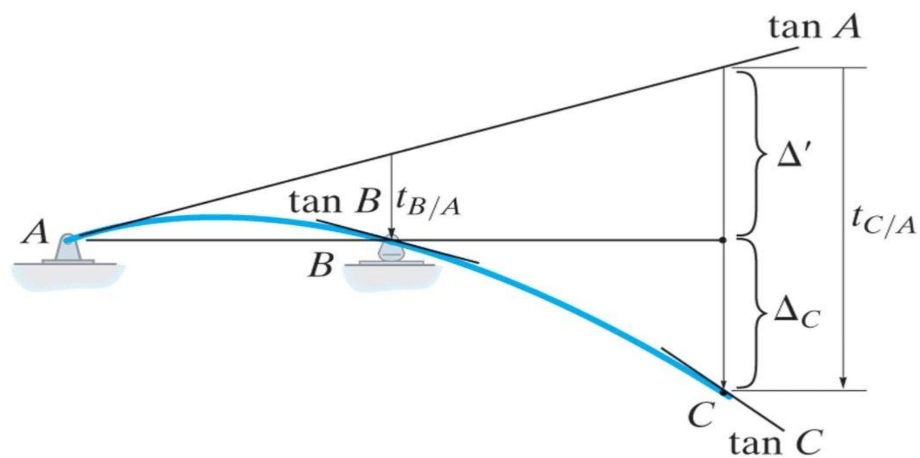
(c)

EXAMPLE 7.17

Determine the deflection at point C of the beam $E = 200$ GPa, $I = 250(10^6)$ mm^4.

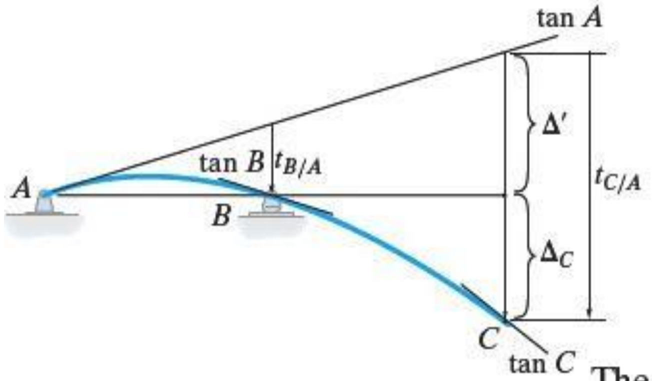






$$\Delta_C = t_{C/A} - 2t_{B/A}$$

EXAMPLE 7.17 (Con.)



(c)

Fig. 8–20

$$t_{C/A} = \left[\frac{3}{4}(8\text{ m})\right]\left[\frac{1}{3}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$+\left[\frac{1}{3}(8\text{ m}) + 8\text{ m}\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$= -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI}$$

The moment of the M/EI diagram between A and B about point B gives

$$t_{B/A} = \left[\frac{1}{3}(8\text{ m})\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] = -\frac{2048\text{ kN}\cdot\text{m}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. (1) yields

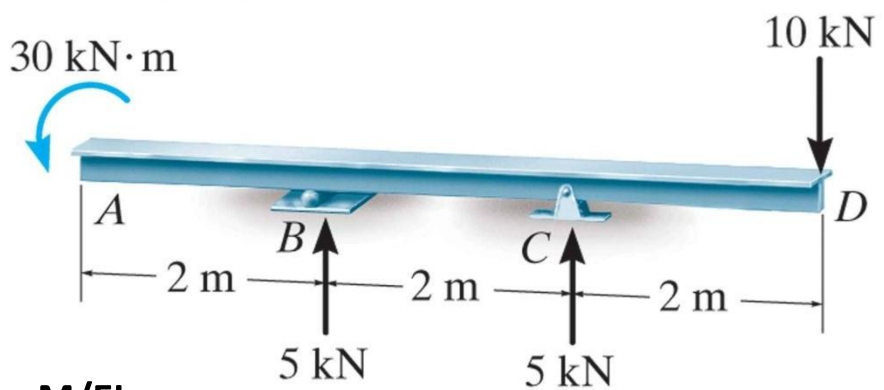
$$\Delta_C = -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI} - 2\left(-\frac{2048\text{ kN}\cdot\text{m}^3}{EI}\right)$$
$$= -\frac{7168\text{ kN}\cdot\text{m}^3}{EI}$$

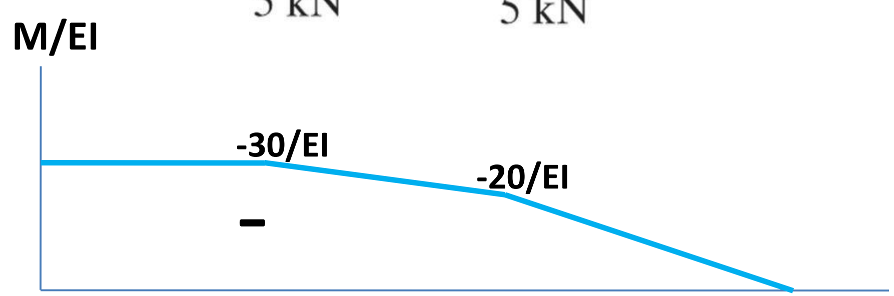
Thus,

$$\Delta_C = \frac{-7168\text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][250(10^6)(10^{-12})\text{ m}^4]}$$
$$= -0.143\text{ m} \qquad \textit{Ans.}$$

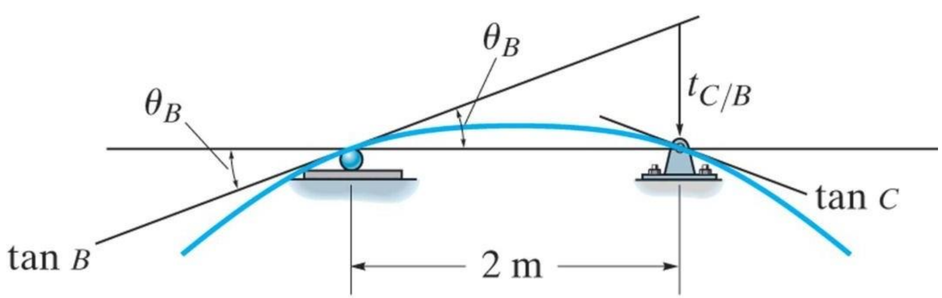
EXAMPLE 7.18

Determine the slope at the roller B of the double overhang beam Take $E = 200$ GPa, $I = 18(10^6)$ mm^4.

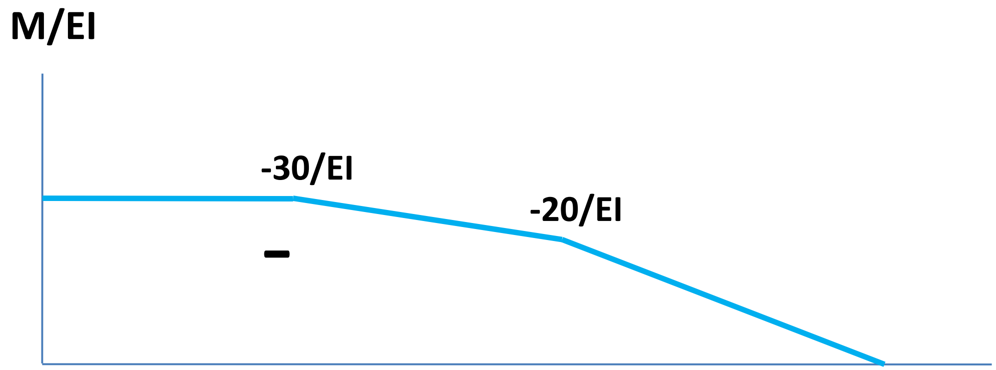




$$\theta_B = \frac{t_{C/B}}{2\text{ m}}$$



EXAMPLE 7.18 (Con.)



Moment Area Theorem. To determine $t_{B/C}$ we apply the moment area theorem by finding the moment of the M/EI diagram between BC about point C. This only involves the shaded area under two of the diagrams in Fig. 8–21b. Thus,

$$t_{C/B} = (1\text{ m})\left[(2\text{ m})\left(\frac{-30\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left(\frac{2\text{ m}}{3}\right)\left[\frac{1}{2}(2\text{ m})\left(\frac{10\text{ kN}\cdot\text{m}}{EI}\right)\right]$$

$$= \frac{53.33\text{ kN}\cdot\text{m}^3}{EI}$$

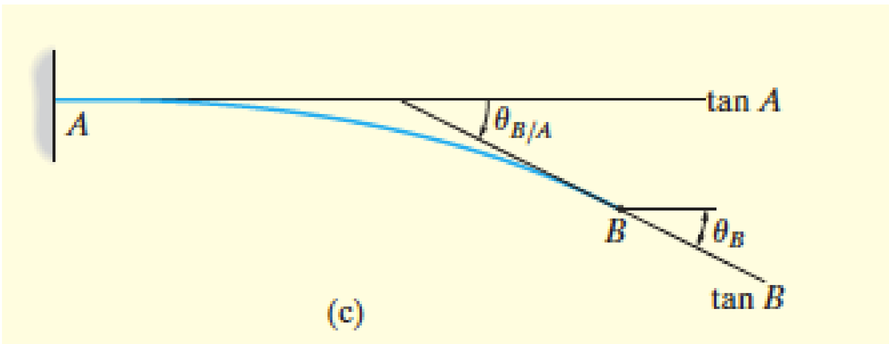
Substituting into Eq. (1),

$$\theta_B = \frac{53.33\text{ kN}\cdot\text{m}^3}{(2\text{ m})[200(10^6)\text{ kN/m}^3][18(10^6)(10^{-12})\text{ m}^4]}$$

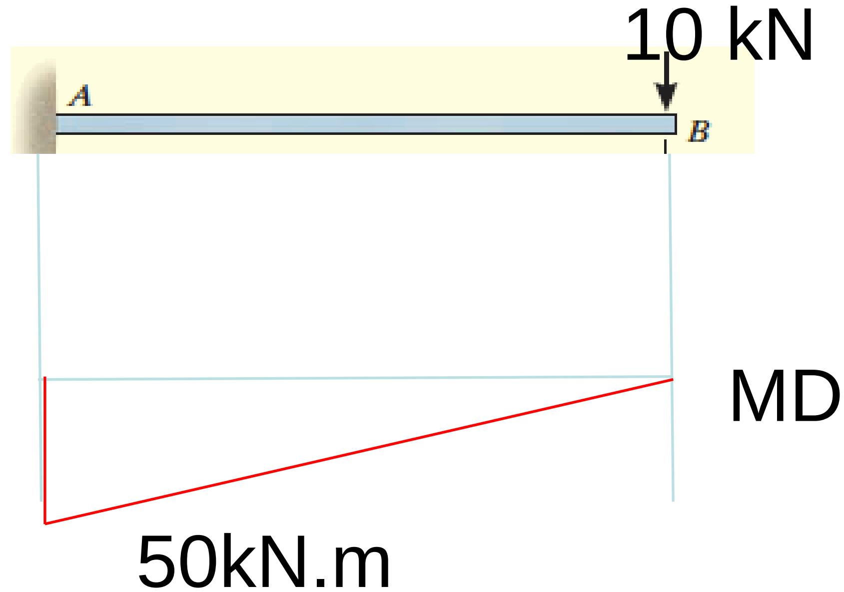
$$= 0.00741\text{ rad}$$

EXAMPLE 7.19

Determine the slope at point B of the 5m long beam shown in Fig. EI=$0.4*10^7 N/m^2$



(c)



The elastic curve is concave downward, since MEI is negative.) The tangent at B is indicated since we are required to find θ_B Also, the tangent at the support (A) is shown. This tangent has known zero slope. By the construction, the angle between tan A and tan B, that is $\theta_{B/A}$, is equivalent to θ_B

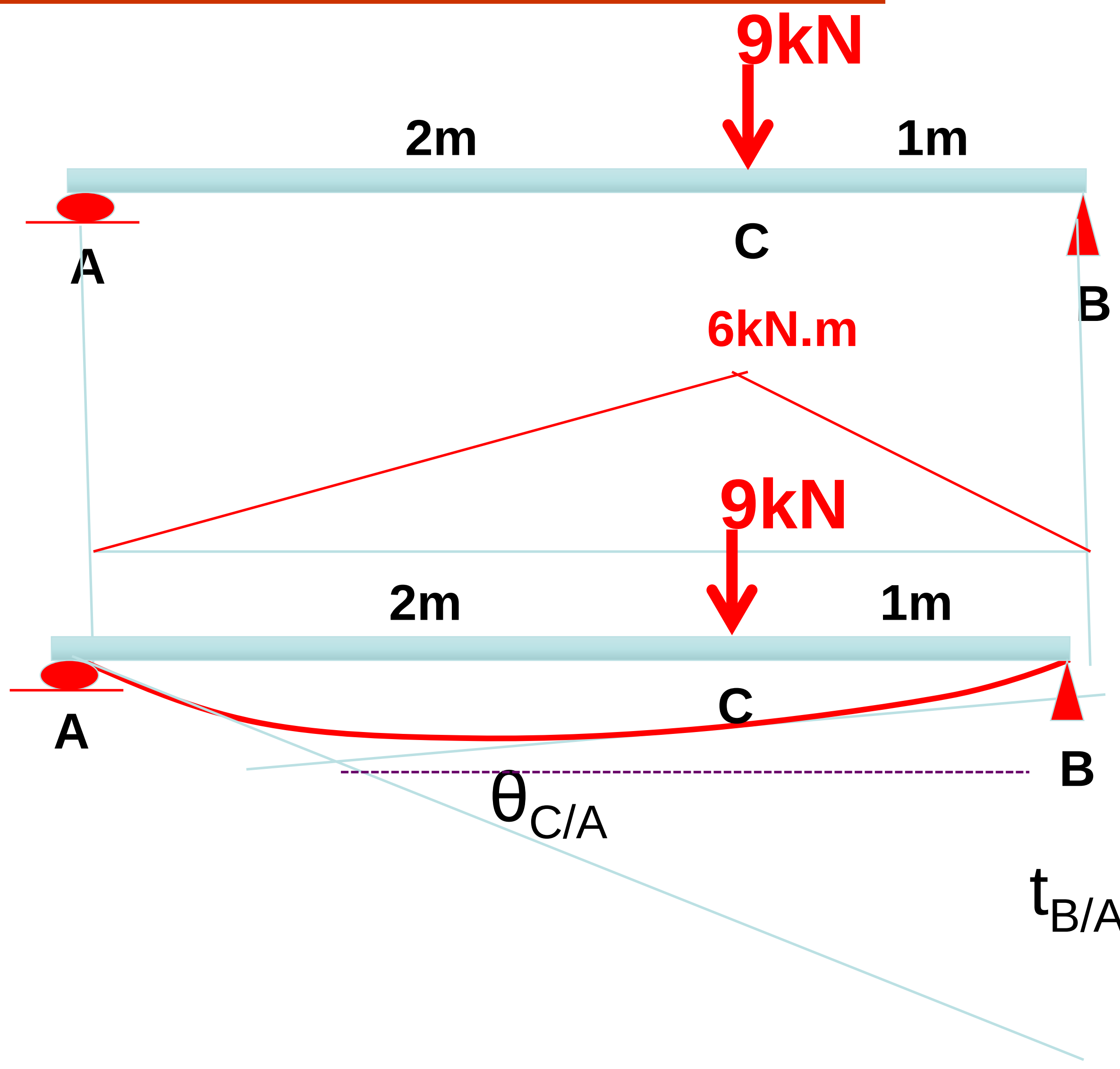
Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M EI diagram between points A and B; that is,

$$q_B = q_{B/A} = (\frac{1}{2}5*(-50)*10^3)/(0.4*10^7) = 0.03125 rad$$ clockwise

EXAMPLE 7.20

Determine slope and Deflection at point C. Use moment area method $EI=62500N.m^2$

Solution



EXAMPLE 7.20 (Con.)

$$t_{B/A} = \frac{1}{2} * 2 * \frac{6}{EI} * (\frac{1}{3} * 2 + 1) + \frac{1}{2} * 1 * \frac{6}{EI} * (\frac{2}{3} * 1)$$

$$= \frac{(10+2) * 10^{3} N.m.m.m}{62500 N.m^{2}} = 192 * 10^{-3}$$

$$t_{C/A} = \frac{1}{2} * 2 * \frac{6}{EI} * (\frac{1}{3} * 2) = \frac{4}{EI} = \frac{4 * 10^{3}}{62500} = 64 * 10^{-3}$$

$$\frac{cc'}{2} = \frac{t_{B/A}}{3} \text{ Þ } cc' = 128 * 10^{-3}$$

$$cc' = dc + t_{C/A} \text{ Þ } dc = 128 * 10^{-3} - 64 * 10^{-3} = 64 * 10^{-3} m$$

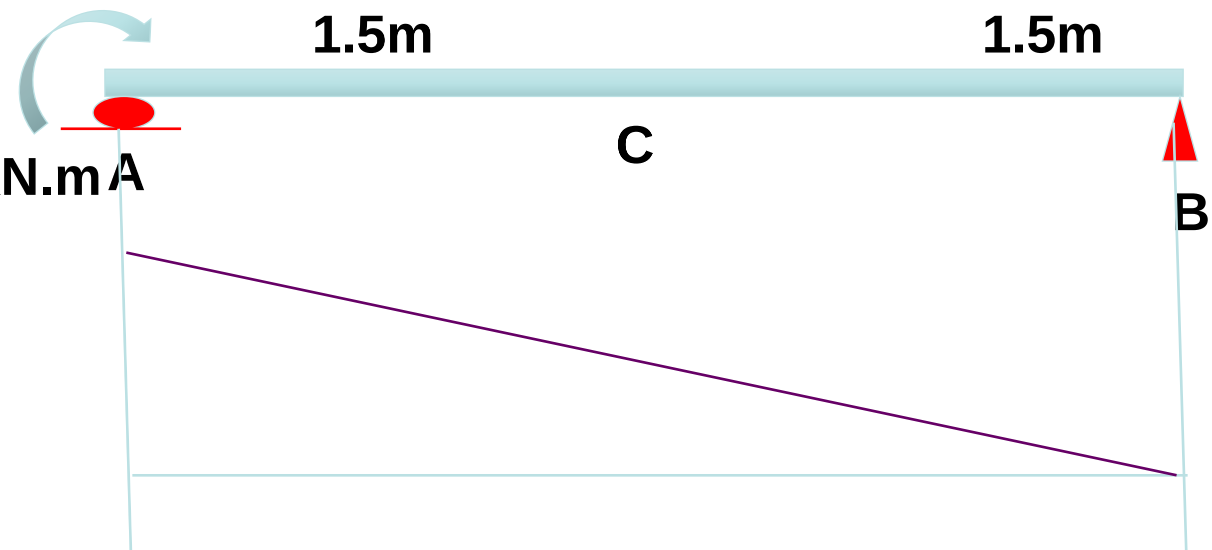
$$q_{C/A} = \frac{1}{2} * 2 * \frac{6}{EI} = \frac{6 * 10^{3} N.m.m}{62500 N.m^{2}} = 96 * 10^{-3} counterclockwise$$

$$q_{A} = \frac{t_{B/A}}{3} = \frac{192 * 10^{-3}}{3} = 64 * 10^{-3} clockwise$$

$$q_{C/A} = q_{C} - q_{A} \text{ Þ } 96 * 10^{-3} = q_{C} - (-64 * 10^{-3}) \text{ Þ } q_{C} = 32 * 10^{-3} rad$$

EXAMPLE 7.21

Determine the displacement at C for the beam shown. EI is 62500N/m².



$$t_{A/B} = \frac{1}{2} * 3 * \frac{12 * 10^3}{62500} * \frac{1}{3} 3 = 0.288$$

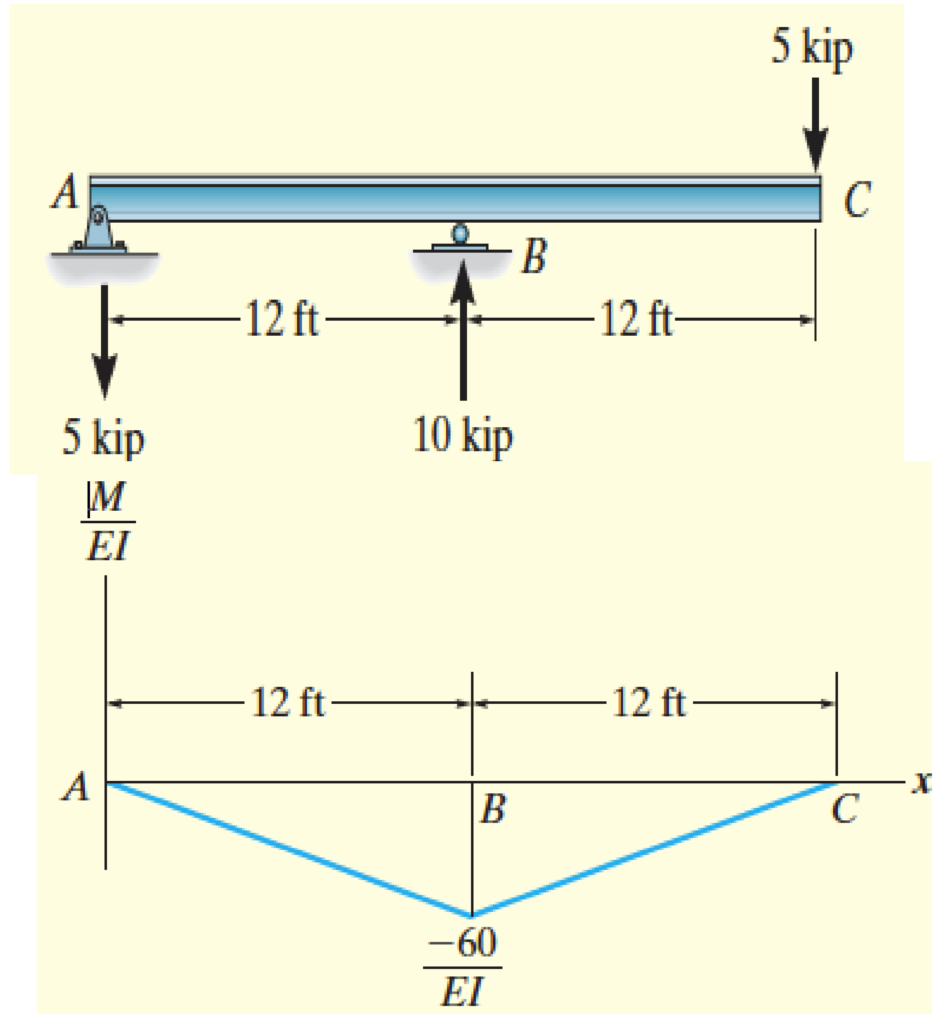
$$t_{C/B} = \frac{1}{2} * 1.5 * \frac{6 * 10^3}{62500} * \frac{1}{3} 1.5 = 0.036$$

$$\frac{0.288}{3} = \frac{cc'}{1.5} \Rightarrow \quad cc' = 0.144$$

$$\delta c = cc' - t_{C/B} = 0.144 - 0.036 = 0.108m$$

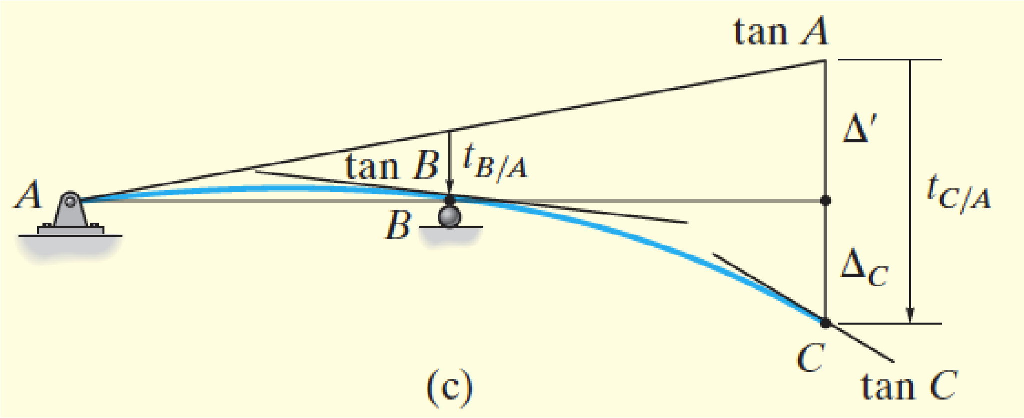
EXAMPLE 7.22

Determine the displacement at point C for the steel overhanging beam $E=29*10^3$ ksi, $I=125in^4$



EXAMPLE 7.22 (Con.)

Solution:



(c)

$$t_{C/A} = \frac{1}{2} * 24 * \frac{-60*10^3*12^3}{3625*10^3*10^3} * (12) = -4.117$$

$$t_{B/A} = \frac{1}{2} * 12 * \frac{60*10^3*12^3}{3625*10^3*10^3} * (\frac{1}{3}12) = -0.686$$

$$cc' = 2t_{B/A} = -1.372$$

$$dc = t_{C/A} - cc' = -4.117 - (-1.372) = 2.75in$$

Thank you!