Chapter 2 : Vector Spaces

In the last chapter we used vectors to represent a quantity with 'magnitude and direction'. In fact, this 'magnitude–direction' definition relates to a specific application of vectors, which is used in the fields of science and engineering.

However, vectors have a much broader range of applications, and they are more generally defined as 'elements of a vector space'.

In this chapter we describe what is meant by a vector space and how it is mathematically defined.

Vector space

Let V be a non-empty set of elements called vectors. We define two operations on the set V– **vector addition** and **scalar multiplication**. Scalars are real numbers.

Let u, v and w be vectors in the set **V**. The set **V** is called a vector space if it satisfies the following 10 axioms.

1. The vector addition $u + v$ is also in the vector space
2. Commutative law: $u + v = v + u$.
3. Associative law: (u + v) + w = u + (v + w).
4. Neutral element. There is a vector called the zero vector in **V** denoted by O which satisfies
$$u + O = u$$
for every vector u in **V**
5. Additive inverse. For every vector u there is a vector $-u$ (minus u) which satisfies the following:
$$u + (-u) = O$$
6. Let k be a real scalar then ku is also in **V**.
7. Associative law for scalar multiplication. Let k and c be real scalars then
$$k(cu) = (kc)u$$
8. Distributive law for vectors. Let k be a real scalar then

$$k(u + v) = ku + k v$$

9. Distributive law for scalars. Let k and c be real scalars then

$$(k + c)u = ku + cu$$

10. Identity element. For every vector u in V we have 1u = u

We say that if the elements of the set **V** satisfy the above 10 axioms then V is called a vector space and the elements are known as vectors.

Examples of vector spaces

1- The Euclidean spaces V = $R^2, R^3, \dots, R^n$ are all examples of vector spaces.
2- The set of matrices $M_{2\times2}$ that are all matrices of size 2 by 2 where matrix addition and scalar multiplication form their own vector space.

Let $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ be matrices in M_{22}.

Example: Let **V** be the set of integers **Z**. Let vector addition be defined as the normal addition of integers, and scalar multiplication by the usual multiplication of integers by a real scalar, which is any real number.

Show that this set is not a vector space with respect to this definition of vector addition and scalar multiplication.

Solution The set of integers **Z** is not a vector space because when we multiply an integer by a real scalar, which is any real number, then the result may not be an integer.

Note: In general if a set forms a vector space with the scalars being real numbers then we say we have a vector space over the real numbers.

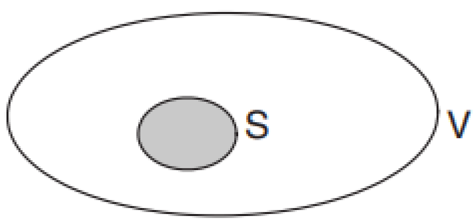
Subspace of a Vector Space

In first section we discussed the whole vector space **V**. In this section we show that parts of the whole vector space also form a vector space in their own right. We will show that a non-empty set within **V**, which is closed under the basic operations of vector addition and scalar multiplication, is also a legitimate vector space.

Let V be a vector space and **S** be a non-empty subset of **V**. If the set **S** satisfies all 10 axioms of a vector space with respect to the same vector addition and scalar multiplication as **V** then **S** is also a vector space. We say **S** is a subspace of **V.**

Definition. A non-empty subset **S** of a vector space **V** is called a subspace of V if it is also a vector space with respect to the same vector addition and scalar multiplication as V.

We illustrate this in figure



Note the difference between subspace and subset. A subset is merely a specific set of elements chosen from V. A subset must also satisfy the 10 axioms of vector space to be called a subspace.

More generally, we will use the following proposition to check if a given subset qualifies as a subspace.

Proposition . Let S be a non-empty subset of a vector space **V**. Then **S** is subspace of V if and only if :

(a) If u and v are vectors in the set S then the vector addition u + v is also in S.

(b) If u is a vector in S then for every scalar k we have, ku is also in S.

Recall from the last section that unit vectors are of length 1, and standard unit vectors in R^n are column vectors with 1 in the kth position of the vector $e_k$and zeros everywhere else.

Example: Let **V** be the set R^2 and vector addition and scalar multiplication be defined as normal. Let **S** be the set of vectors of the form $\binom{0}{y}$. Show that **S** is a subspace of **V.**

Solution We only need to check conditions (a) and (b) of Proposition above

Let $u = \binom{0}{a}$ and $v = \binom{0}{b}$ be vectors in S. Then

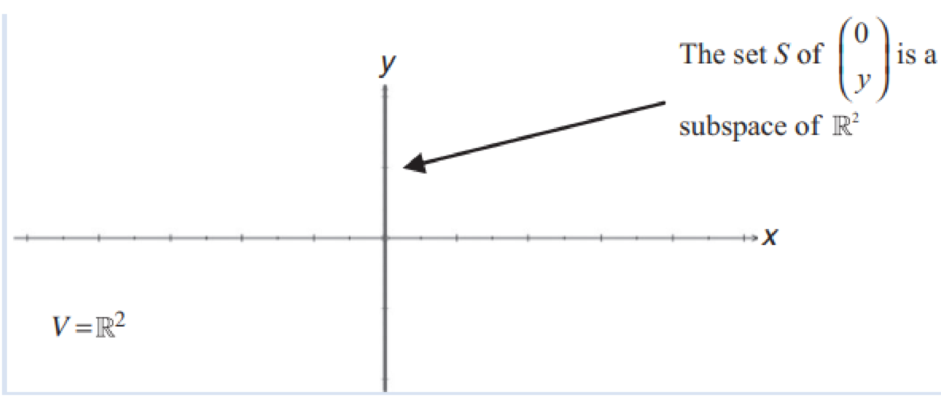
$$u + v = \binom{0}{a} + \binom{0}{b} = \binom{0}{a+b}$$

which is in the set **S** and

$$ku = k\binom{0}{a}\binom{0}{ka}$$

which is in S as well.

Conditions (a) and (b) are satisfied, therefore the given set S is a subspace of the vector space R^2.



Example: Let **S** be the subset of vectors of the form $\binom{x}{y}$ where $x \geq 0$ in the vector space R^2 is not a subspace of R^2

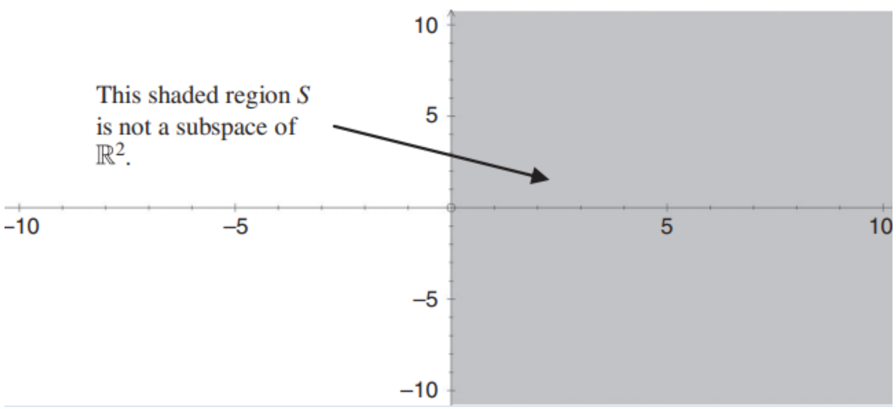
Solution: If we can show that we do not have closure under vector addition or scalar multiplication of vectors in S, then we can conclude that S is not a subspace.

Consider the scalar k = −1 and the vector $u = \binom{1}{2}$

then clearly $\boldsymbol{u}$ is in the set **S**, but the scalar multiplication

$$ku = (-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

is not in **S.**



linear combination

Linear combination combines the two fundamental operations of linear algebra – vector addition and scalar multiplication. In the last chapter we introduced linear combination in Rn.

For example, we had

$$\mathbf{u} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_n\mathbf{e}_n$$

which is a linear combination of the standard unit vectors.

$$\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots \text{ and } \mathbf{e}_n.$$

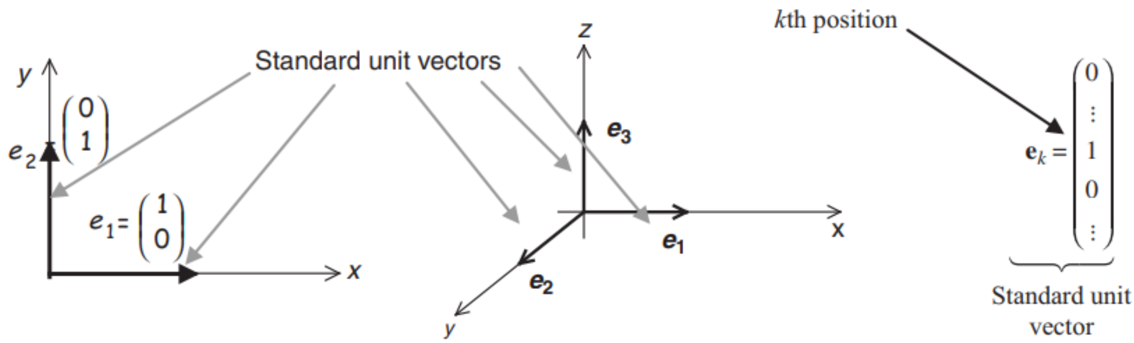
Similarly for general vector spaces we define linear combination as:

Definition Let $\mathbf{v}_1,\ \mathbf{v}_2, \ldots$ and $\mathbf{v}_n$ be vectors in a vector space. If a vector $\mathbf{x}$ can be expressed as

$$\mathbf{x} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \text{ (where } k\text{'s are scalars)}$$

then we say $\mathbf{x}$ is a **linear combination** of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots$ and $\mathbf{v}_n$.

Recall from the last section that unit vectors are of length 1, and standard unit vectors in R^n are column vectors with 1 in the kth position of the vector e_k and zeros everywhere else



Why are these standard unit vectors important?

Because we can write any vector $\boldsymbol{u}$ of R^n in terms of scalars and standard unit vectors as we seen in previous chapter like.

$$\mathbf{u} = \underbrace{x_1}_{\text{scalar}}\ \underbrace{\mathbf{e}_1}_{\text{unit vector}} + \underbrace{x_2}_{\text{scalar}}\ \underbrace{\mathbf{e}_2}_{\text{unit vector}} + \cdots + \underbrace{x_k}_{\text{scalar}}\ \underbrace{\mathbf{e}_k}_{\text{unit vector}} + \cdots + \underbrace{x_n}_{\text{scalar}}\ \underbrace{\mathbf{e}_n}_{\text{unit vector}}$$

For example, the vector $\boldsymbol{u} = \binom{2}{3}$ in R^2 can be written as

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\mathbf{e}_1 + 3\mathbf{e}_2$$

[In this case the scalars $x_1 = 2$ and $x_2 = 3$.]

Note: that the scalars $x_1 = 2$ and $x_2 = 3$ are the coordinates of the vector $\boldsymbol{u}$.

This representation

$$\mathbf{u} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

is a linear combination of the scalars and standard unit vectors

$$\mathbf{e}_1, \mathbf{e}_2, \ldots \text{ and } \mathbf{e}_n.$$

We can write this

$$\mathbf{u} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

in matrix form as

$$\mathbf{u} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \cdots \quad \mathbf{e}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ where } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

Proposition. Let $\boldsymbol{u}$ be any vector in R^n , then the linear combination

$$\mathbf{u} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_k\mathbf{e}_k + \cdots + x_n\mathbf{e}_n$$

is unique.

What does this proposition mean?

It means for any vector $\boldsymbol{u}$ the scalars in the above linear combination are unique.

Linear independence

Definition We say vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ in $\mathbb{R}^n$ are **linearly independent** $\Leftrightarrow$ the only real scalars $k_1,\ k_2,\ k_3, \ldots$ and k_n which satisfy:

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O} \text{ are } k_1 = k_2 = k_3 = \cdots = k_n = 0$$

What does this mean?

The only solution to the linear combination

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

occurs when all the scalars

$k_1, k_2, k_3, \ldots$ and k_n are equal to zero.

In other words, you cannot make any one of the vectors vj, say, by a linear combination of the others.

We can write the linear combination

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

in matrix form as

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

The first column of the matrix

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$$

is given by the entries in v_1, the second column is given by the entries in v_2 and the nth column by entries in v_n.

The standard unit vectors are not the only vectors in R^n which are linearly independent.

In the following example, we show another set of linearly independent vectors

Example: Show that $\boldsymbol{u} = \binom{-1}{1}$ and $\boldsymbol{v} = \binom{2}{3}$ are linearly independent in $R^2$and plot them.

Solution: Consider the linear combination:

$$ku + cv = 0 \text{ where (} k \text{ and } c \text{ are scalar)}$$

Substituting the given vectors u and v into this eq. we have

$$k\binom{-1}{1} + c\binom{2}{3} = 0$$

So we need solve an equation $AX = 0$

Where $A = (u\ v) = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} k \\ c \end{pmatrix}$ and $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Carrying out row operations on the augmented matrix (A | O)

$$\begin{array}{cc} & \begin{array}{cc} k & c \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & 3 & 0 \end{array}\right) \end{array} \Rightarrow \begin{array}{cc} & \begin{array}{cc} k & c \end{array} \\ \begin{array}{c} R_1 \\ R_2 + R_1 \end{array} & \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 5 & 0 \end{array}\right) \end{array}$$

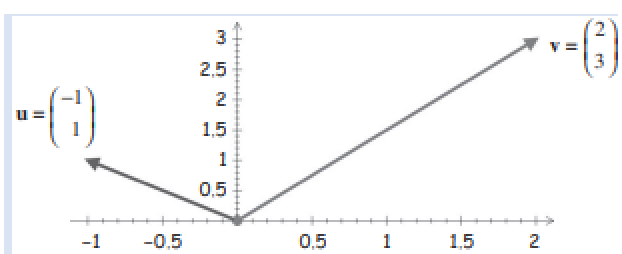
From the bottom row, $R_2 + R_1$, we have $5c = 0$, which gives c = 0.

Substituting this into the first row yields $k = 0$.

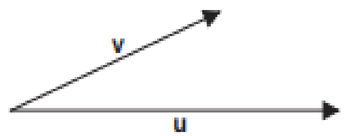
This means that the only values of the scalars are k = 0 and c = 0. Hence the linear combination

$ku + cv = 0$ yields $k = 0$ and $c = 0$,

therefore the given vectors u and v are linearly independent, because all the scalars, k and c, are equal to zero.



Note: When the vectors u and v are linearly independent, it means that they are not scalar multiples of each other. Arbitrary linear independent vectors u and v in R^2 can be illustrated as shown in Figure



Linearly independent vectors **u** and **v**

Linear dependence

Definition. Conversely we have: the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ in $\mathbb{R}^n$ are linearly **dependent** $\Leftrightarrow$ the scalars $k_1, k_2, k_3, \ldots$ and k_n are not all zero and satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

Linear dependence of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \text{ and } \mathbf{v}_n$$

means that there are non-zero scalars k's which satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

Example: Show that $\boldsymbol{u} = \binom{-3}{1}$ and $\boldsymbol{v} = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$ are linearly independent in $R^2$and plot them.

Solution: Consider the linear combination:

$$ku + cv = 0 \text{ where (} k \text{ and } c \text{ are scalar)}$$

Substituting the given vectors u and v into this eq. we have

$$k\binom{-3}{1} + c\begin{pmatrix}1\\ \frac{1}{3}\end{pmatrix} = 0$$

So we need solve an equation $AX = 0$

Where $A = (u\ v) = \begin{pmatrix}-3 & 1\\ 1 & 1l3\end{pmatrix}$ and $X = \binom{k}{c}$ and $0 = \binom{0}{0}$

Carrying out row operations on the augmented matrix (A | O)

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left(\begin{array}{cc|c} -3 & 1 & 0\\ 1 & -1/3 & 0\end{array}\right) \Rightarrow \begin{array}{cc} & \begin{array}{cc} k & c \end{array} \\ \begin{array}{c} R_1 \\ 3R_2 + R_1 \end{array} & \left(\begin{array}{cc|c} -3 & 1 & 0\\ 0 & 0 & 0\end{array}\right) \end{array}$$

From the first row, $-3R_1 + R_1$, we have $-3k + c = 0$, which gives $c = 3k$. Let $k = 1$, then $c = 3$

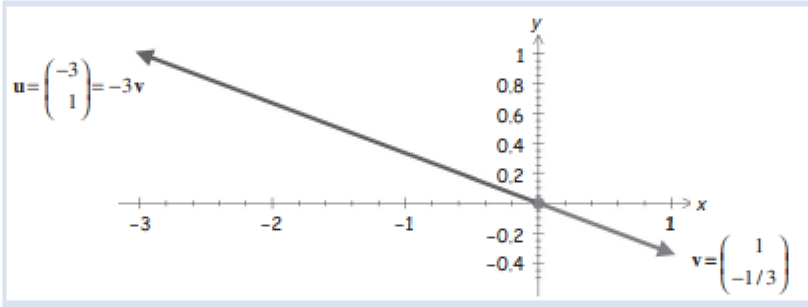
This means that the only values of the scalars are k = 0 and c = 0. Hence the linear combination

$ku + cv = 0$ yields $k = 0$ and $c = 0$,

Substituting our values $k = 1$ and $c = 3$ into

$ku + cv = 0$ gives $u + 3v = 0$ or $u = -3v$

We have found non-zero scalars, $k = 1$ and $c = 3$, which satisfy $ku + cv = 0$, therefore the given vectors u and v are linearly dependent, and $u = -3v$. Plotting the given vectors u and v we have



Note that $u = -3v$ means that the vector u is a scalar multiple (-3) of the vector v. Hence the size of vector u is three times the size of vector v, but in the opposite direction.

If vectors u and v in R^2 are linearly dependent, then we have

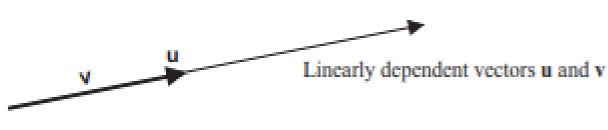
$$ku + cv = 0 \text{ where } k \neq 0 \text{ or } c \neq 0$$

That is, at least one of scalars is not zero.

Suppose $k \neq 0$ then

$$ku = -cv \implies u = \frac{-c}{k} v$$

This means that the vector u is a scalar multiple of the other vector v, which suggests that u is in the same (or opposite) direction as vector v. Plotting these we have



Example: Test the vectors $\boldsymbol{u} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in R^4 form linear independence.

Solution: Consider the linear combination

$$k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w} + k_4\boldsymbol{x} = \boldsymbol{0}$$

We need to determine the values of the scalars, k_i's. Substituting the given vectors into this linear combination:

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} + k_4\mathbf{x} = k_1\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + k_3\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + k_4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix of this is given by

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{cccc|c} -3 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 3 & 0 \end{array}\right)$$

Carrying out the row operation $R_2 + R_3$gives one extra zero in the middle row:

$$\begin{array}{cc} & \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 + \mathrm{R}_3 \\ \mathrm{R}_3 \end{array} & \left(\begin{array}{cccc|c} -3 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 & 0 \end{array}\right) \end{array}$$

From the bottom row, we have

$$-k_2 + 3k_4 = 0 \implies k_2 = 3k_4$$

Let $k_4 = 1 \implies k_2 = 3$

From the middle row we have

$$k_1 + 5k_4 = 0 \implies k_1 = -5k_4 = -5$$

The top row gives

$$-3k_1 + k_2 + k_4 = 0 \implies -3(-5) + 2k_3 + 1 = 0 \implies k_3 = -8$$

Our scalars are $k_1 = -5$, $k_2 = 3$, $k_3 = -8$ and $k_4 = 1$.

Substituting these into the above linear combination

$$k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w} + k_4\boldsymbol{x} = \boldsymbol{0}$$

gives the relationship between the vectors:

$$-5\begin{pmatrix}-3\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\-1\end{pmatrix} - 8\begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}15\\-5\\0\end{pmatrix} + \begin{pmatrix}0\\3\\-3\end{pmatrix} + \begin{pmatrix}-16\\0\\0\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Since we have non-zero scalars (k’s) the given vectors are linearly dependent.

So, the linear combination

$$\boldsymbol{x} = 5\boldsymbol{u} - 3\boldsymbol{v} + 8\boldsymbol{w}$$

$$\boldsymbol{x} = 5\begin{pmatrix}-3\\1\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\\-1\end{pmatrix} + 8\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-15\\5\\0\end{pmatrix} + \begin{pmatrix}0\\-3\\3\end{pmatrix} + \begin{pmatrix}16\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

means that we can make the vector x out of the vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$.

Note: In the next proposition we will explain, if there are more vectors than the value of **n** in the n-space then the vectors are linearly dependent.

As in the above Example we had four vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{x}$ in R^3 and $4 > 3$, therefore the given vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{x}$ were linearly dependent.

Proposition. Let v_1, v_2, v_3, ... and v_m be different vectors in R^n.If $n < m$, that is the value of n in the n-space is less than the number m of vectors, then the vectors v_1, v_2, v_3, ... and v_m are linearly dependent.

Exercise:

Determine whether the following vectors are linearly dependent in $\mathbb{R}^2$:

(a) $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (b) $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$

(c) $\mathbf{u} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$ (d) $\mathbf{u} = \begin{pmatrix} \pi \\ -2\pi \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

(e) $\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Basis and Spanning Set

To describe a vector in R^n we need a coordinate system. A basis is a coordinate system or framework which describes the Euclidean n-space.

For example, there are infinitely many vectors in the plane R^2 but we can describe all of these by using the standard unit vectors $e_1 = (1, 0)$ in the x direction and $e_2 = (0, 1)$ in the y direction.

Spanning sets:

From the last section we know that we can write any vector in R^n in terms of the standard unit vectors

$e_1, e_2, \ldots$ and e_n.

Example: Let (a, b, c) be any vector in R^3. Write this vector $\boldsymbol{v}$ in terms of the unit vectors:

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

Vectors e_1, e_2 and e_3 specify x, y and z directions respectively

Solution: We have

$$v = (a, b, c) = a(1, 0,0) + b(0, 1, 0) + c(0, 0, 1) = ae_1 + be_2 + ce_3$$

That is, we can write the vector v as a linear combination of vectors e_1, e_2 and e_3

We say that the vectors e1, e2 and e3 span or generate R3, because the linear combination:

$$ae_1 + be_2 + ce_3$$

produces any vector in R^3. We define the term ‘span’ as follows:

Definition Consider the n vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ in the n- space, $\mathbb{R}^n$. If every vector in $\mathbb{R}^n$ can be produced by a linear combination of these vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ then we say these vectors **span** or **generate** the n-space, $\mathbb{R}^n$.

This set S = {**v1**, **v2**, **v3**, ... , **vn**} is called the spanning set. We also say that the set S spans the n-space or S spans R^n.

For example: the standard unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ span R^2.

Note: The unit vectors are not the only vectors which span R^2, there are other vectors which also span R^2, as the next example demonstrates.

Example: Consider the vectors $\boldsymbol{u} = (1, 2)$ and $\boldsymbol{v} = (-1, 1)$ in R^2.

i- Show that the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ span R^2.

ii- Write the vector $\boldsymbol{z} = (3\ 2)$ in terms of the given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ **.**

Solution: Let $w = (a, b)$ be an arbitrary vector in R^2. Consider the linear combination:

$$k\boldsymbol{u} + c\boldsymbol{v} = \boldsymbol{w} \quad \text{where k and c are scalars}$$

$$k\binom{-3}{1} + c\binom{1}{\frac{1}{3}} = \binom{a}{b}$$

So We can write this eq. in matrix form as

$$AX = w$$

Where $A = \begin{pmatrix} -3 & 1 \\ 1 & 1l3 \end{pmatrix}$ and $X = \binom{k}{c}$ and $w = \binom{a}{b}$

Where $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\binom{k}{c} = \binom{a}{b}$

Writing this as an augmented matrix, we have

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left(\begin{array}{cc|c} 1 & -1 & a \\ 2 & 1 & b \end{array}\right)$$

We execute row operations so that we can find the values for scalars k and c:

$$\begin{array}{cc} & \begin{array}{cc} k & c \end{array} \\ \begin{array}{c} R_1 \\ R_2 - 2R_1 \end{array} & \left(\begin{array}{cc|c} 1 & -1 & a \\ 0 & 3 & b - 2a \end{array}\right) \end{array}$$

From the bottom row we have

$$3c = b - 2a \quad \Rightarrow \quad c = \frac{b - 2a}{3}$$

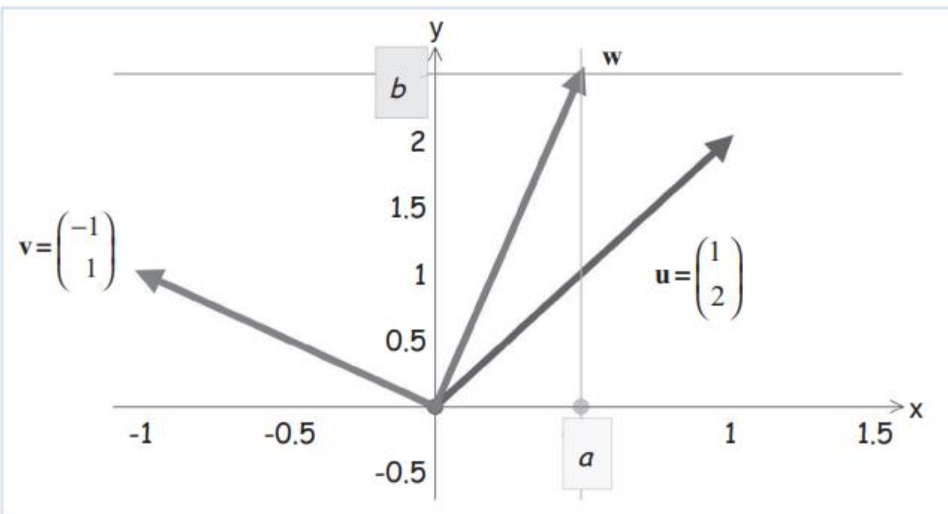
Substituting this into the top row gives:

$$k - \frac{b - 2c}{3} = a \quad \Rightarrow \quad k = \frac{b - 2c}{3} + a = \frac{b - 2c + 3a}{3} = \frac{b + a}{3}$$

So, we have found the scalars

$$k = \frac{b+a}{3} \text{ and } c = \frac{b-2a}{3}$$

Therefore $w = ku + cv$, which means that these vectors $\boldsymbol{u}$ and **v** span or generate R^2. We can illustrate these vectors as shown in Figure



Hence any vector in R^2 can be written as:

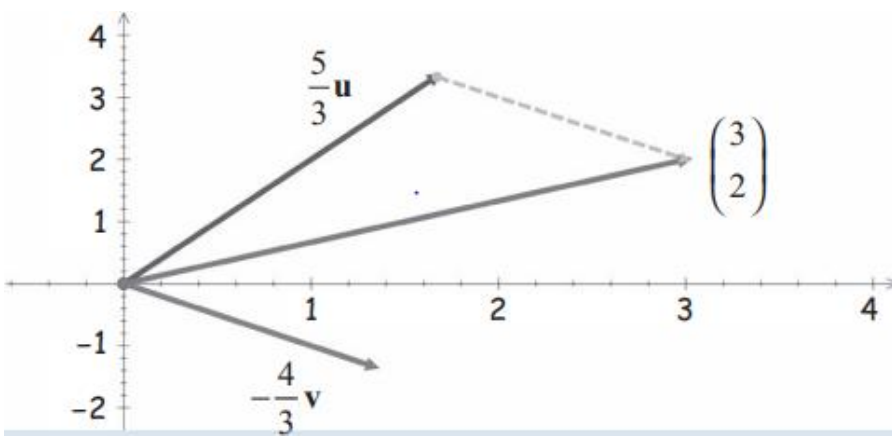
$$\begin{pmatrix} a \\ b \end{pmatrix} = \left(\frac{a+b}{3} \right) \mathbf{u} + \left(\frac{b-2a}{3} \right) \mathbf{v}$$

iii- How do we write the vector $\boldsymbol{z} = (3\ 2)$ in terms of $\boldsymbol{u} = (1, 2)$ and $\boldsymbol{v} = (-1, 1)$ in R^2.

We can use part (i) with $a = 3$ and $b = 2$, because we have shown above that any vector in R^2 can be generated by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

Substituting these, $a = 3$ and $b = 2$, into a above, we obtain

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \left(\frac{3+2}{3}\right)\mathbf{u} + \left(\frac{2-2(3)}{3}\right)\mathbf{v} = \frac{5}{3}\mathbf{u} - \frac{4}{3}\mathbf{v}$$



Therefore, the vector $\boldsymbol{z}$ is made by adding $\frac{5}{3}\boldsymbol{u}$ and $\frac{4}{3}\boldsymbol{v}$

Note: If we cannot write an arbitrary vector $\boldsymbol{w}$ as a linear combination of vectors $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ then these vectors do not span R^n.

Example: Show that the vectors $\boldsymbol{u} = \binom{\mathbf{1}}{\mathbf{-2}}$ and $\boldsymbol{v} = \binom{\mathbf{0}}{\mathbf{0}}$ do not span R^2

Solution: Let $w = \binom{a}{b}$ be an arbitrary vector in R^2. Consider the linear combination:

$$k\boldsymbol{u} + c\boldsymbol{v} = \boldsymbol{w} \quad \text{where k and c are scalars}$$

This linear combination can be written in matrix form as

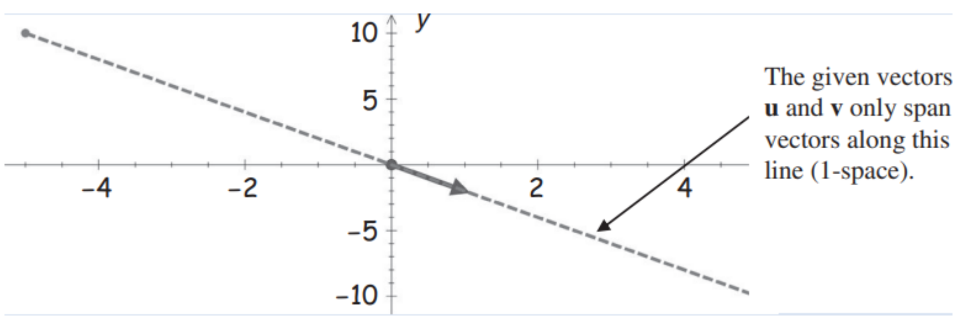
$$\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} k \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} k \\ -2k \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

We have the simultaneous equations

$$k = a \text{ and } -2k = b \quad \Rightarrow \quad b = -2a$$

This case only works if $b = -2a$, that is for the vector $w = \binom{a}{b} = \binom{a}{-2a} = a\binom{1}{-2}$

Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ only span (generate) vectors in the direction of $\binom{1}{-2}$



Any vector away from the dashed line in Figure above cannot be made by a linear combination of the given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

We conclude that the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ do not span R^2.

Determine whether the following vectors span $\mathbb{R}^2$:

(a) $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (b) $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(c) $\mathbf{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (d) $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 10 \end{pmatrix}$

Basis

We want a simple way to write down our vectors.

How can we do this?

Given some vectors we can generate others by a linear combination. We need just enough vectors to build all other vectors from them through linear combination. This set of just enough vectors is called a basis.

An example is the standard unit vectors $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$ for R^2.

This is the basis which forms the x and y axes of beR^2.

Each additional basis vector introduces a new direction.

Definition Consider the n vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ in the n space, $\mathbb{R}^n$. These vectors form a **basis** for $\mathbb{R}^n$ $\Leftrightarrow$

(i) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ span $\mathbb{R}^n$ and

(ii) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ are linearly independent

We can write the vectors $\boldsymbol{v_1}, \ \boldsymbol{v_2}, \boldsymbol{v_3}, \ldots$ and $\boldsymbol{v_n}$ as a set

$$B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \ldots, \boldsymbol{v_n}\}.$$

These are called the basis vectors – independent vectors which span R^n

Any vector in R^n can be constructed from the basis vectors. Bases (plural of basis) are the most efficient spanning sets.

There are many sets of vectors that can span a space. However, in these sets some of the vectors might be redundant in spanning the space (because they can be 'made' from the other vectors in the set). A basis has no redundant vectors. This is exactly what is captured by demanding linear independence in the definition.

Example: Show that the vectors $\boldsymbol{u} = \binom{1}{1}$ and $\boldsymbol{v} = \binom{1}{-1}$ form a basis for R^2.

Solution: We are required to show two things: that $\boldsymbol{u}$ and $\boldsymbol{v}$

(i) Span R^2 and
(ii) Are linearly independent.

Solution: (i) How do we verify that vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ span R^2.

Let $w = \binom{a}{b}$ be an arbitrary vector in R^2. Consider the linear combination:

$$k\boldsymbol{u} + c\boldsymbol{v} = \boldsymbol{w} \quad \text{where k and c are scalars}$$

Substituting the given vectors $\boldsymbol{u}$,$\boldsymbol{v}$ and $\boldsymbol{w}$ yields:

$$k\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

with the augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left(\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right)$$

Carrying out the row operation R2 + R1:

$$\begin{array}{c c} & \begin{array}{cc} k & c \end{array} \\ \begin{array}{c} R_1 \\ R_2 + R_1 \end{array} & \left(\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b+a \end{array}\right) \end{array}$$

Solving this for scalars gives

$$k = \frac{1}{2}(a+b) \text{ and } c = \frac{1}{2}(a-b)$$

Since $k\boldsymbol{u} + c\boldsymbol{v} = \boldsymbol{w}$ we can write any vector $\boldsymbol{w}$ in R^2 as

$$\mathbf{w} = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}(a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore, we conclude that the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ span R^2

(iii) What else do we need to show for vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ to be a basis?

We need to verify that they are linearly independent. To show linearly independence of two vectors we just need to check that they are not scalar multiples of each other.

To check that we need only show that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq c\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ for all nonzero number c (c is scalar)}$$

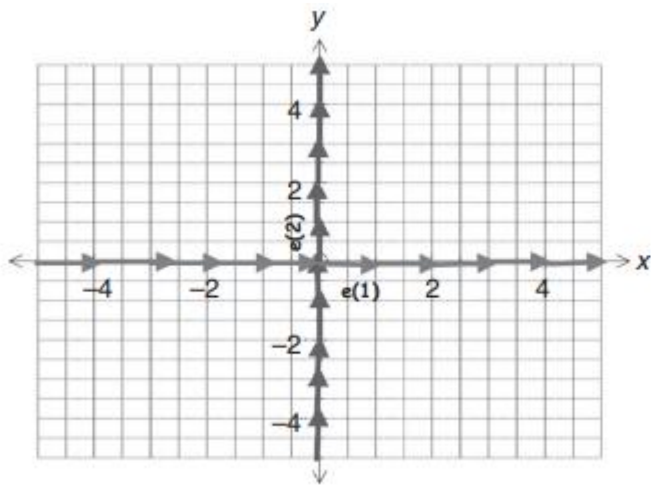
are not scalar multiples of each other.

Suppose that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ -c \end{pmatrix} \Longrightarrow c = 1 \text{ and } c = -1$

Which contradiction. So, they are linearly independent.

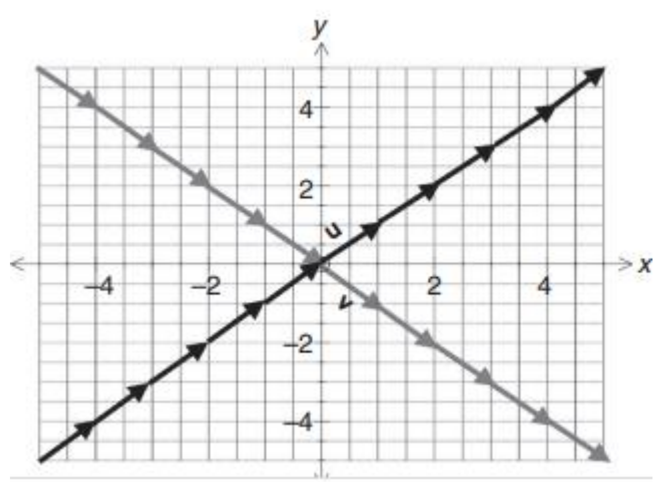
Therefore,Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ both span R^2 and are linearly independent so they are a basis for R^2.

Note: The standard unit vectors $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ (illustrated in Figure) are another basis for R^2. This is generally called the standard basis for R^2.



(b)

Figure shows some scalar multiplies of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ of the above example. These basis vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ form another coordinate system for R2 as shown. Figure shows the natural basis $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$, which form our normal **x-y** coordinate system for R^2.



Properties of bases

Proposition. Any **n** linearly independent vectors in R^n form a basis for R^n.

Note that this is an important result because it means that given **n** vectors in the n-space, R^n, it is enough to show that they are linearly independent to form a basis.

We don't need to show that they span R^n as well.

For example, we only need:

(i) 3 linearly independent vectors in R^3 to form a basis for R^3.
(ii) 10 linearly independent vectors in R^{10} to form a basis for R^{10}.

Proposition . Any **n** vectors which span R^n form a basis for R^n.

Note: again, we only need to show that **n** vectors span R^n to prove that they are a basis for R^n.

For example, we need 4 vectors that span R^4 to form a basis for R^4.

Both these results, make a lot easier because if we have **n** vectors in R^n then we only need to check one of the conditions, either independence or span.

Example: Show that the vectors $\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ form a basis for R^3.

Solution: By above result , only Required to prove that the given vectors are linearly independent in R^3.

Consider the linear combination:

$$k_1 u + k_2 v + k_3 w = 0$$

Using row operations on the augmented matrix

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \end{array}$$

$-R_1 + R_3$ we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$R_2 + R_3$ we get

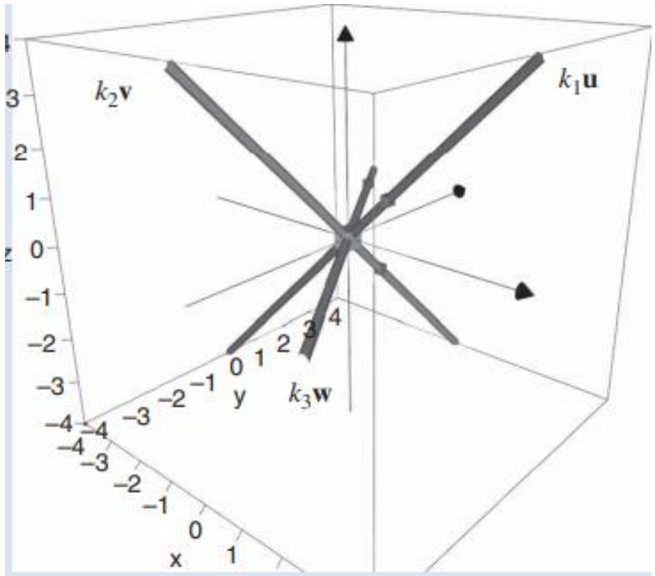
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

So, we have

$$k_1 = k_2 = k_3 = 0$$

Therefore, the given vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent because all the scalars are zero.

By the above Proposition the given three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ form a basis for R^3.



By linearly combining these vectors, we can make any vector $\boldsymbol{x}$ in R^3:

$$\boldsymbol{x} = k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w}$$

Proposition. Let the vectors $\{v_1, v_2, v_3, \ldots$ and $v_n\}$ be a basis for R^n. Every vector in R^n can be written uniquely as a linear combination of the vectors in this basis.

What does this proposition mean?

There is only one way of writing any vector as a linear combination of the basis vectors.

Proposition. Every basis of R^n contains exactly n vectors.