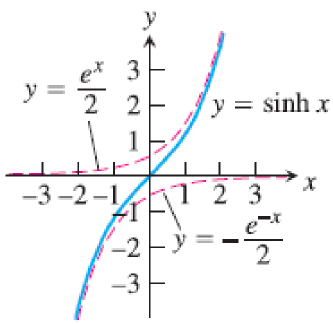
Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}.

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

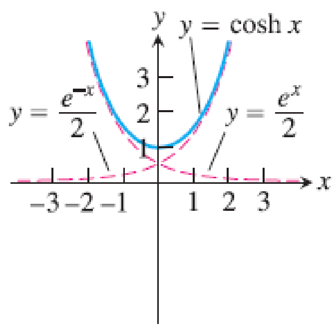
Table 1 The six basic hyperbolic functions



(a)

Hyperbolic sine:

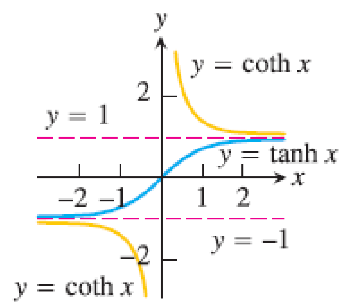
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



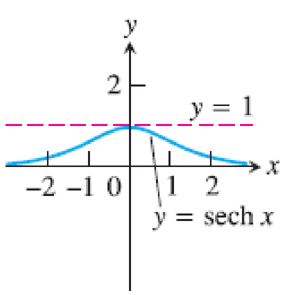
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

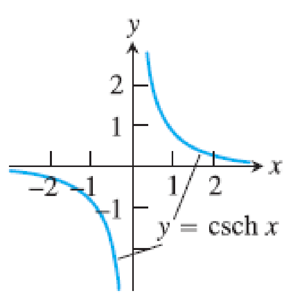
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic functions satisfy the identities in Table **2**. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions, as we show here for the second one:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \sinh 2x. \end{aligned}$$

TABLE 2 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Adding and subtracting,

$$\begin{aligned} \sinh(x \pm y) &= \sinh x \cosh y \pm \sinh y \cosh x \\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y \end{aligned}$$

Dividing,

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

From the definitions of the hyperbolic sine and cosine, we can derive the following identities:

$$e^{x} = \cosh x + \sinh x$$

and

$$e^{-x} = \cosh x - \sinh x.$$

It can be seen that cosh *x* and sech *x* are even functions; the others are odd functions.

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$

Hence:

$$\tanh(-x) = -\tanh x$$
$$\coth(-x) = -\coth x$$
$$\text{sech}(-x) = \text{sech } x$$
$$\text{csch}(-x) = -\text{csch } x$$

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Figure **1** a.

The function $y = \cosh x$ is not one-to-one because its graph in Table **1** does not pass the horizontal line test. The restricted function $y = \cosh x, x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1, y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x, x \geq 0$, and $y = \cosh^{-1} x$ are shown in Figure **1** b.

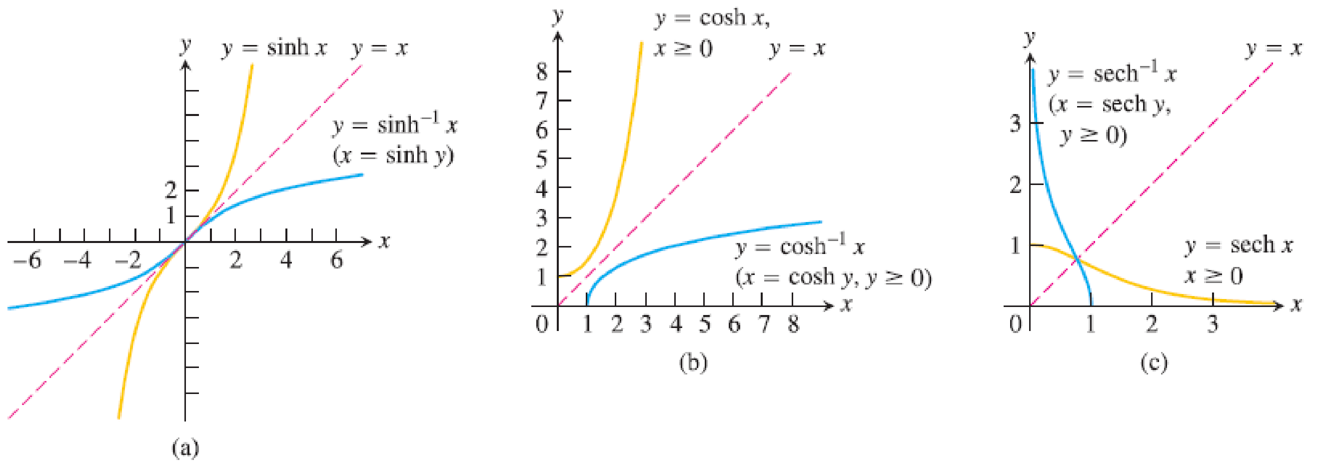


FIGURE 1 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one,

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Figure 1 c.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure 2 .

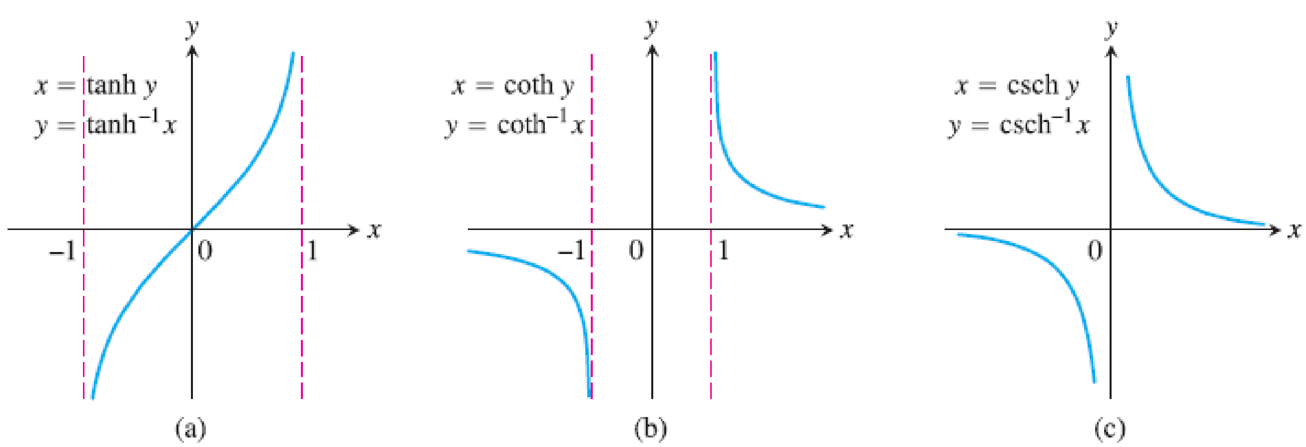


FIGURE 2 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

Identities for inverse hyperbolic functions

TABLE 3 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Useful Identities

We use the identities in Table 3 to calculate the values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ on calculators that give only $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$. These identities are direct consequences of the definitions. For example, if $0 < x \le 1$, then

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x.$$

We also know that $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$, so because the hyperbolic secant is one-to-one on $(0, 1]$, we have

$$\cosh^{-1}\left(\frac{1}{x}\right) = \operatorname{sech}^{-1} x.$$

Inverses of the hyperbolic functions and their formulae:

$y=\sinh^{-1}x$ iff $\sinh y=x$ $\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$ $-\infty < x < \infty$	$y=\cosh^{-1}x$ iff $\cosh y=x$ $\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$ $x \geq 1$
$y=\tanh^{-1}x$ iff $\tanh y=x$ $\tanh^{-1}x=\frac{1}{2}\ln\left\lvert\frac{1+x}{1-x}\right\rvert$ $\lvert x\rvert < 1$	$y=\coth^{-1}x$ iff $\coth y=x$ $\coth^{-1}x=\frac{1}{2}\ln\left\lvert\frac{x+1}{x-1}\right\rvert$ Also: $\coth^{-1}(x)=\tanh^{-1}\left(\frac{1}{x}\right)$ $\lvert x\rvert > 1$
$y=\operatorname{sech}^{-1}x$ iff $\operatorname{sech} y=x$ $\operatorname{sech}^{-1}x=\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right)$ Also: $\operatorname{sech}^{-1}(x)=\cosh^{-1}\left(\frac{1}{x}\right)$ $0 < x \leq 1$	$y=\operatorname{csch}^{-1}x$ iff $\operatorname{csch} y=x$ $\operatorname{csch}^{-1}x=\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right)$ Also: $\operatorname{csch}^{-1}(x)=\sinh^{-1}\left(\frac{1}{x}\right)$ $x \neq 0$