Tishk International University
Mechatronics Engineering Department
Finite Element Method ME 323
Lecture 9: 19/04 /2021
week

Derivation of stiffness matrix for Beam Element

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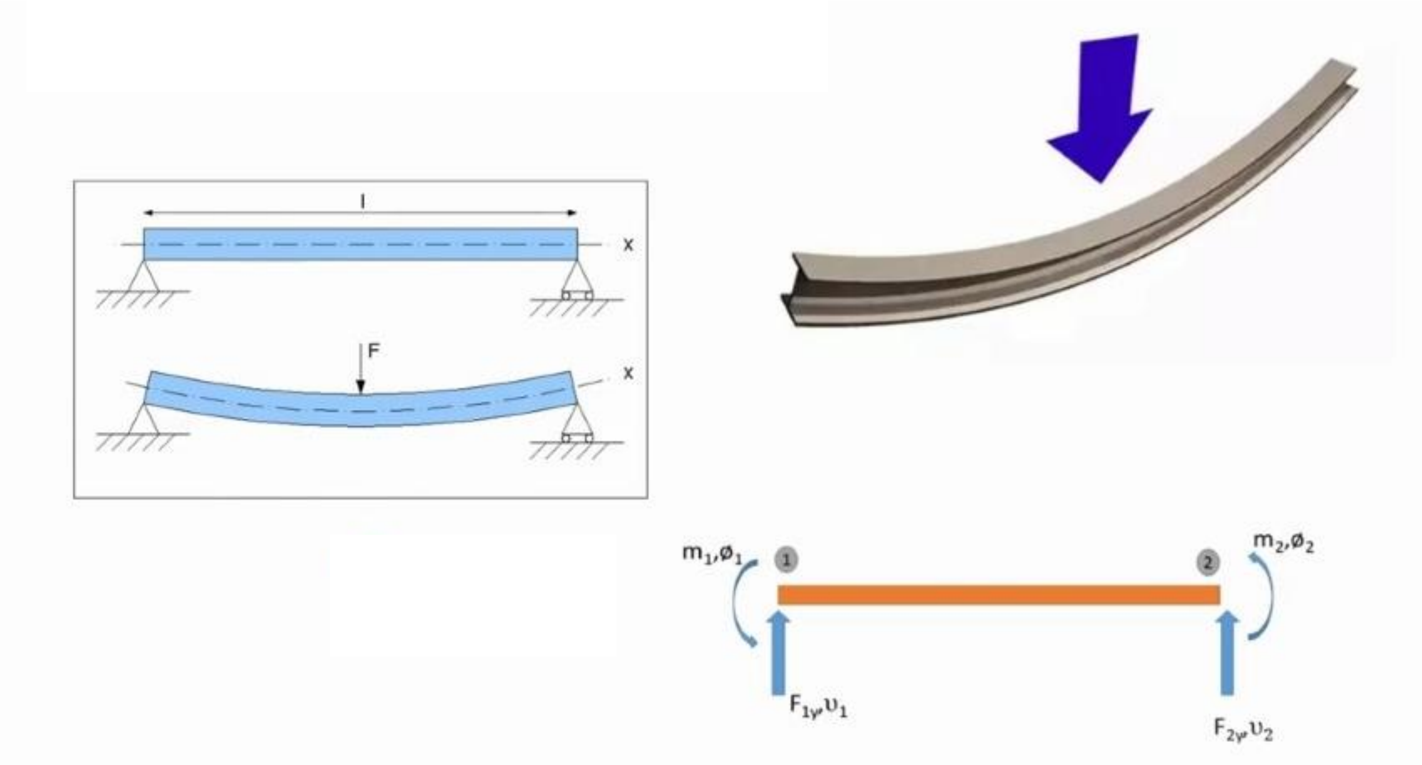
Objectives

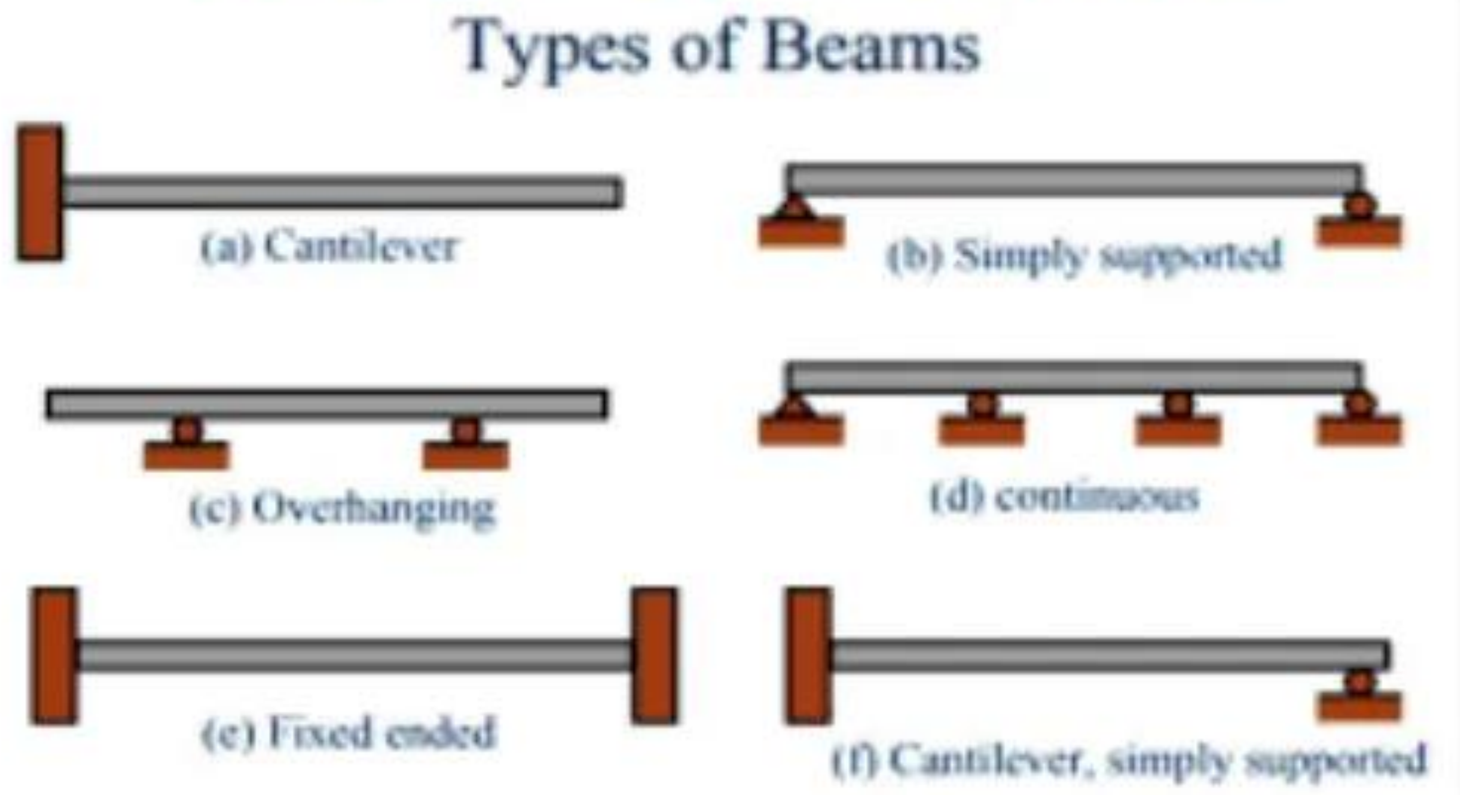
- To Derive Stiffness matrix for beam element
- To apply beam analysis using direct stiffness method

Beam

Beam is a long slender structural member subjected to transverse loading that produces significant bending effects.



TYPES OF BEAMS



Types of Beams

(a) Cantilever

(b) Simply supported

(c) Overhanging

(d) continuous

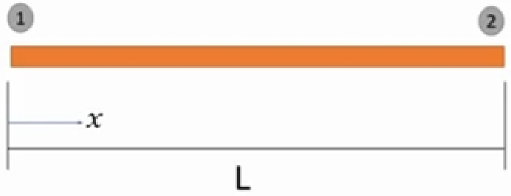
(e) Fixed ended

(f) Cantilever, simply supported

Displacement Function

- $v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$

- $v(0) = v_1 = a_4$
- $\frac{dv(0)}{dx} = \emptyset_1 = a_3$
- $v(L) = v_2 = a_1L^3 + a_2L^2 + \emptyset_1 L + v_1$ .... (1)
- $\frac{dv(L)}{dx} = \emptyset_2 = 3a_1L^2 + 2a_2L + \emptyset 1$ ....(2)



Boundary conditions

Solving Equation 1 &2

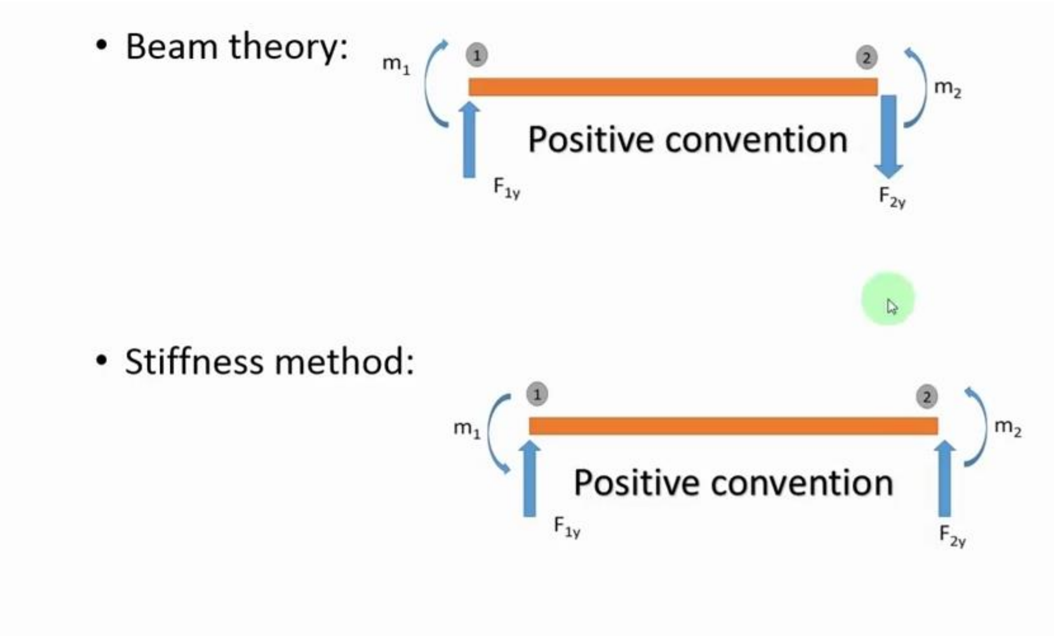
$$a_1 = \frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\emptyset_1 + \emptyset_2)$$

$$a_2 = -\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(\emptyset_1 + \emptyset_2)$$

All together:

$$v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\emptyset_1 + \emptyset_2)\right]x^3 + \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(\emptyset_1 + \emptyset_2)\right]x^2 + \emptyset_1 x + v_1$$

Sign Convention



- Beam theory:

Positive convention

- Stiffness method:

Positive convention

- Note that * $m(x) = \frac{d^2v(x)}{dx^2}EI$ and $V(x) = \frac{d^3v(x)}{dx^3}EI$

$$F_{1y} = V(0) = EI\frac{d^3v(0)}{dx^3} = \frac{EI}{L^3}(12v_1 + 6L\emptyset_1 - 12v_2 + 6L\emptyset_2)$$

$$\mathrm{m}_1 = -m(0) = -EI\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}(6Lv_1 + 4L^2\emptyset_1 - 6Lv_2 + 2L^2\emptyset_2)$$

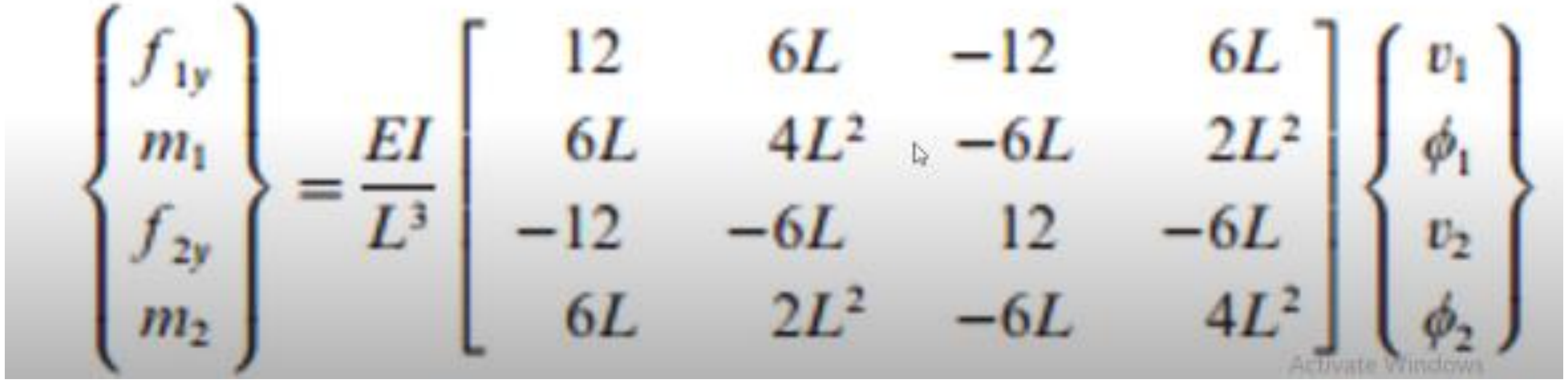
$$F_{2y} = -V(L) = -EI\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}(-12v_1 - 6L\emptyset_1 + 12v_2 - 6L\emptyset_2)$$

$$\mathrm{m}_2 = m(L) = EI\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}(6Lv_1 + 2L^2\emptyset_1 - 6Lv_2 + 4L^2\emptyset_2)$$

Stiffness Matrix for the beam element

- $K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

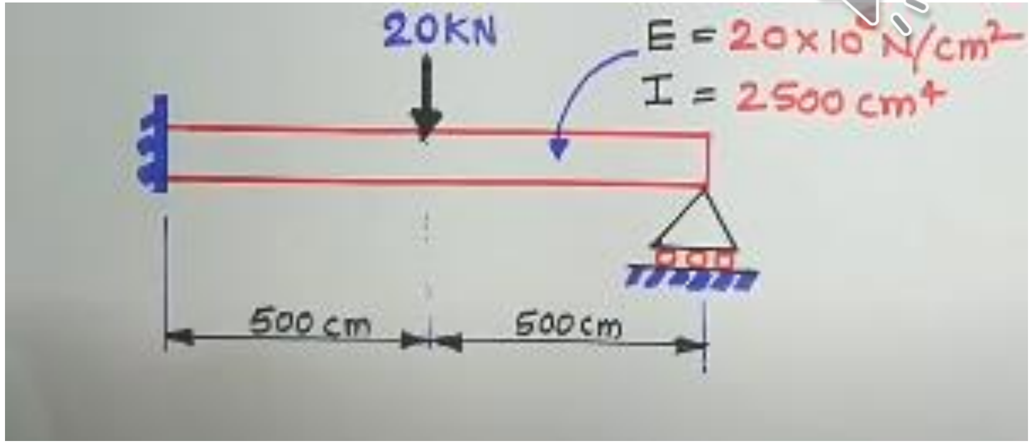
Derived Force ,stiffness and Displacement matrix of beam

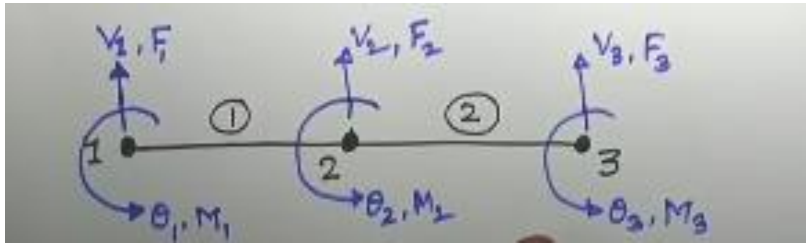


$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Example

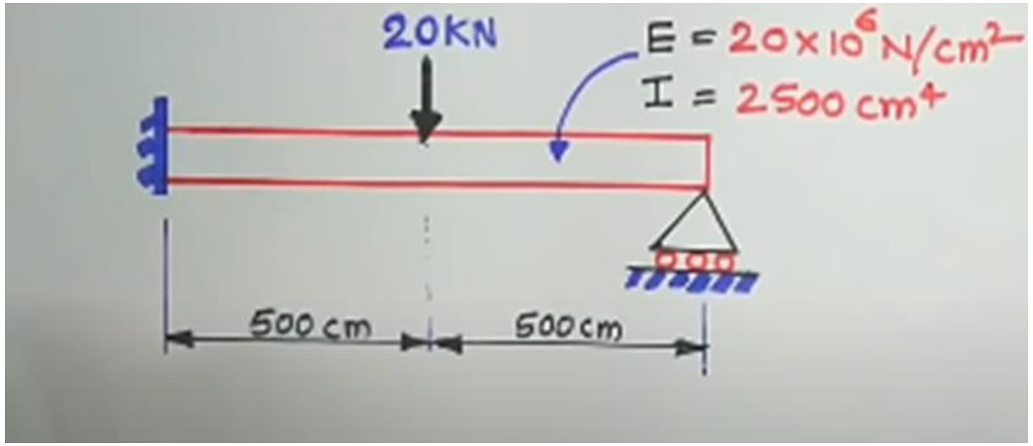
- A beam fixed at one end and supported by a roller at the other end, has 20 KN concentrated load applied at the center of the span, as shown in the figure determine deflection under the load.

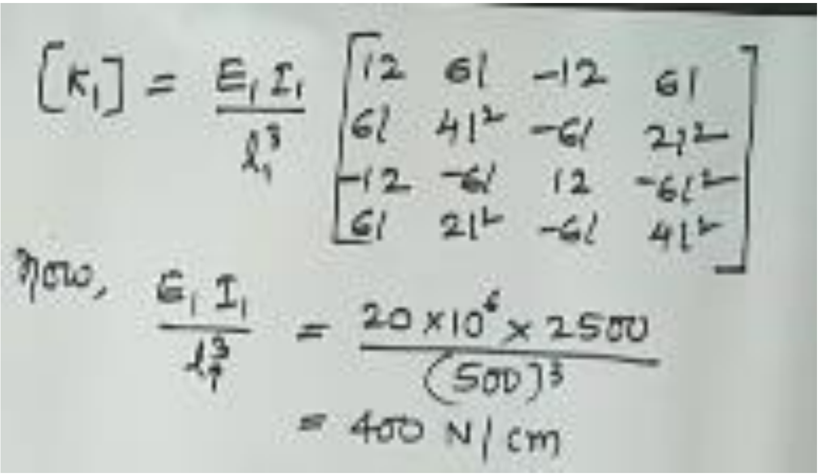




Element 1

- K=$\frac{EI}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

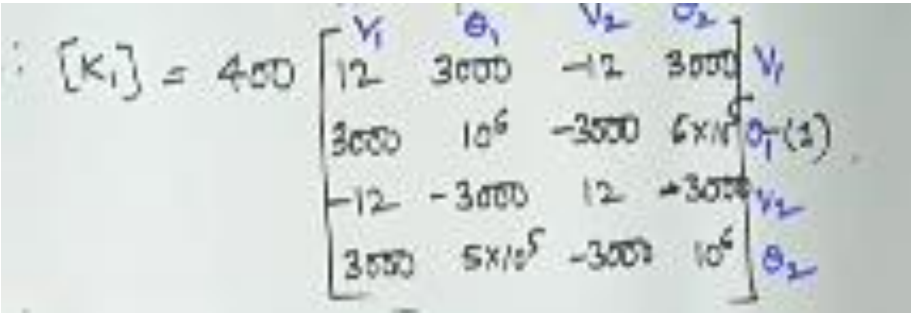




$$[K_1] = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l^2 \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Now, $\frac{E_1 I_1}{l_1^3} = \frac{20\times10^6\times2500}{(500)^3}$

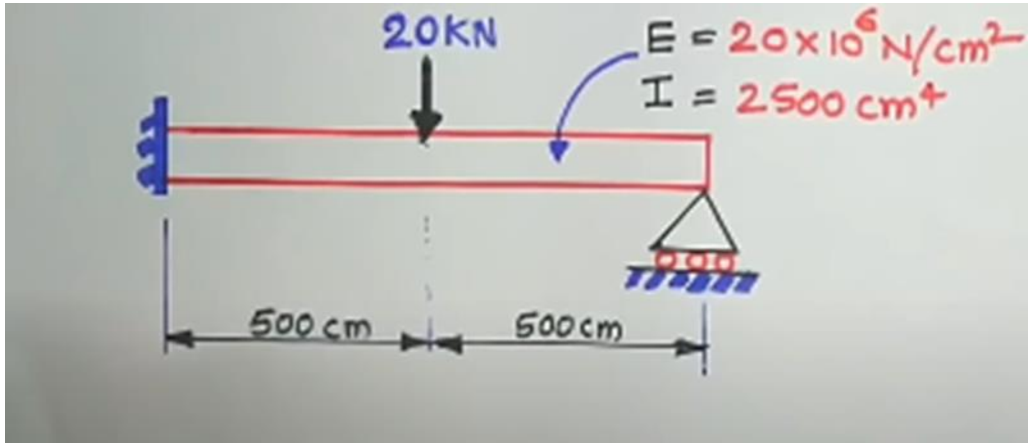
$= 400$ N/cm

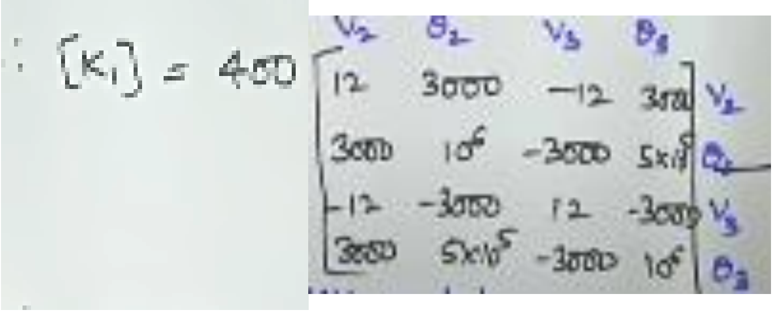


$$\therefore [K_1] = 400\begin{array}{c} \\ \left[\begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \\ 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 6\times10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5\times10^5 & -3000 & 10^6 \end{array}\right] \end{array}\begin{array}{l} \\ v_1 \\ \theta_1 \quad (1) \\ v_2 \\ \theta_2 \end{array}$$

Element 2

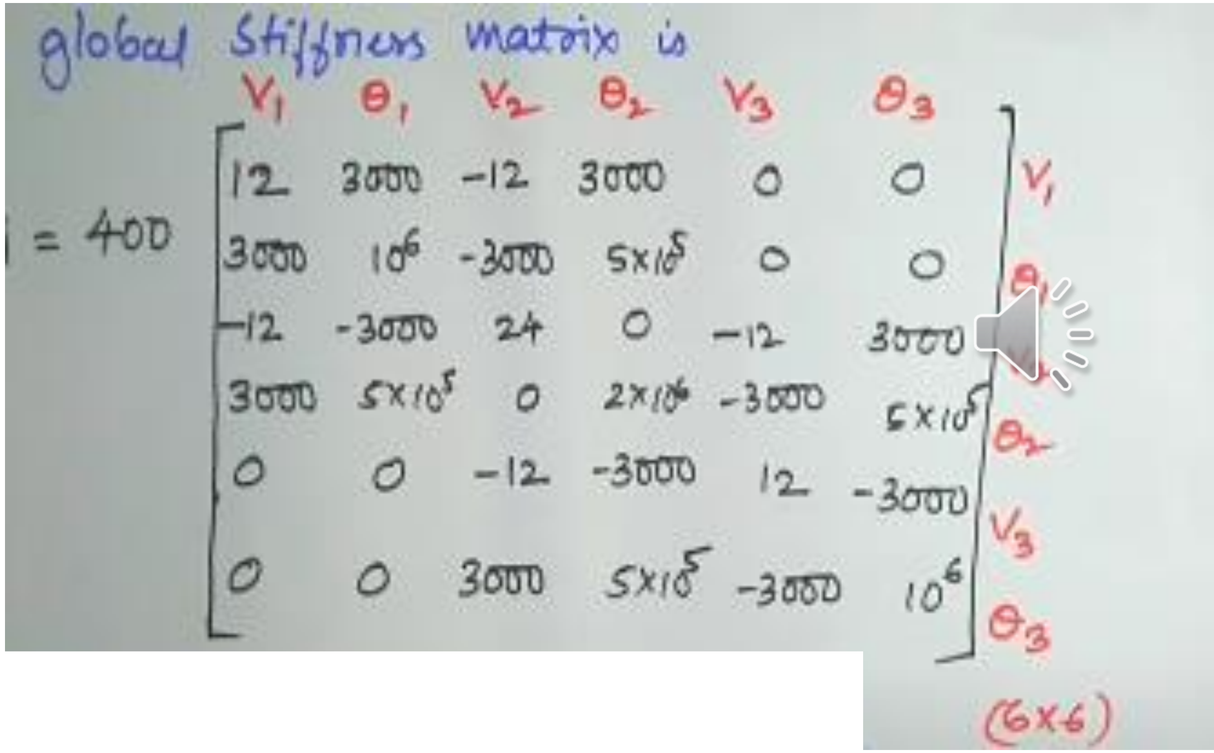
- K=$\frac{EI}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$





$$[K_1] = 400\begin{array}{cccc|c} V_2 & \theta_2 & V_3 & \theta_3 & \\ 12 & 3000 & -12 & 3000 & V_2 \\ 3000 & 10^6 & -3000 & 5\times10^5 & \theta_2 \\ -12 & -3000 & 12 & -3000 & V_3 \\ 3000 & 5\times10^5 & -3000 & 10^6 & \theta_3 \end{array}$$

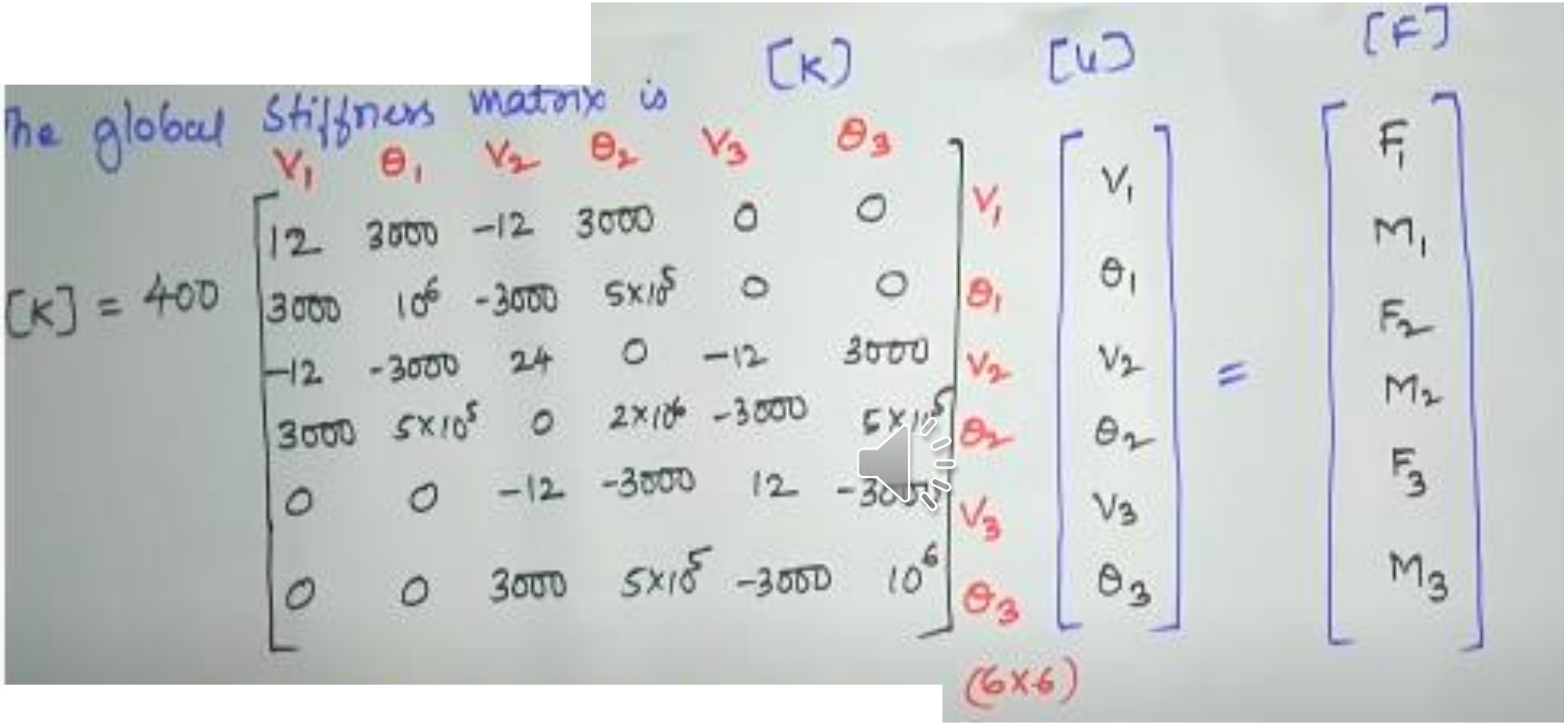
Global stiffness matrix



global stiffness matrix is

$$= 400 \begin{array}{c|cccccc|c} & V_1 & \theta_1 & V_2 & \theta_2 & V_3 & \theta_3 & \\ \hline & 12 & 3000 & -12 & 3000 & 0 & 0 & V_1 \\ & 3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 & \theta_1 \\ & -12 & -3000 & 24 & 0 & -12 & 3000 & \\ & 3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 & \theta_2 \\ & 0 & 0 & -12 & -3000 & 12 & -3000 & V_3 \\ & 0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6 & \theta_3 \end{array}$$

(6×6)



The global Stiffness matrix is

$$[K] = 400 \overset{[K]}{\left[\begin{array}{cccccc|c} V_1 & \theta_1 & V_2 & \theta_2 & V_3 & \theta_3 & \\ 12 & 3000 & -12 & 3000 & 0 & 0 & V_1 \\ 3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 & \theta_1 \\ -12 & -3000 & 24 & 0 & -12 & 3000 & V_2 \\ 3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 & \theta_2 \\ 0 & 0 & -12 & -3000 & 12 & -3000 & V_3 \\ 0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6 & \theta_3 \end{array}\right]}_{(6\times6)} \overset{[u]}{\begin{bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{bmatrix}} = \overset{[F]}{\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{bmatrix}}$$

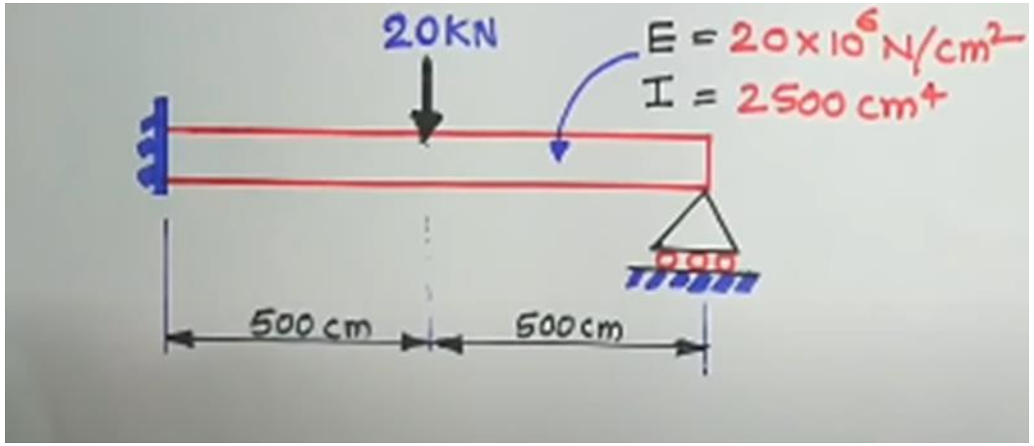
Boundary (Condition and force)

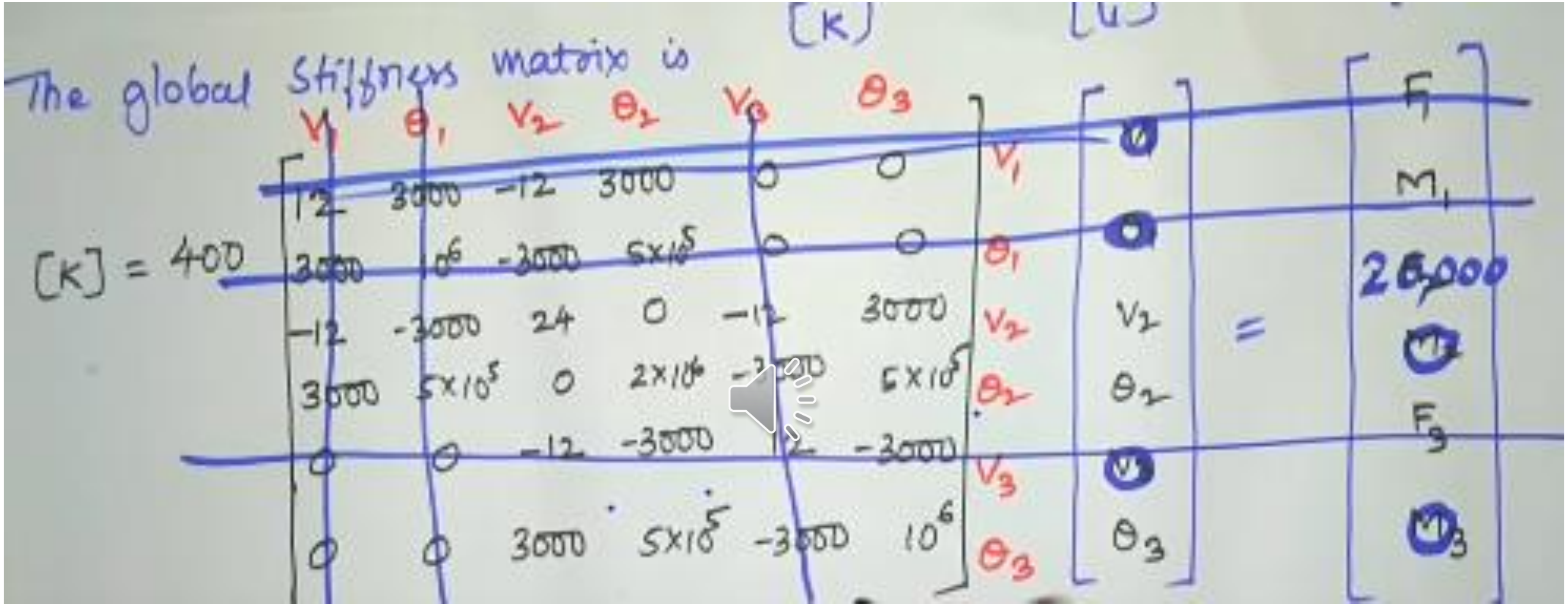
- Boundary condition
- V1=$\theta 1 = 0 \; node \; 1$
- V2 , $\theta 2 = ?$ Node2
- V3=0

Boundary force

F=-20 M2=0

M3=0



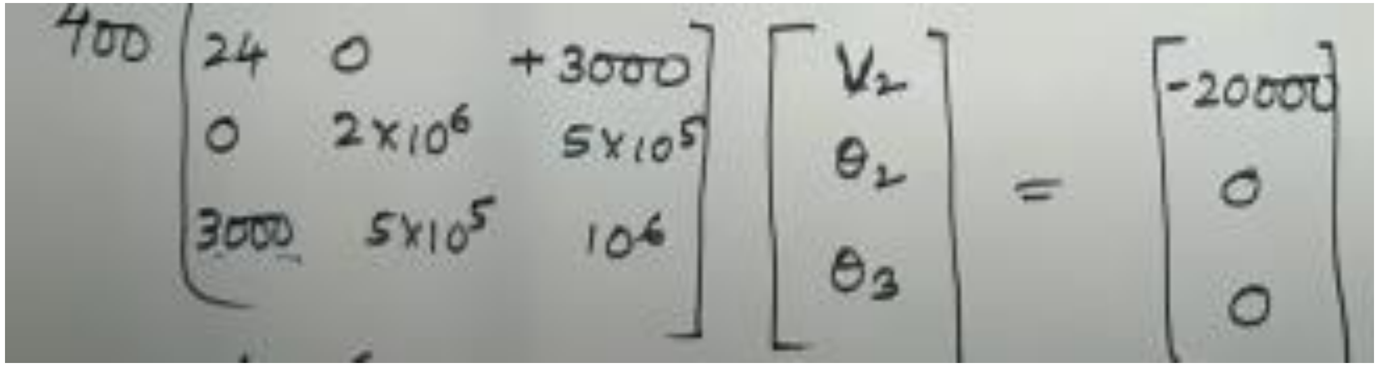


The global stiffness matrix is

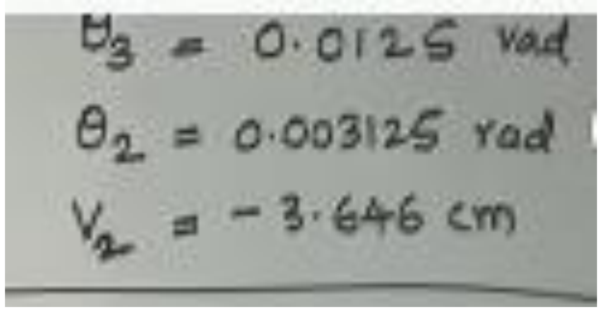
$[K]$ $[U]$

$$[K] = 400 \begin{array}{cccccc|c} V_1 & \theta_1 & V_2 & \theta_2 & V_3 & \theta_3 & \\ 12 & 3000 & -12 & 3000 & 0 & 0 & V_1 \\ 3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 & \theta_1 \\ -12 & -3000 & 24 & 0 & -12 & 3000 & V_2 \\ 3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 & \theta_2 \\ 0 & 0 & -12 & -3000 & 12 & -3000 & V_3 \\ 0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6 & \theta_3 \end{array}$$

$$\begin{bmatrix} 0 \\ 0 \\ V_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ 26{,}000 \\ 0 \\ F_3 \\ 0 \end{bmatrix}$$



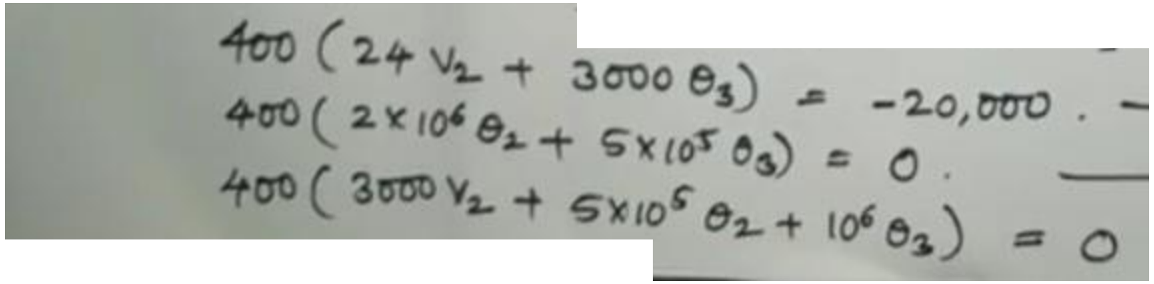
$$400\begin{bmatrix} 24 & 0 & +3000 \\ 0 & 2\times10^6 & 5\times10^5 \\ 3000 & 5\times10^5 & 10^6 \end{bmatrix}\begin{bmatrix} V_2 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -20000 \\ 0 \\ 0 \end{bmatrix}$$



$\theta_3 = 0.0125$ rad

$\theta_2 = 0.003125$ rad

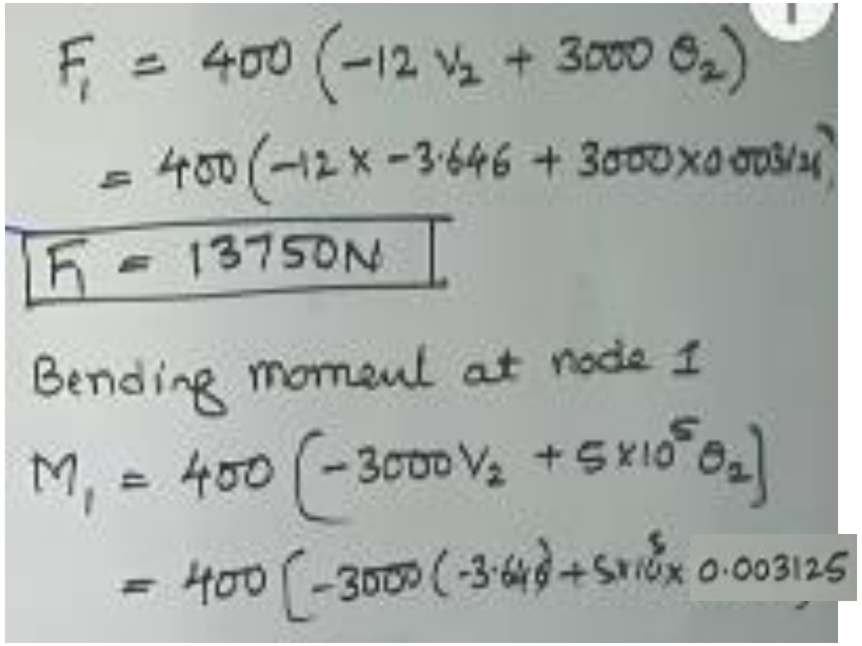
$V_2 = -3.646$ cm



$400(24V_2 + 3000\theta_3) = -20{,}000.$

$400(2\times10^6\theta_2 + 5\times10^5\theta_3) = 0.$

$400(3000V_2 + 5\times10^5\theta_2 + 10^6\theta_3) = 0$



$F_1 = 400(-12V_2 + 3000\theta_2)$

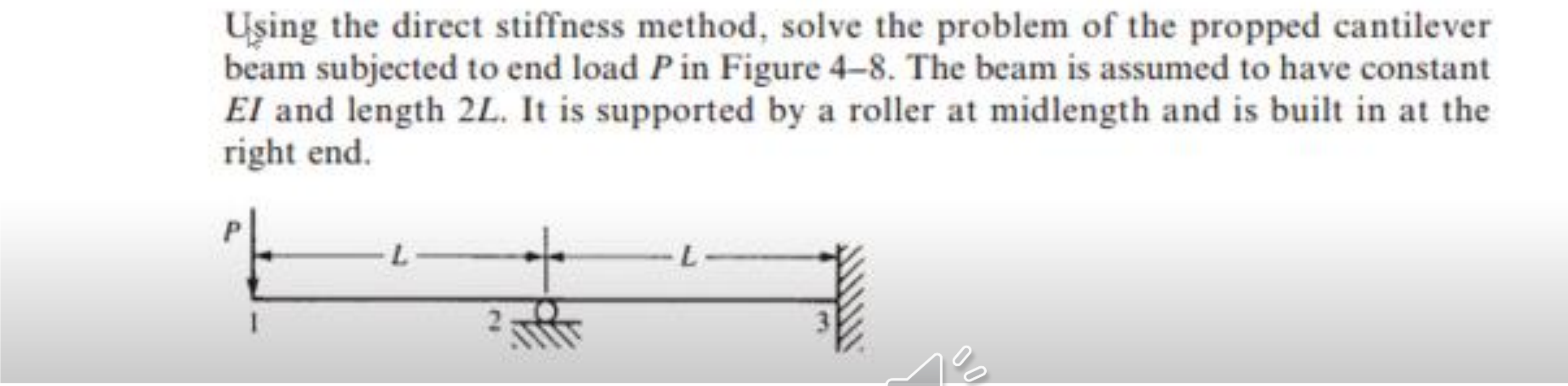
$= 400(-12\times-3.646 + 3000\times0.003125)$

$\boxed{F_1 = 13750N}$

Bending moment at node 1

$M_1 = 400\left[-3000V_2 + 5\times10^5\theta_2\right]$

$= 400\left[-3000(-3.646) + 5\times10^5\times0.003125\right.$



Using the direct stiffness method, solve the problem of the propped cantilever beam subjected to end load P in Figure 4–8. The beam is assumed to have constant EI and length $2L$. It is supported by a roller at midlength and is built in at the right end.

k1= $\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$ k2= $\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

Global Force displacement matrix

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

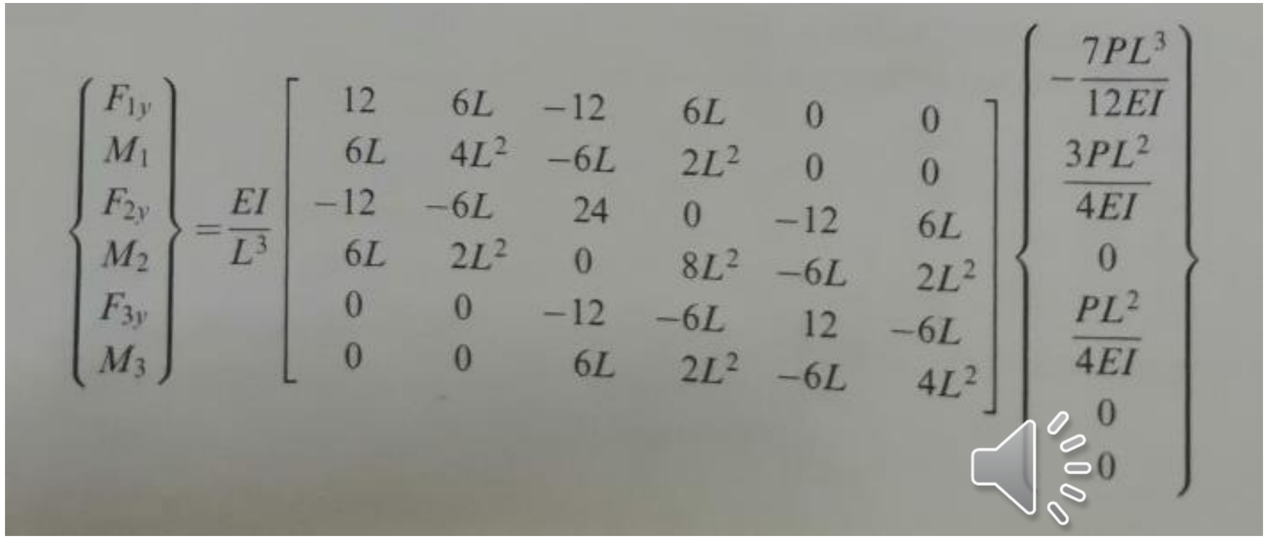
Boundary Condition

$$v_2 = 0 \qquad v_3 = 0 \qquad \phi_3 = 0$$

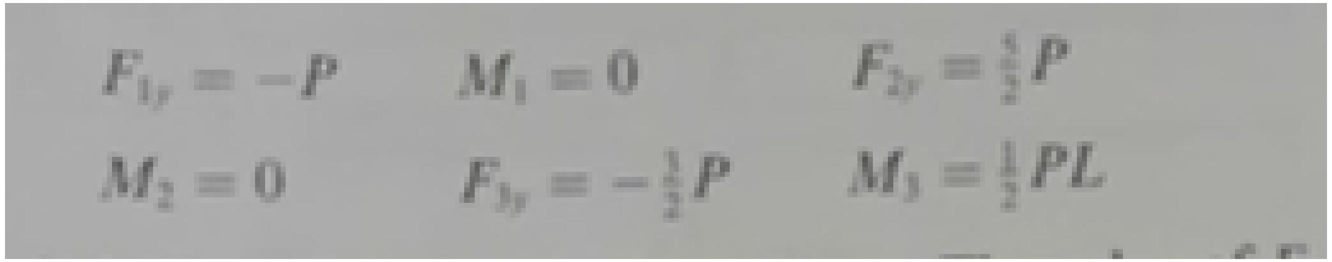
$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

v1 $= -\dfrac{7PL^3}{12EI}$

$$\phi_1 = \frac{3PL^2}{4EI} \qquad \phi_2 = \frac{PL^2}{4EI}$$



$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\dfrac{7PL^3}{12EI} \\ \dfrac{3PL^2}{4EI} \\ 0 \\ \dfrac{PL^2}{4EI} \\ 0 \\ 0 \end{Bmatrix}$$



$$F_{1y} = -P \qquad M_1 = 0 \qquad F_{2y} = \tfrac{3}{2}P$$

$$M_2 = 0 \qquad F_{3y} = -\tfrac{1}{2}P \qquad M_3 = \tfrac{1}{2}PL$$

Local force

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\frac{7PL^3}{12EI} \\ \frac{3PL^2}{4EI} \\ 0 \\ \frac{PL^2}{4EI} \end{Bmatrix}$$

$\hat{f}_{1y} = -P$

$\hat{m}_1 = 0$

$\hat{f}_{2y} = P$

$\hat{m}_2 = -PL$

Next Lecture

- Distributed load beam analysis