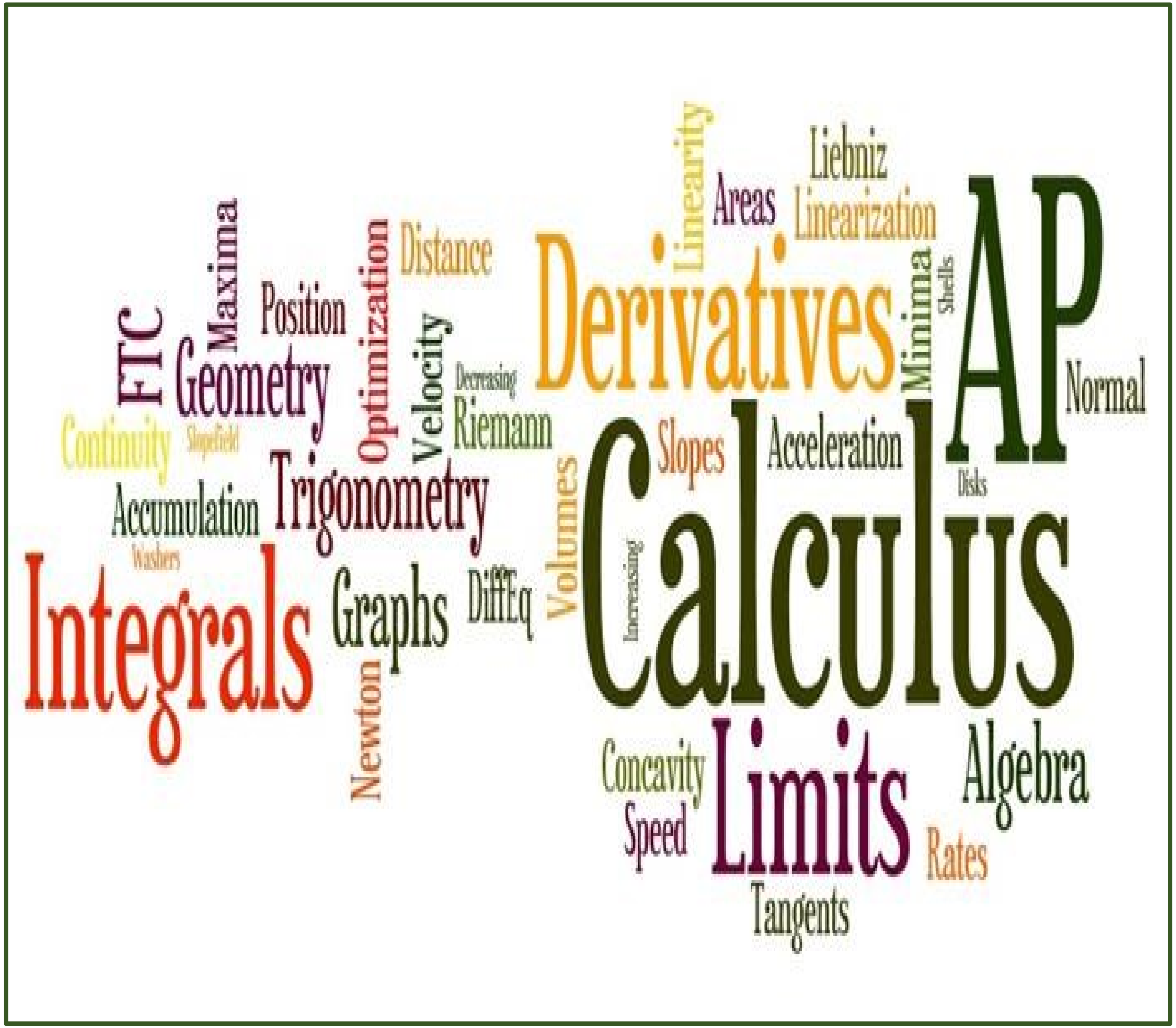


Petroleum and Mining Department
First Grade- Fall Semester

Calculus- Functions and Their Graphs (Lecture 3)

Lecturer: Ms. Jwan Khaleel M.

Lecture content

- **Functions and Their Graphs**
- **Common Functions; Mathematical Models**
- **Composite Function**

Learning Outcomes

At the end of this lecture you will be able to:

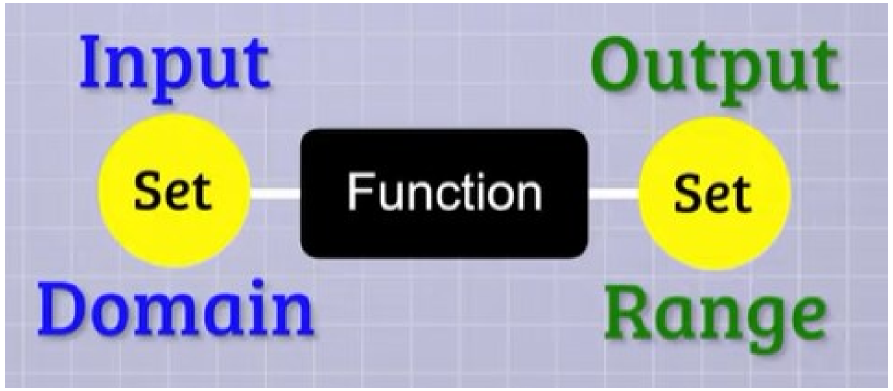
- Evaluate the concept of domain and range
- Draw the graph of functions
- Represent a Function Numerically
- Show the properties of Even Functions and Odd Functions graphs.
- Evaluate the concept of Sums, Differences, Products, and Quotients
- Evaluate the composite of functions
- Use the techniques to solve examples.

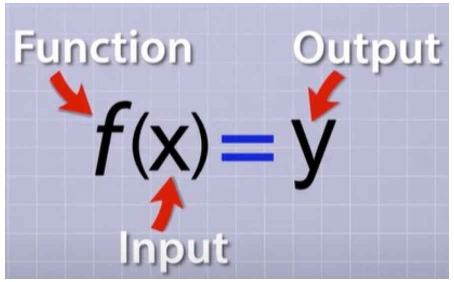
What is Function?

- Functions are a tool for describing the real world in mathematical terms.
- A function can be represented by an equation, a graph, a numerical table, or a verbal description.

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.





Example

For the function $f(x) = 3x^2 + 2x - 1$, evaluate

a. $f(-2)$

b. $f(\sqrt{2})$

c. $f(a + h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$

b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$

c. $f(a + h) = 3(a + h)^2 + 2(a + h) - 1 \quad = 3\left(a^2 + 2ah + h^2\right) + 2a + 2h - 1$

$$= 3a^2 + 6ah + 3h^2 + 2a + 2h - 1$$

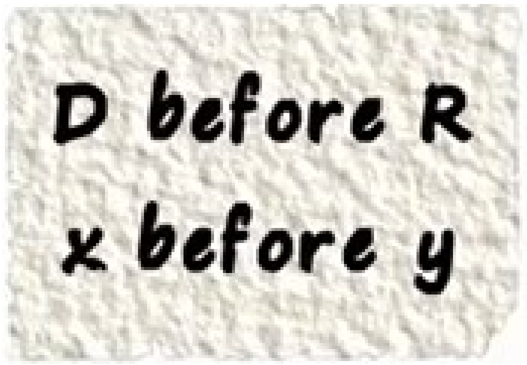
EXAMPLE If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$

SOLUTION We first evaluate $f(a+h)$ by replacing x by $a+h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

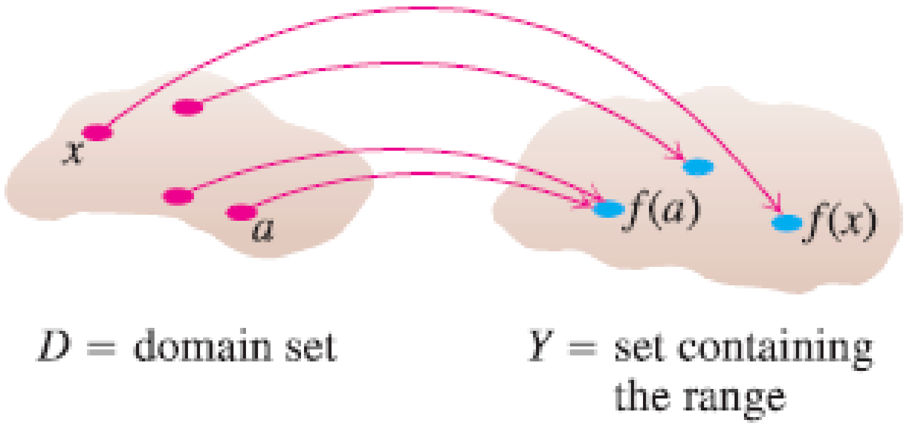
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$



- **Functions; Domain and Range**

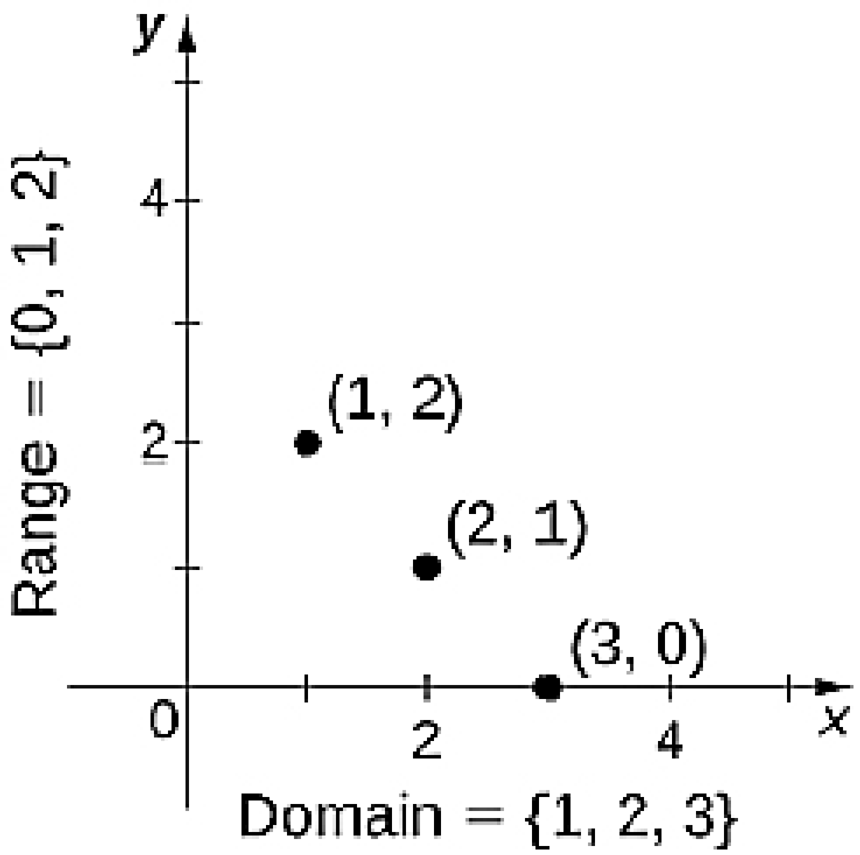
$y = f(x)$ ("y equals f of x").

- The set D of all possible input values is called the **domain** "D" of the function.
- The set of all values of $f(x)$ as x varies throughout D is called the **range** "R" of the function.
- The symbol f represents the function, the letter x is the independent variable representing the input value of f, and y is the dependent variable or output value of f at x.



Note: Range is the set of nonnegative real numbers, i.e. division by zero and $\sqrt{-ve}$ is not allowed. $f(x) = \frac{1}{x-3}$ so the domain should be any number except 3 also $f(x) = \sqrt{x-3}$.

For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$

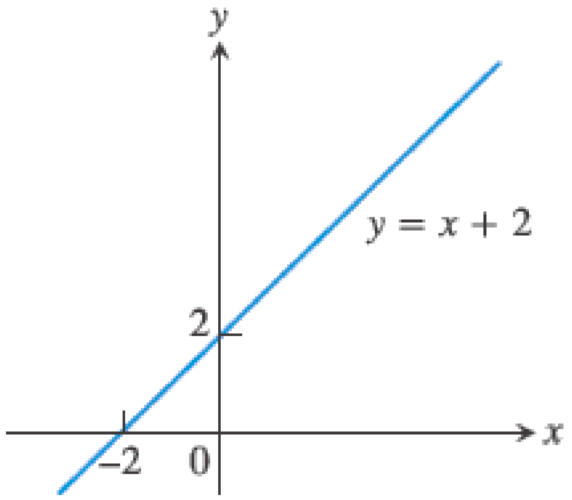


Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

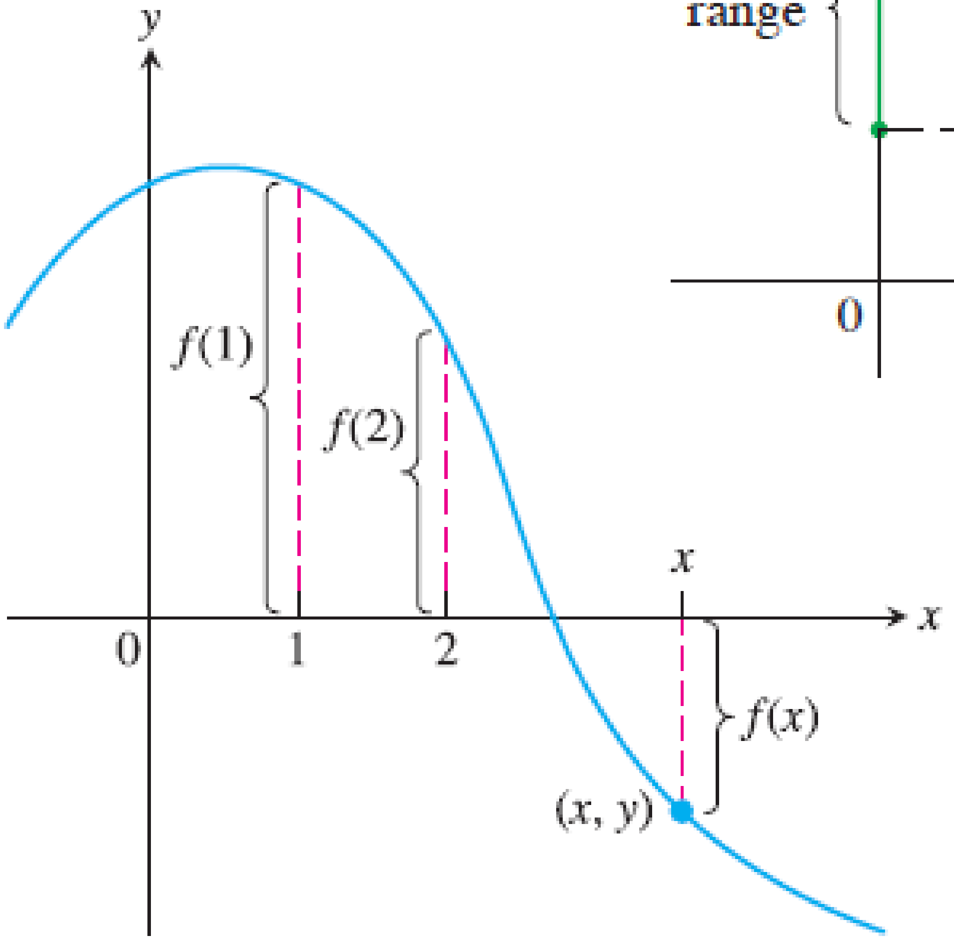
• Graphs of Function:

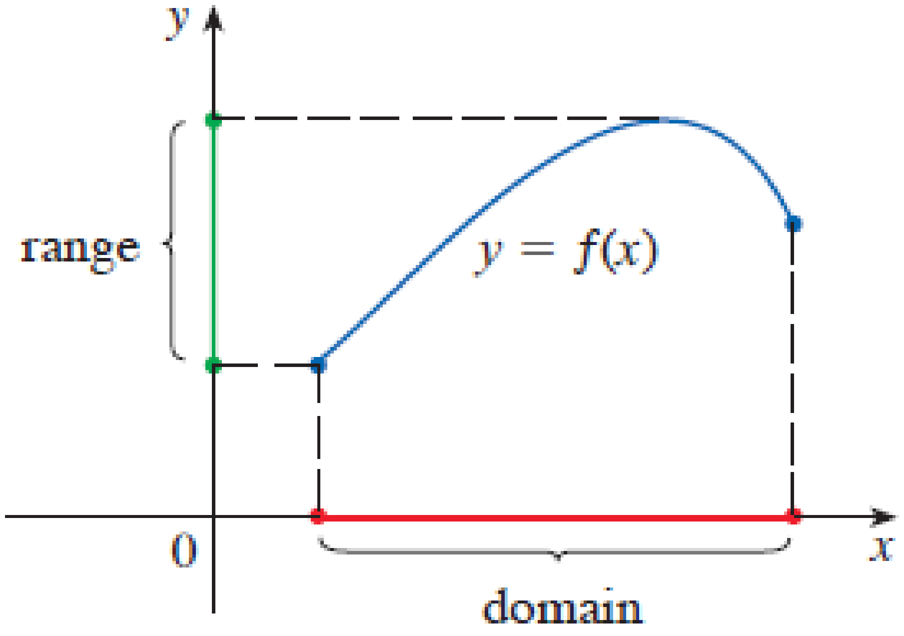
If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation. the graph is:

$$\{(x, f(x)) \mid x \in D\}$$



The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

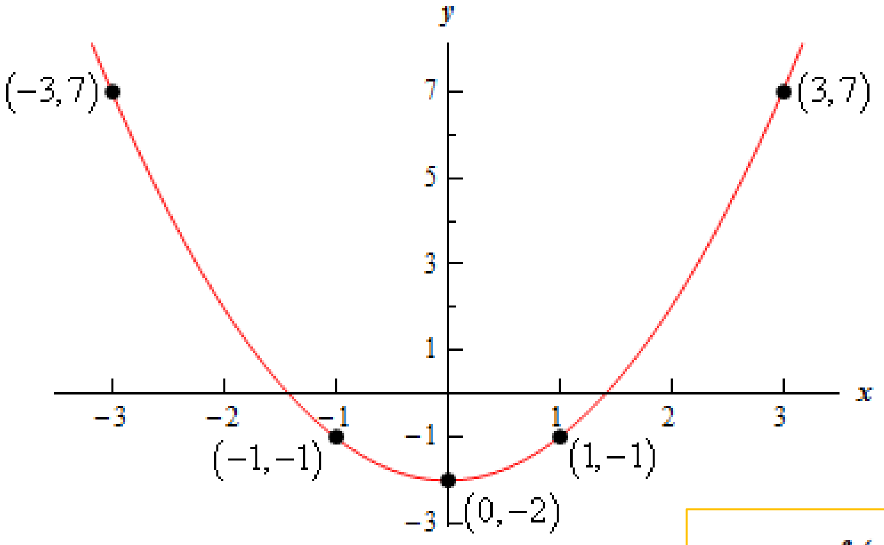




Example: Sketch the graph of the following function.

$f(x) = x^2 - 2$

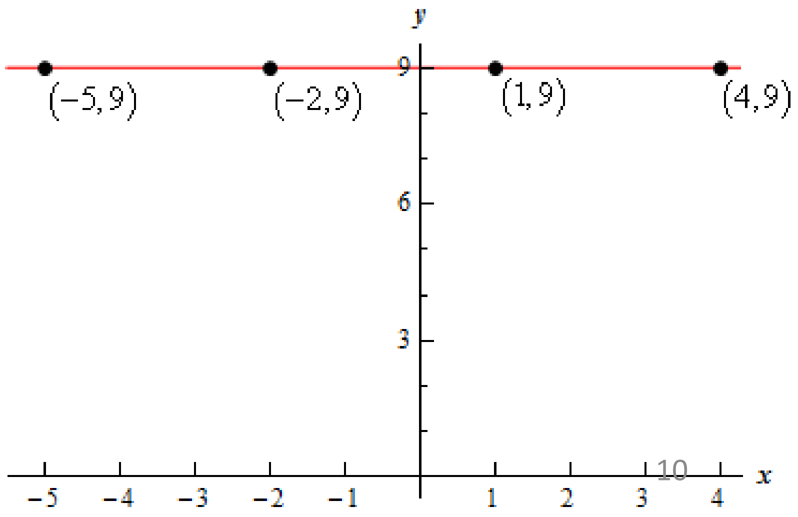
x	$f(x)$	(x, y)
-3	7	$(-3, 7)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
3	7	$(3, 7)$



Domain is $(-\infty, \infty)$ *and*
Range is $[-2, \infty)$

$f(x) = 9$

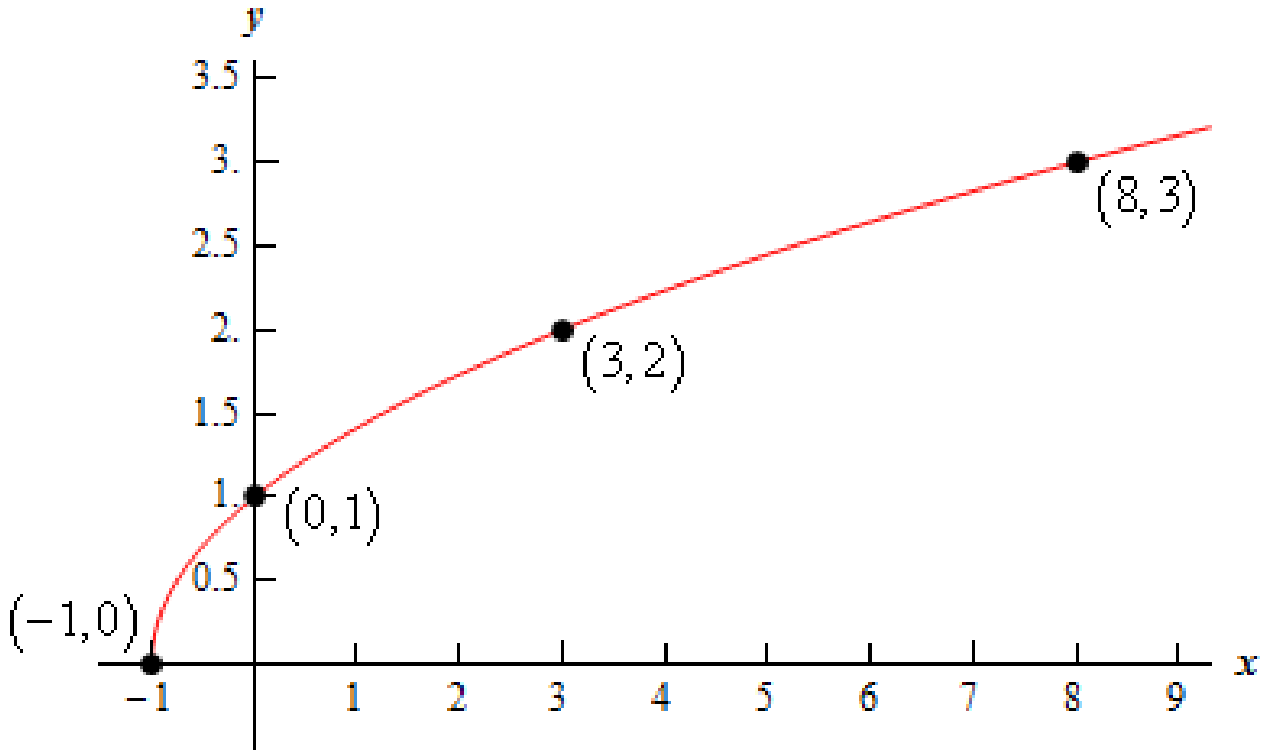
x	$f(x)$	(x, y)
-5	9	$(-5, 9)$
-2	9	$(-2, 9)$
1	9	$(1, 9)$
4	9	$(4, 9)$



Example: Sketch the graph of the following function.

$$f(x) = \sqrt{x+1}$$

x	$f(x)$	(x, y)
-1	0	$(-1, 0)$
0	1	$(0, 1)$
3	2	$(3, 2)$
8	3	$(8, 3)$



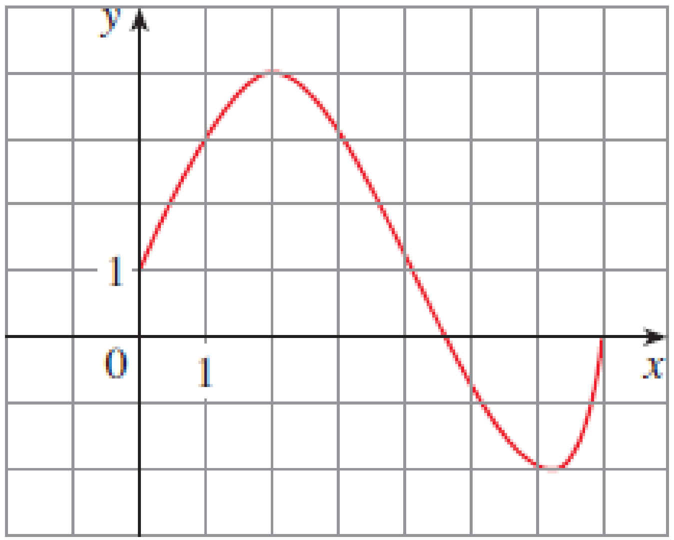
Domain is $[-1, \infty)$ *and*

Range is $[0, \infty)$

EXAMPLE // The graph of a function f is shown in Figure shown.

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?



SOLUTION

(a) We see from Figure the value of f at 1 is $f(1) = 3$.
(the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 units below the x-axis,
so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$

Example

Find the domains and ranges of each of the following are:

(a) $y = x^3 \quad -5 \le x < 4$ (b) $y = x^4$ (c) $y = \dfrac{1}{(x-1)(x+2)} \quad 0 \le x \le 6$

Solution

(a) $y = x^3 \quad -5 \le x < 4$

domain $-5 \le x < 4$, range $-125 \le y < 64$

(b) $y = x^4$

domain $-\infty < x < \infty$, range $0 \le y < \infty$

(c) $y = \dfrac{1}{(x-1)(x+2)}, \quad 0 \le x \le 6$

domain $0 \le x < 1$ and $1 < x \le 6$,

range $-\infty < y \le -0{\cdot}5,\ 0{\cdot}25 \le y < \infty$

$$(x-1)(x+2) \ge 0$$
$$x^2 + x - 2 \ge 0$$

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Example// Find the Domain and Range of each function

1. $f(x) = 2x + 3$
2. $f(x) = x^2 + 4$
3. $f(x) = \frac{1}{x}$
4. $f(x) = \sqrt{x - 4}$
5. $f(x) = \sqrt{4 - x}$
6. $f(x) = \frac{1}{\sqrt{x-4}}$
7. $f(x) = x^2 + 3x + 1$
8. $f(x) = \frac{2x+1}{x^2+5x+6}$
9. $f(x) = \frac{2}{x^2+3}$
10. $f(x) = \sqrt{x^2 + 5x + 6}$
11. $f(x) = \frac{x^2+2x+3}{\sqrt{x+1}}$
12. $f(x) = \sqrt{1 - x^2}$

- ## Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value functions**.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

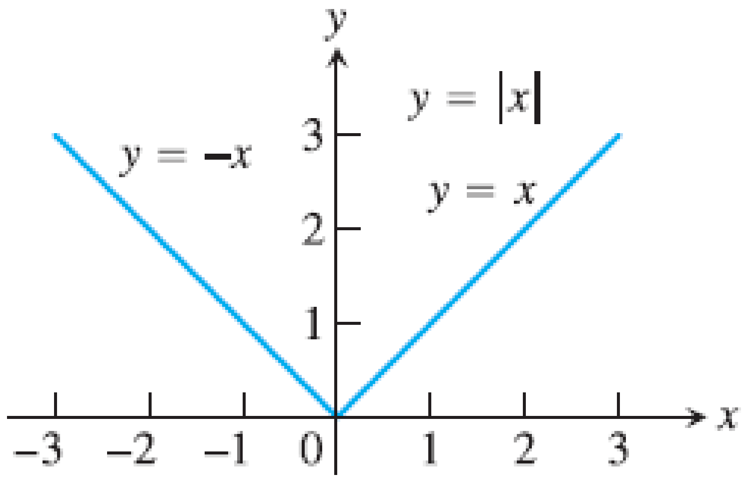
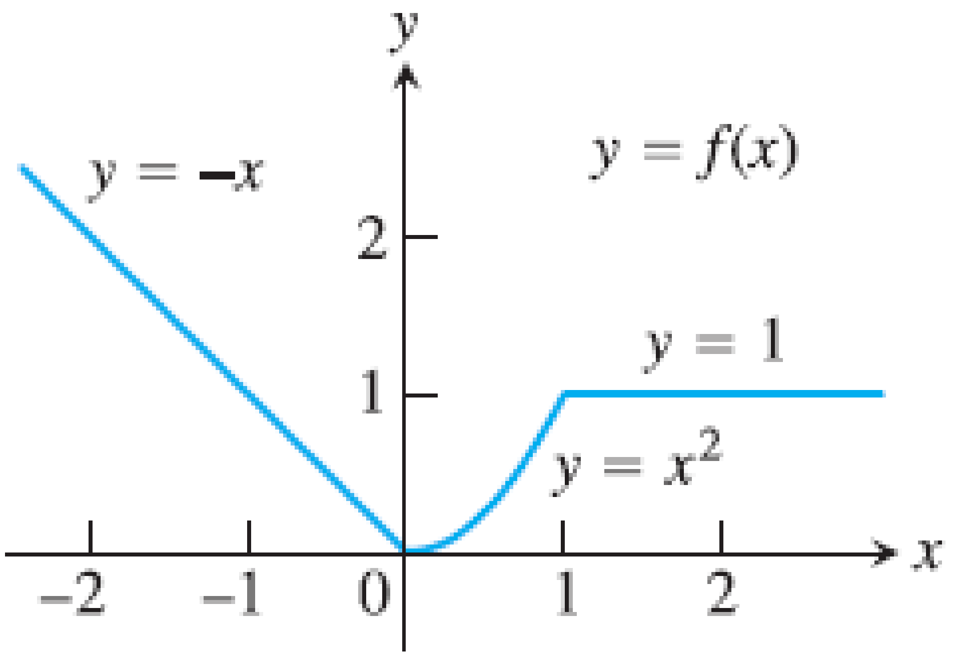


FIGURE The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

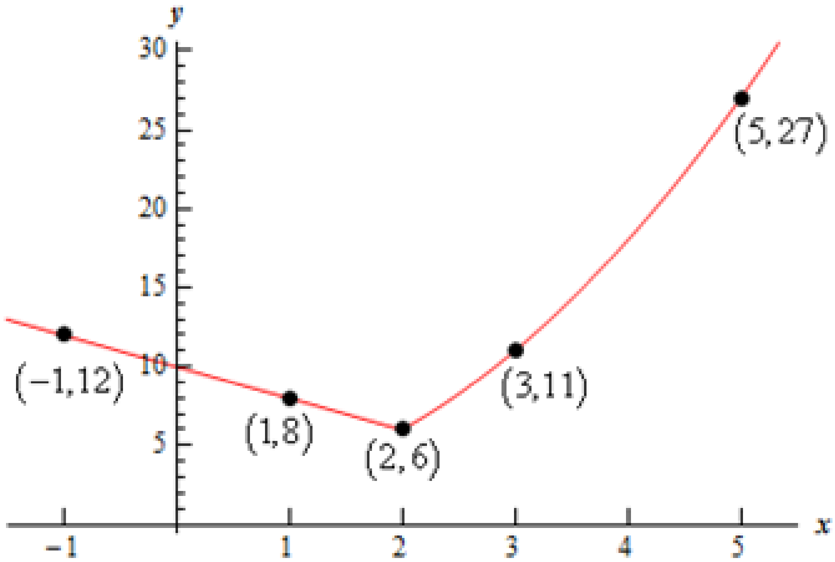


Example: Sketch the graph of the following piecewise function.

$$f(x) = \begin{cases} 10 - 2x & \text{if } x < 2 \\ x^2 + 2 & \text{if } x \geq 2 \end{cases}$$

x	$10 - 2x$	(x, y)
-1	12	$(-1, 12)$
1	8	$(1, 8)$
2	6	$(2, 6)$

x	$x^2 + 2$	(x, y)
2	6	$(2, 6)$
3	11	$(3, 11)$
5	27	$(5, 27)$

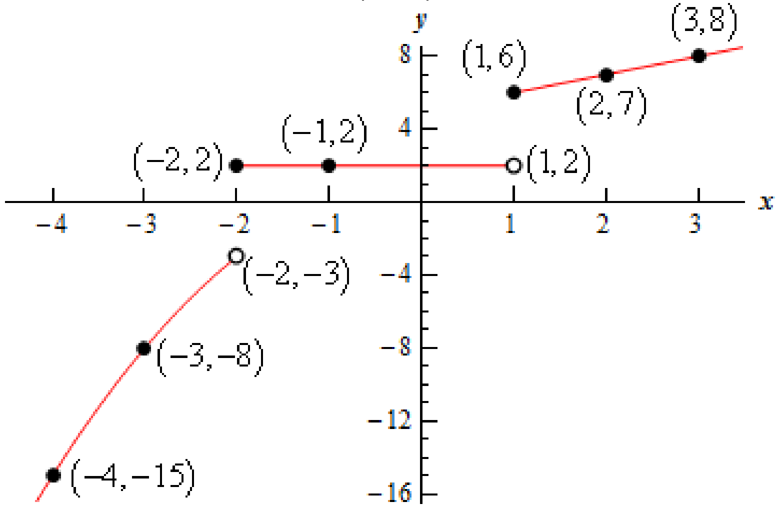


$$f(x) = \begin{cases} 5 + x & \text{if } x \geq 1 \\ 2 & \text{if } -2 \leq x < 1 \\ 1 - x^2 & \text{if } x < -2 \end{cases}$$

x	$1 - x^2$	(x, y)
-4	-15	$(-4, -15)$
-3	-8	$(-3, -8)$
-2	-3	$(-2, -3)$

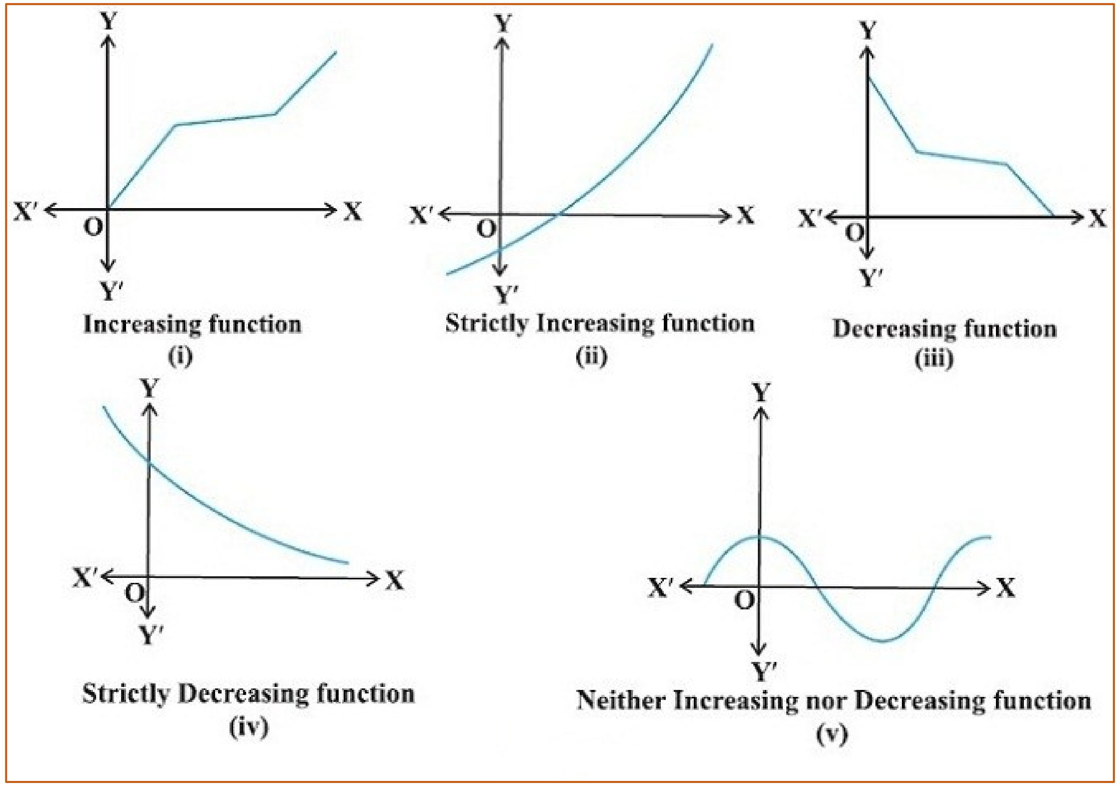
x	2	(x, y)
-2	2	$(-2, 2)$
-1	2	$(-1, 2)$
1	2	$(1, 2)$

x	$5 + x$	(x, y)
1	6	$(1, 6)$
2	7	$(2, 7)$
3	8	$(3, 8)$



• Increasing and Decreasing Functions

- If the graph of a function climbs or rises as you move from left to right, function is **increasing.**
- If the graph descends or falls as you move from left to right, the function is **decreasing.**



DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

- **Increasing and Decreasing Functions**

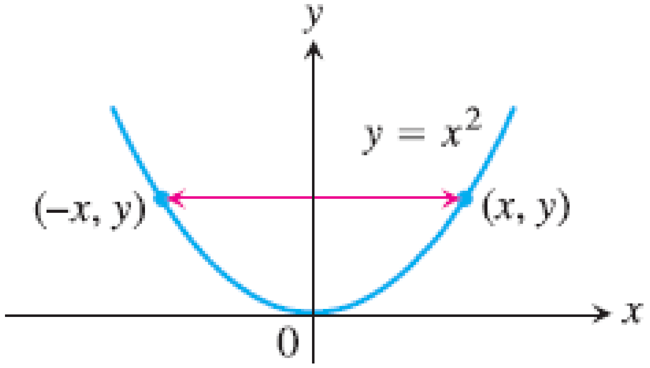
Function	Where increasing	Where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \le x < \infty$	Nowhere
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

• Even Functions and Odd Functions: Symmetry

DEFINITIONS A function $y = f(x)$ is an

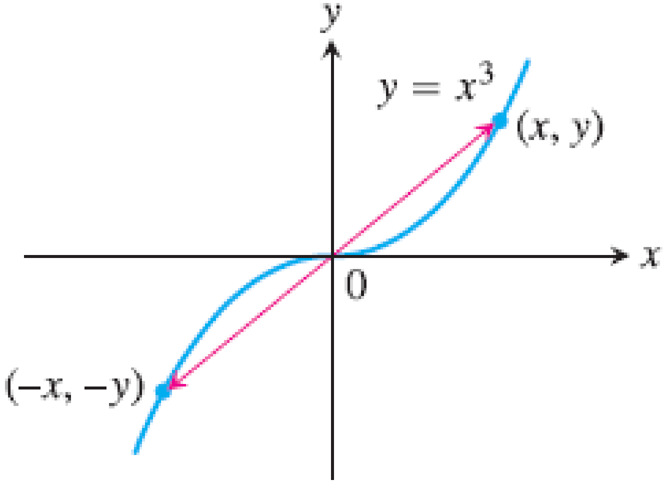
even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.



(a)

(a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis.

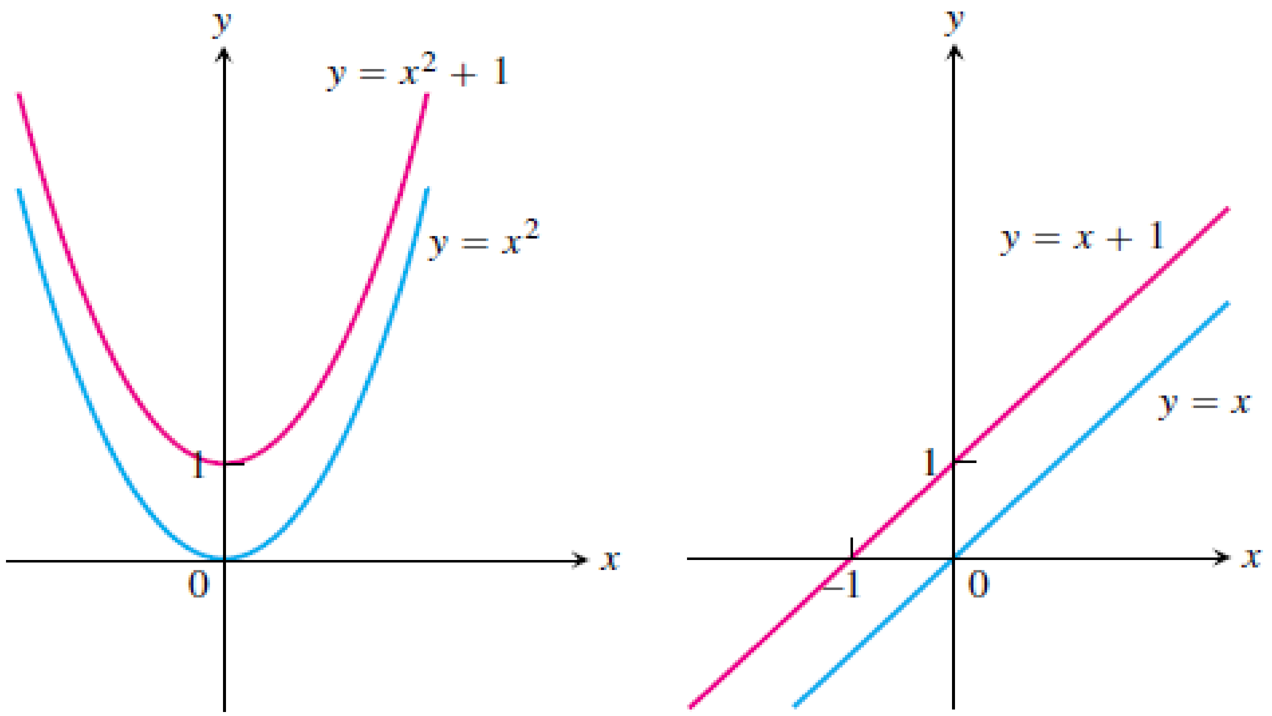


(b)

(b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

EXAMPLE Recognizing Even and Odd Functions

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 'a).
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure b).



(a) (b)

EXAMPLE Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)

$$f(-x) = (-x)^5 + (-x) = (-1)^5x^5 + (-x)$$
$$= -x^5 - x = -(x^5 + x)$$
$$= -f(x)$$

Therefore f is an odd function.

(b)

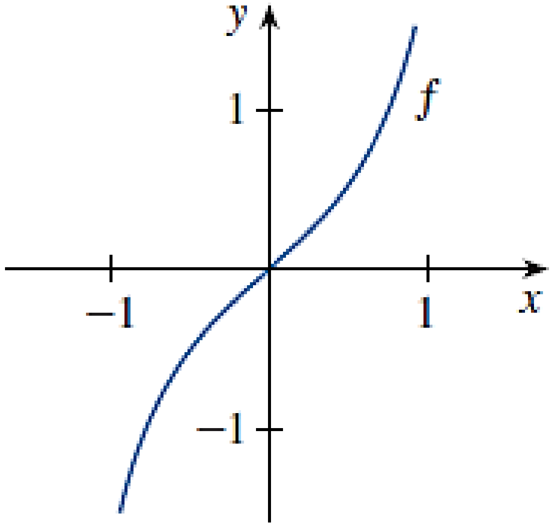
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

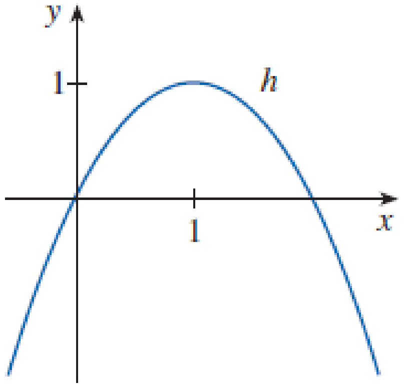
(c)

$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

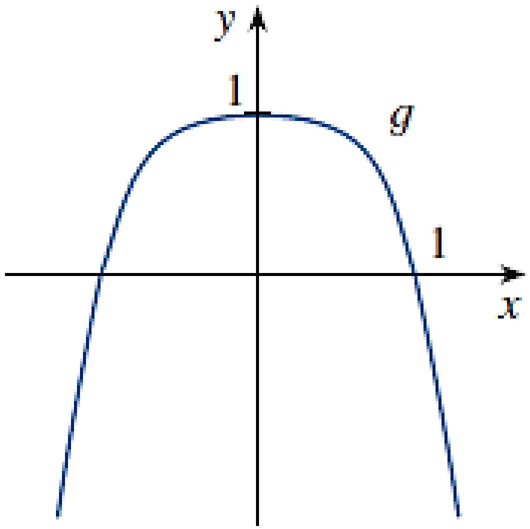
Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.



(a)



(c)

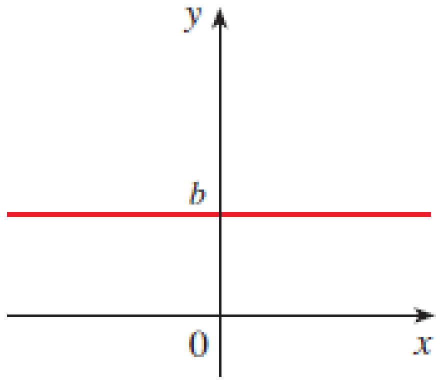


(b)

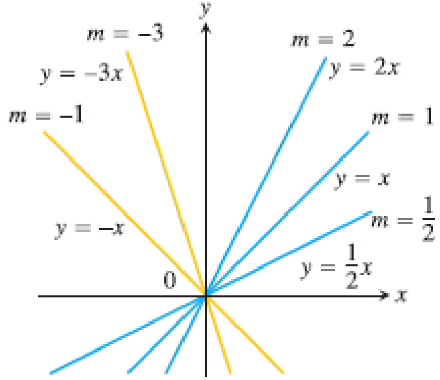
Common functions

Linear Functions

$f(x) = mx + b$



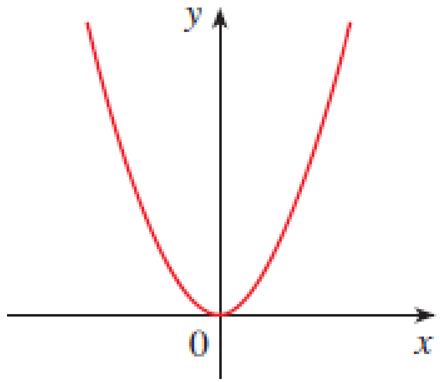
$f(x) = b$



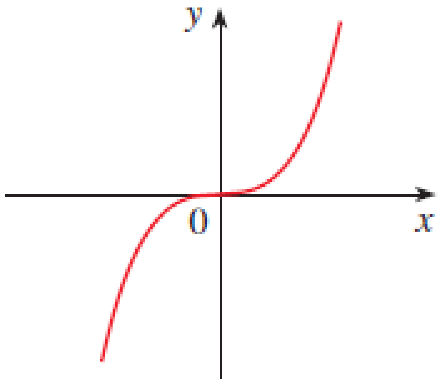
$f(x) = mx + b$

Power Functions

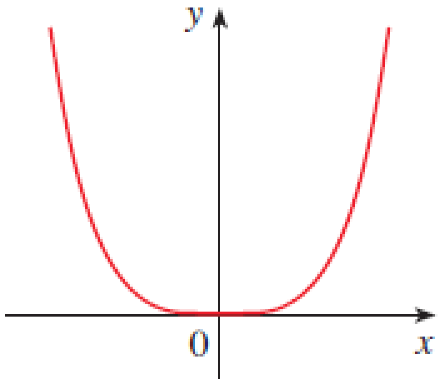
$f(x) = x^n$



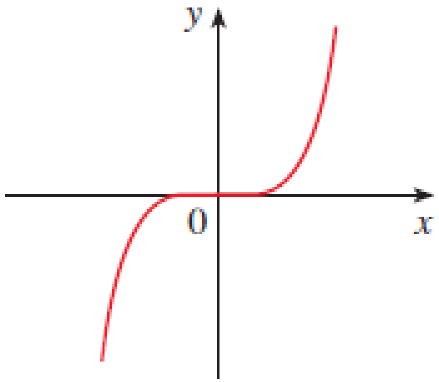
$f(x) = x^2$



$f(x) = x^3$



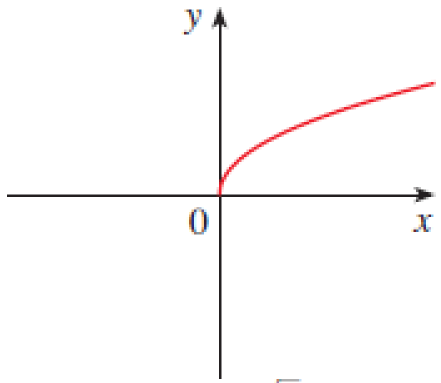
$f(x) = x^4$



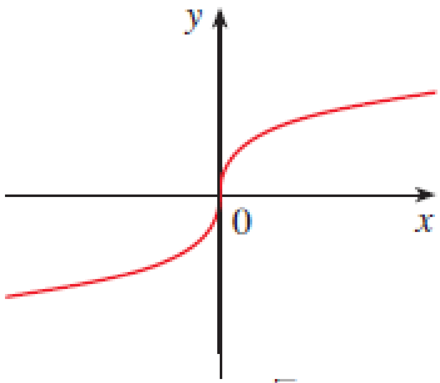
$f(x) = x^5$

Root Functions

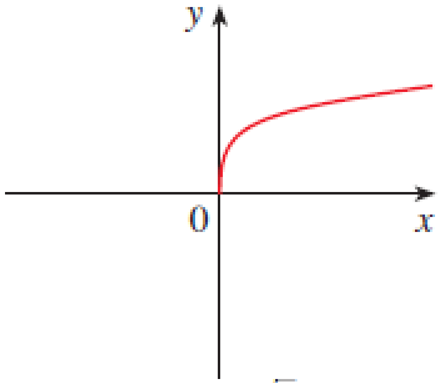
$f(x) = \sqrt[n]{x}$



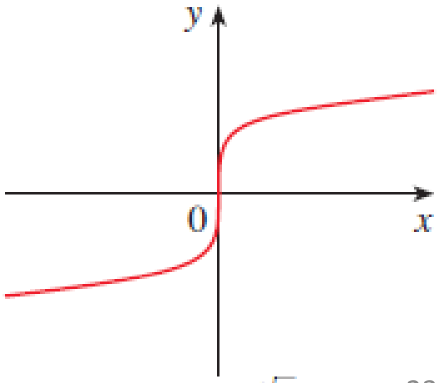
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



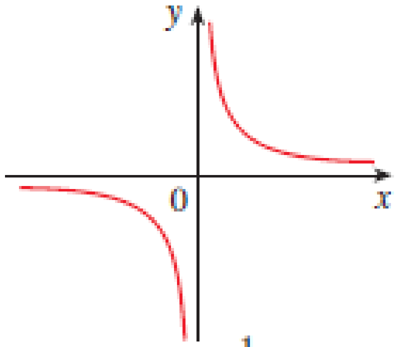
$f(x) = \sqrt[4]{x}$



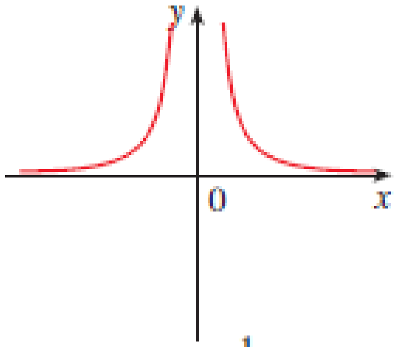
$f(x) = \sqrt[5]{x}$

Reciprocal Functions

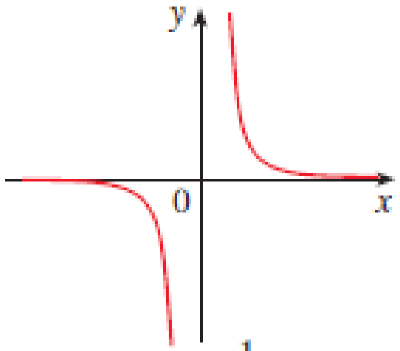
$f(x) = \frac{1}{x^n}$



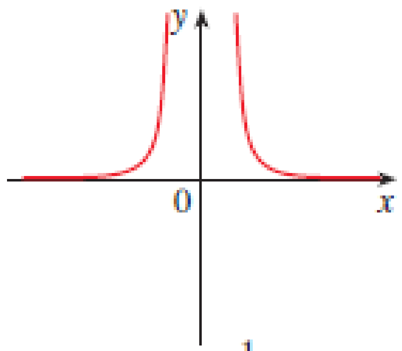
$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$



$f(x) = \frac{1}{x^3}$

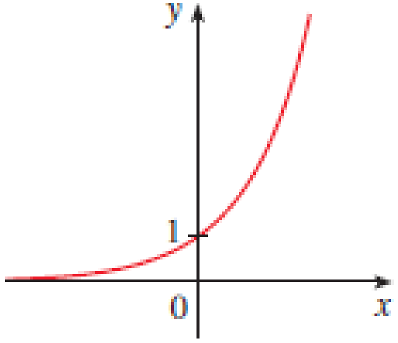


$f(x) = \frac{1}{x^4}$

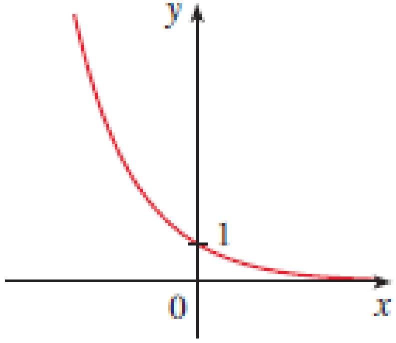
Exponential and Logarithmic Functions

$f(x) = b^x$

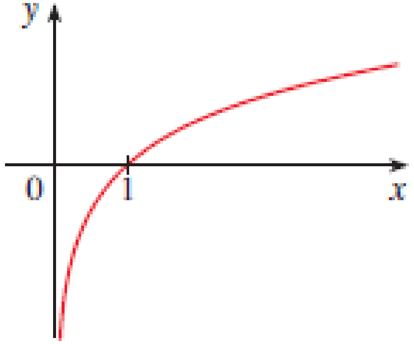
$f(x) = \log_b x$



$f(x) = b^x \ (b > 1)$



$f(x) = b^x \ (b < 1)$



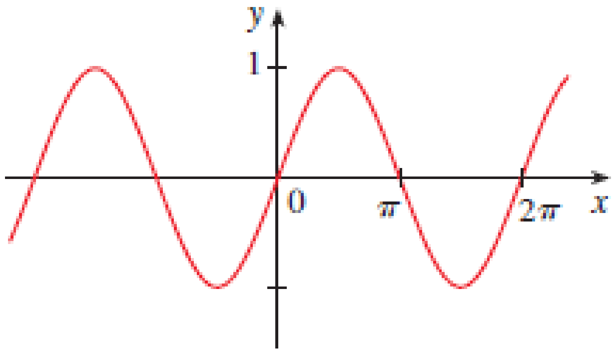
$f(x) = \log_b x \ (b > 1)$

Trigonometric Functions

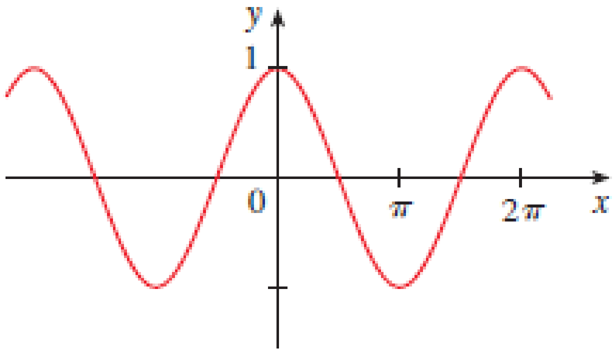
$f(x) = \sin x$

$f(x) = \cos x$

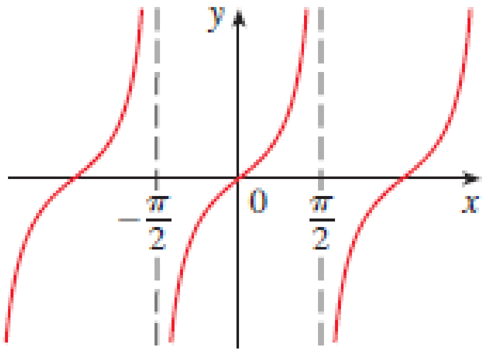
$f(x) = \tan x$



$f(x) = \sin x$



$f(x) = \cos x$



$f(x) = \tan x$

Lecturer: Jwan Khaleel M.

- **Sums, Differences, Products, and Quotients**

We define functions $f + g$, $f - g$, and fg by the formulas

$$x \in D(f) \cap D(g)$$

Definition Given two functions f and g, the **sum, difference, product**, and **quotient** functions are defined by

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\,g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Functions can also be multiplied by constants: If c is a real number, then the function $c\,f$ is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a. $(f+g)(x)$ b. $(f-g)(x)$ c. $(f \cdot g)(x)$ d. $\left(\frac{f}{g}\right)(x)$

Solution

a. $(f+g)(x) = (2x-3) + (x^2-1) = x^2 + 2x - 4$. The domain of this function is the interval $(-\infty, \infty)$.

b. $(f-g)(x) = (2x-3) - (x^2-1) = -x^2 + 2x - 2$. The domain of this function is the interval $(-\infty, \infty)$.

c. $(f \cdot g)(x) = (2x-3)(x^2-1) = 2x^3 - 3x^2 - 2x + 3$. The domain of this function is the interval $(-\infty, \infty)$.

d. $\left(\frac{f}{g}\right)(x) = \frac{2x-3}{x^2-1}$. The domain of this function is $\{x|x \neq \pm 1\}$.

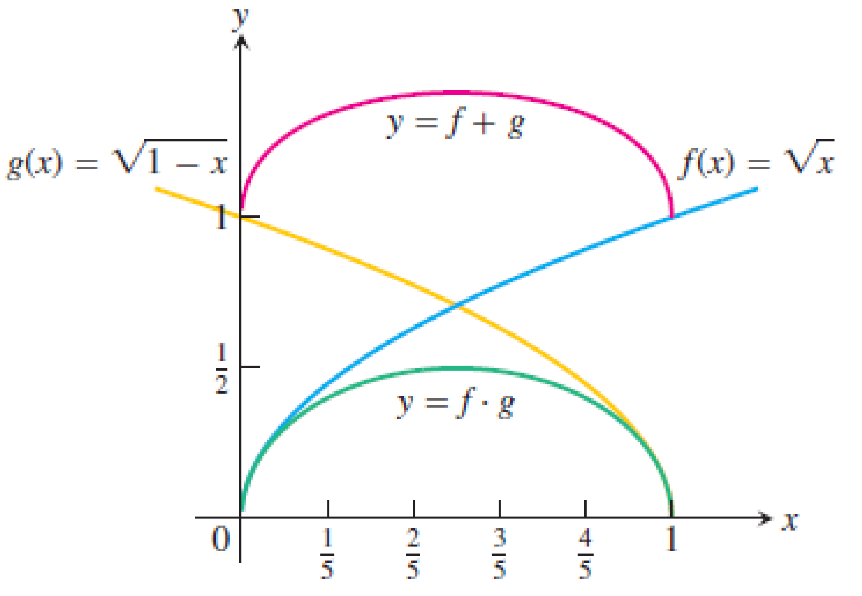
EXAMPLE The functions defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)



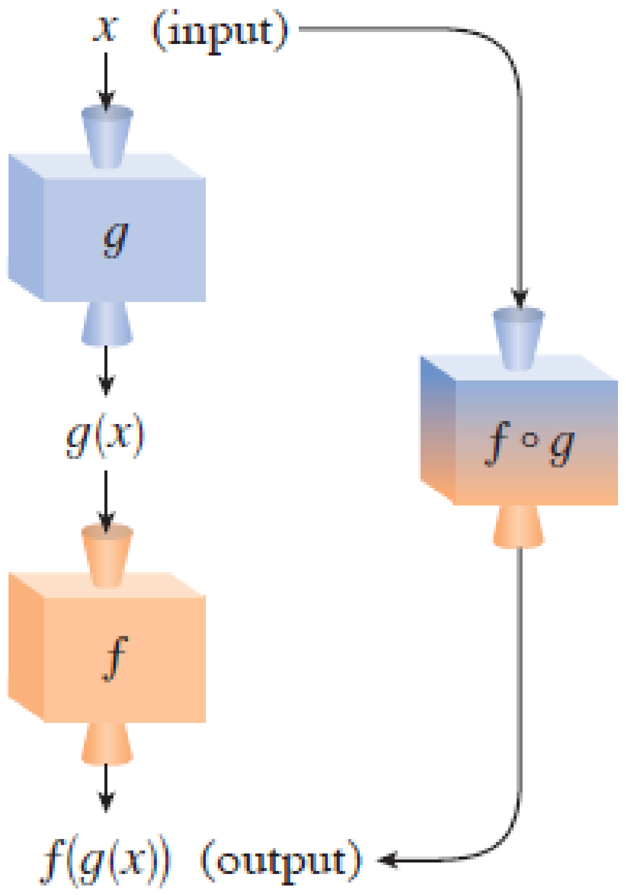
• Composite Functions

Composition is another method for combining functions.

DEFINITION If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

EXAMPLE If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

EXAMPLE Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 3))$$

$$= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}$$

EXAMPLE Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.

b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.

Solution

a. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

b. $(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$

$$(g \circ f)\left(-\tfrac{1}{2}\right) = g\left(f\left(-\tfrac{1}{2}\right)\right) = g\left(\left(-\tfrac{1}{2}\right)^2 + 1\right) = g\left(\tfrac{5}{4}\right) = \frac{4}{5}$$

EXAMPLE Consider the functions f and g described by **Table 1** and **Table 2**

Table 1

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

Table 2

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

a. Evaluate $(g \circ f)(3)$, $(g \circ f)(0)$.

b. State the domain and range of $(g \circ f)(x)$.

c. Evaluate $(f \circ f)(3)$, $(f \circ f)(1)$.

d. State the domain and range of $(f \circ f)(x)$.

SOLUTION

a. $(g \circ f)(3) = g(f(3)) = g(-2) = 0$

$(g \circ f)(0) = g(4) = 5$

b. The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$.

Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.

c. $(f \circ f)(3) = f(f(3)) = f(-2) = 4$

$(f \circ f)(1) = f(f(1)) = f(-2) = 4$

d. The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$.

Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

EXAMPLE If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

Composite	**Domain**
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$.

- **Shifting a Graph of a Function**

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

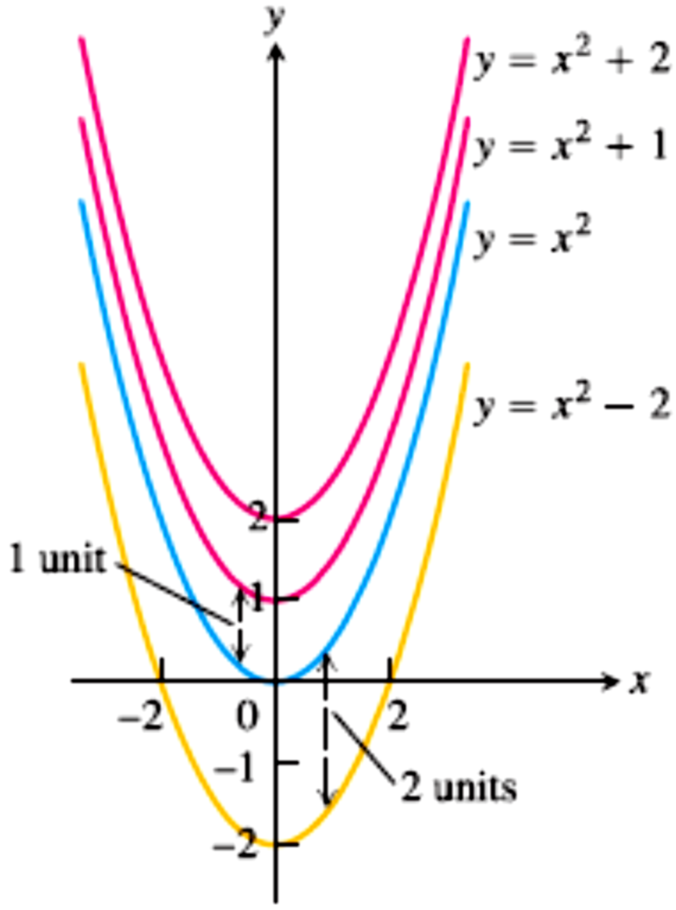
Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h \rvert$ units if $h < 0$

EXAMPLE

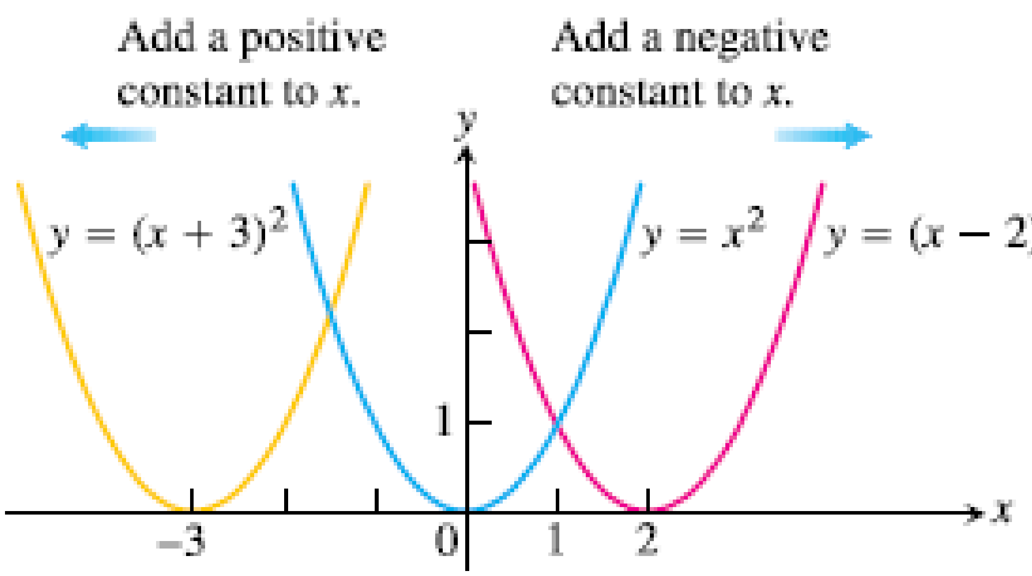
(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure).

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure).



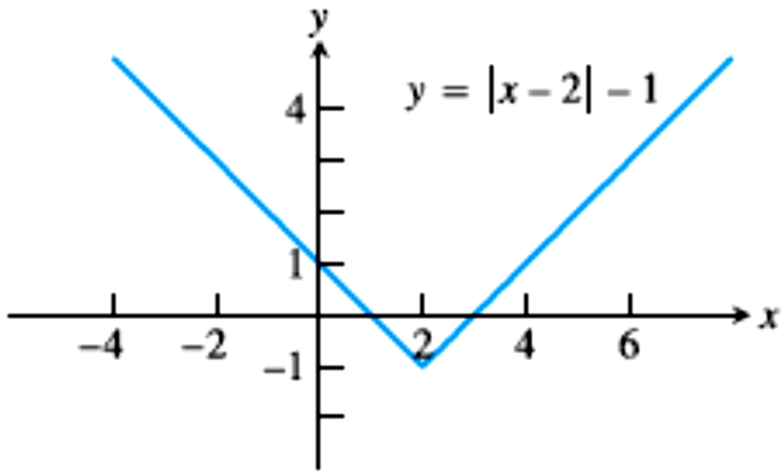
To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure).



To shift the graph of $y = x^2$ to the left, we add a positive constant to x
To shift the graph to the right, we add a negative constant to x.

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure).



Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down

- **Scaling and Reflecting a Graph of a Function**

Vertical and Horizontal Scaling and Reflecting Formulas

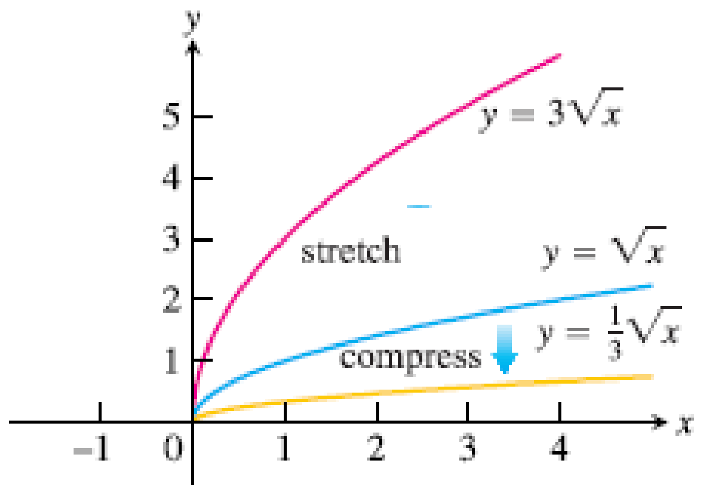
For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

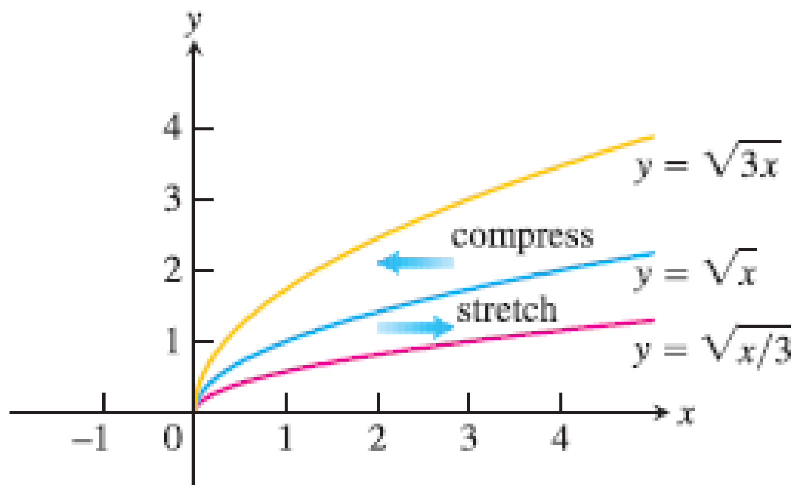
For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

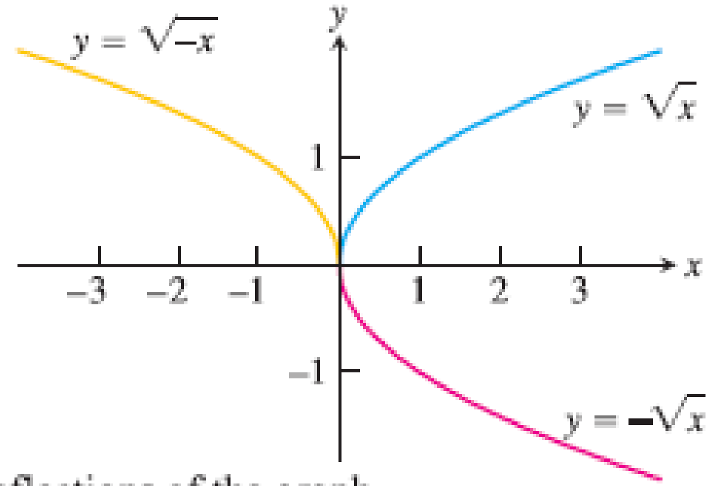
EXAMPLE scale and reflect the graph of $y = \sqrt{x}$.



Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3



Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3

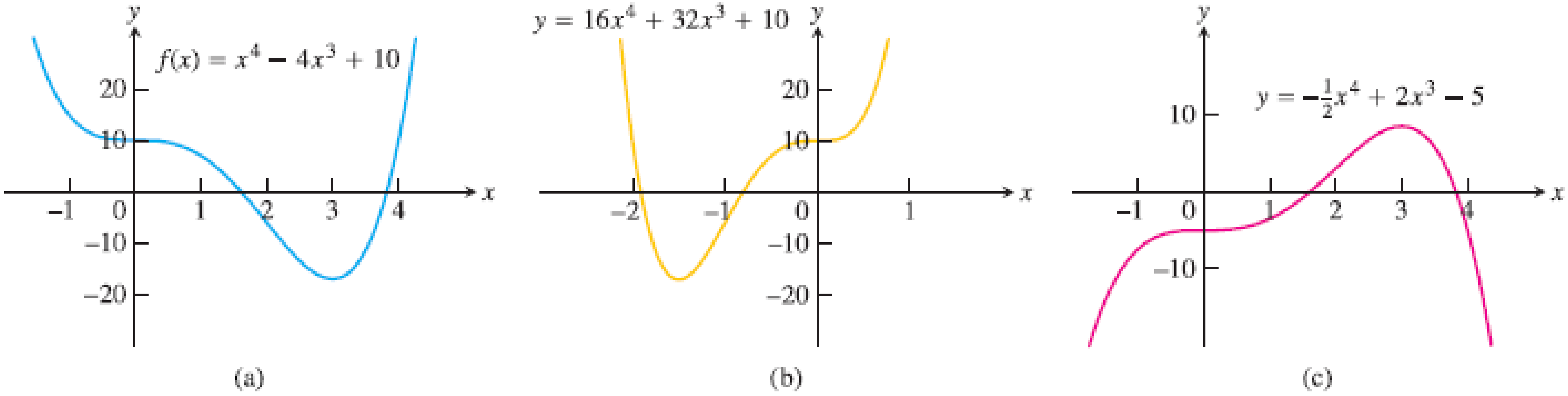


Reflections of the graph $y = \sqrt{x}$ across the coordinate axes

EXAMPLE Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure b).

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure c).



Solution

(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f:

$$y = f(-2x) = (-2x)^4 - 4(-2x)^3 + 10$$
$$= 16x^4 + 32x^3 + 10.$$

(b) The formula is

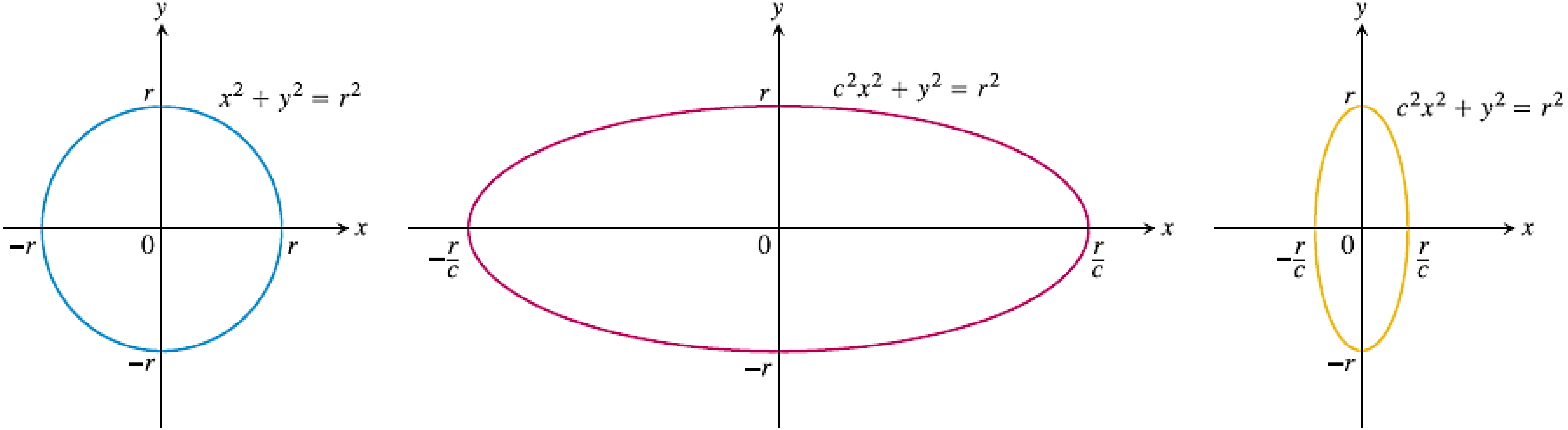
$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$

- **Ellipses**

$$x^2 + y^2 = r^2$$

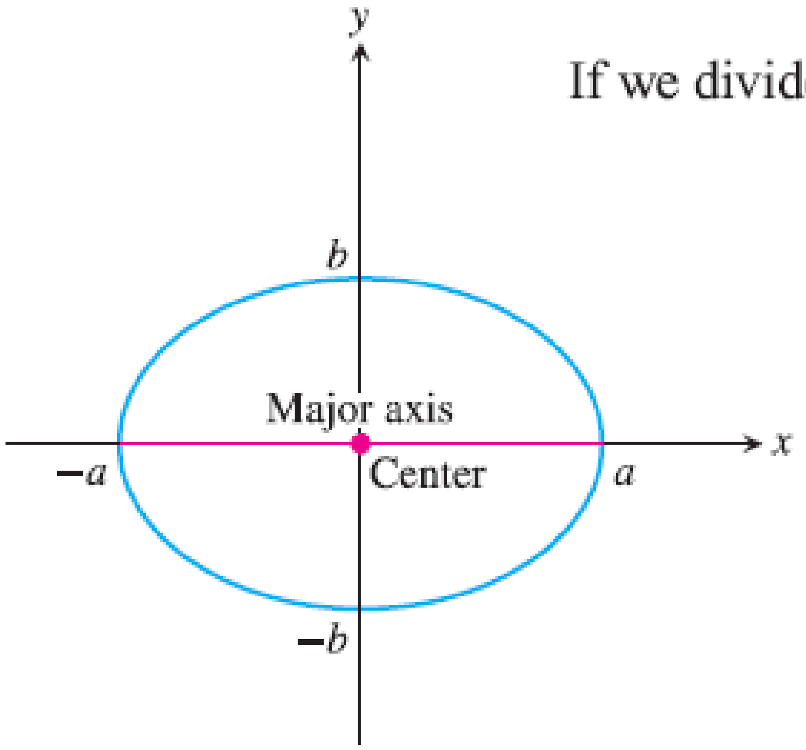
Substituting cx for x in the standard equation for a circle gives

$$c^2x^2 + y^2 = r^2 \qquad (1)$$



(a) circle (b) ellipse, $0 < c < 1$ (c) ellipse, $c > 1$

If $0 < c < 1$, the graph of Equation (1) horizontally stretches the circle; if $c > 1$ the circle is compressed horizontally. In either case, the graph of Equation (1) is an ellipse



Graph of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b,$$

where the major axis is horizontal.

If we divide both sides of Equation (1) by r^2, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2)$$

where $a = r/c$ and $b = r$. If $a > b$, the major axis is horizontal;

if $a < b$, the major axis is vertical.

The **center** of the ellipse given by Equation (2) is the origin

Substituting $x - h$ for x, and $y - k$ for y, in Equation (2) results in

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \qquad (3)$$

Equation (3) is the **standard equation of an ellipse** with center at (h, k).

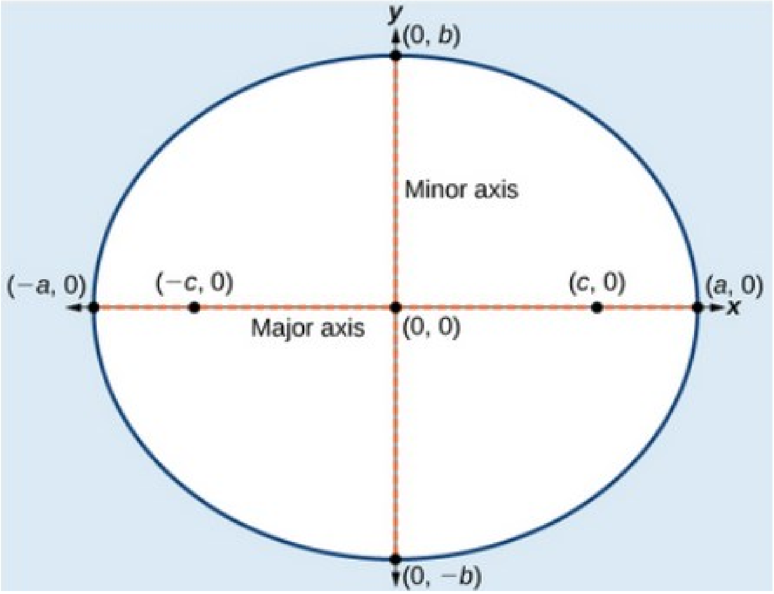
Example

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

The foci are on the x-axis, so the major axis is the x-axis.

Thus the equation will have the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The vertices are $(\pm 8, 0)$, so $a = 8$ and $a^2 = 64$.

The foci are $(\pm 5, 0)$, so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$.

Solving for b^2 we have

$$c^2 = a^2 - b^2$$

$$25 = 64 - b^2 \quad \text{Substitute for } c^2 \text{ and } a^2.$$

$$b^2 = 39 \quad \text{Solve for } b^2.$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ into the standard form of the equation.

The equation of the ellipse is

$$\frac{x^2}{64} + \frac{y^2}{39} = 1$$

Example

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

The *x*-coordinates of the vertices and foci are the same, so the major axis is parallel to the *y*-axis.

Thus, the equation of the ellipse will have the form $$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

First, we identify the center, (h,k). The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have:

$$(h,k)=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)=(-2,-3)$$

Next, we find a^2. The length of the major axis, $2a$, is bounded by the vertices. We solve for a by finding the distance between the *y*-coordinates of the vertices.

$$2a=2-(-8)$$
$$2a=10$$
$$a=5$$

So $a^2=25$.

Now we find c^2. The foci are given by $(h, k \pm c)$. So, $(h, k - c) = (-2, -7)$ and $(h, k + c) = (-2, 1)$. We substitute $k = -3$ using either of these points to solve for c.

$$k + c = 1$$

$$-3 + c = 1$$

$$c = 4$$

So $c^2 = 16$.

Next, we solve for b^2 using the equation $c^2 = a^2 - b^2$.

$$^2 = a^2 - b^2$$

$$16 = 25 - b^2$$

$$b^2 = 9$$

Finally, we substitute the values found for h, k, a^2, and b^2 into the standard form equation for an ellipse: $\dfrac{(x+2)^2}{9} + \dfrac{(y+3)^2}{25} = 1$

Assignments

A// Evaluate the difference quotient for the given function. Simplify your answer:

1. $f(x) = 4 + 3x - x^2, \qquad \dfrac{f(3+h) - f(3)}{h}$

2. $f(x) = x^3, \qquad \dfrac{f(a+h) - f(a)}{h}$

3. $f(x) = \dfrac{1}{x}, \qquad \dfrac{f(x) - f(a)}{x - a}$

4. $f(x) = \sqrt{x+2}, \qquad \dfrac{f(x) - f(1)}{x - 1}$

B// Find the Domain and Range of each function

1. $f(x) = \frac{3}{\sqrt{x^2-4}}$

2. $f(x) = 3x^2 + 6x - 2$

3. $f(x) = 2 + \sqrt{x-1}$

4. $f(x) = \sqrt{x-1}$

5. $f(x) = \frac{\sqrt{x+1}}{x^2-4}$

C// Evaluate $f(-3), f(0), and\ f(2)$ for the pricewise defined function. Then sketch the graph of the function.

1. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$

2. $f(x) = \begin{cases} 5 & \text{if } x < 2 \\ \frac{1}{2}x - 3 & \text{if } x \geqslant 2 \end{cases}$

3. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

4. $f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

D// say whether the function is even, odd, or neither. Give reason for your answer.

1. $f(x) = 3$
2. $f(x) = x^{-5}$
3. $f(x) = x^2 + 1$
4. $f(x) = x^2 + x$
5. $g(x) = x^3 + x$
6. $g(x) = x^4 + 3x^2 - 1$
7. $g(x) = \dfrac{1}{x^2 - 1}$
8. $g(x) = \dfrac{x}{x^2 - 1}$
9. $h(t) = \dfrac{1}{t - 1}$
10. $h(t) = |t^3|$
11. $h(t) = 2t + 1$
12. $h(t) = 2|t| + 1$

E//

1. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$

c. $f(g(x))$ **d.** $g(f(x))$

e. $f(f(-5))$ **f.** $g(g(2))$

g. $f(f(x))$ **h.** $g(g(x))$

2. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

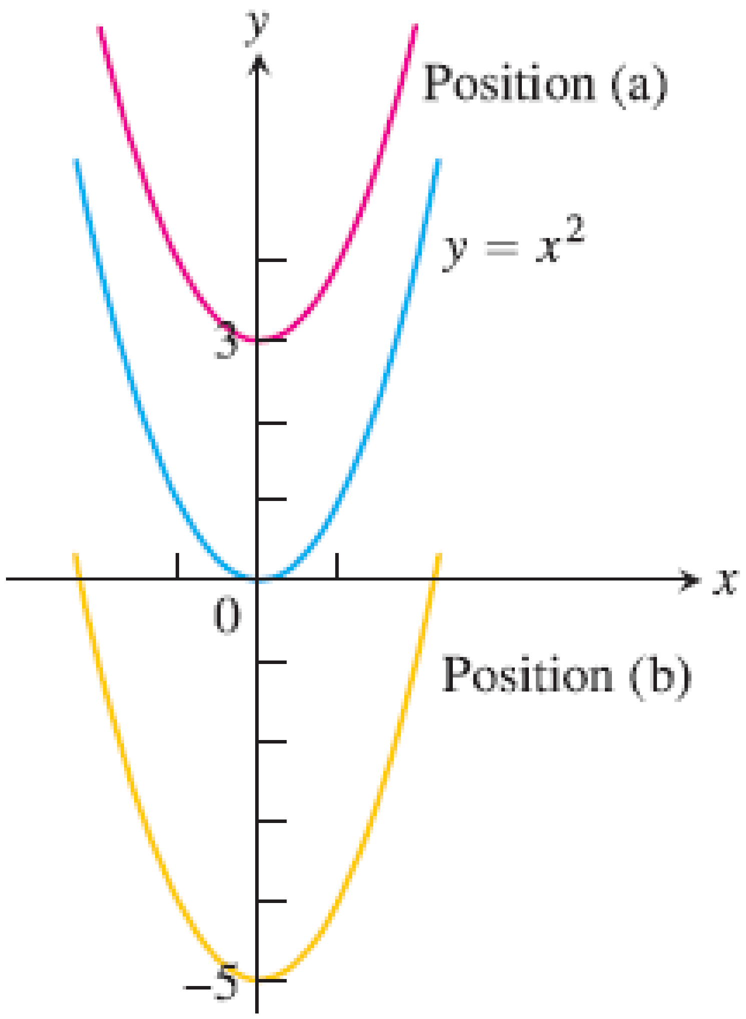
a. $f(g(1/2))$ **b.** $g(f(1/2))$

c. $f(g(x))$ **d.** $g(f(x))$

e. $f(f(2))$ **f.** $g(g(2))$

g. $f(f(x))$ **h.** $g(g(x))$

F// The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



G// Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2. \end{cases}$$

a. $f(g(0))$ **b.** $g(f(3))$ **c.** $g(g(-1))$

d. $f(f(2))$ **e.** $g(f(0))$ **f.** $f(g(1/2))$

H// Give equations of ellipses. Put each equation in standard form and sketch the ellipse.

4. $9x^2 + 25y^2 = 225$

5. $3x^2 + (y - 2)^2 = 3$

References

1. Engineering Mathematics by K.A. Stroud& Dexter J. Booth (8^{th} edition)
2. Calculus, by George B. Thomas, 12^{th} Edition
3. Calculus, Volume-1, by Gilbert Strang, Edwin “Jed” Herman
4. Calculus by James Stewart (2019)

The end of the lecture

Enjoy your time