Hardy Cross Method for Solving Pipe Network Problems

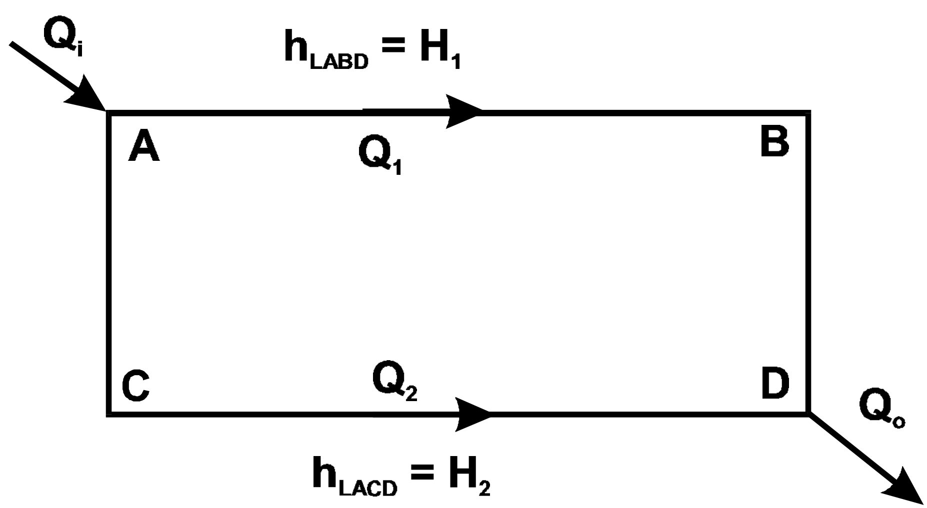
From previous manipulations of the Hazen Williams Eq.:

$$Q = \frac{0.278\,C\,D^{2.63}}{L^{0.54}} H^{0.54} = K\,H^{0.54} \qquad D\ \text{m},\ Q\ \text{m}^3/\text{s},\ H\ \text{m}$$

$$H = \frac{L}{\left[0.278\,C\,D^{2.63}\right]^{1.85}} Q^{1.85} = k\,Q^{1.85} \qquad \text{note: } k = \left(\frac{1}{K}\right)^{1.85}$$

The Hardy Cross method is an iterative procedure for analysis for pipe networks in which corrections are applied to initial estimates of flow (or headloss) until a hydraulic "balance" criteria is met.

Derivation for a simple pipe network is shown here. The method is equally valid for complex networks



There are two "balance" criteria

1. flow into a junction (node) = flow out of a junction (node)

   i.e. $Q_i = Q_1 + Q_2 = Q_o$

2. Headloss by any pathway between two junctions must be equal

i.e. $H_1 = H_2$

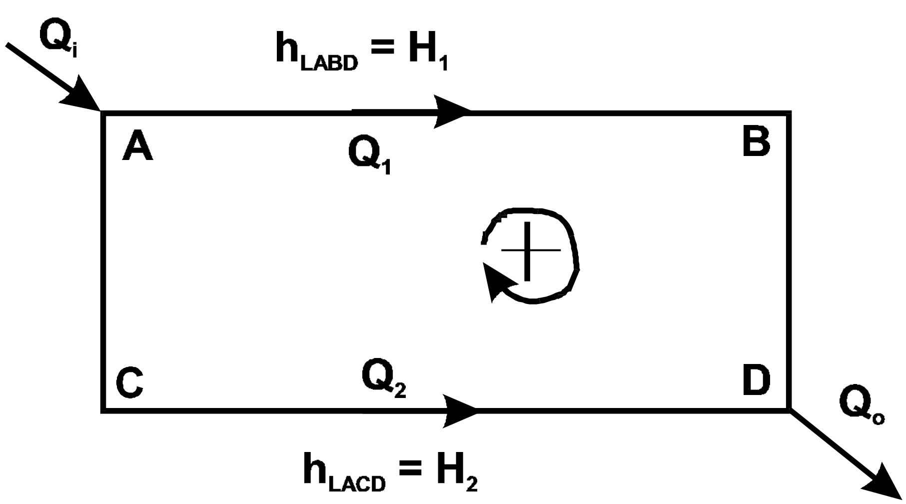
There are two methods of analysis and solution because there are two "balance" criteria:

i) balancing flows

ii) balancing headlosses

Method of Balancing Headlosses around Loops

Establish a sign convention for Q's and H's for balancing



clockwise Q's and H's are designated positive

counter clockwise Q's and H's are designated negative

For a balanced network

$$\sum H = H_1 + H_2 = 0$$

Note: the sign convention will account for + or - H

Make initial flow estimates for Q_1 and Q_2 observing continuity at each node in the system.

If the initial flow estimates are in error,(i.e. $\sum H \neq 0$) then each flow in the loop must be corrected by an amount q

i.e. $Q_1' = Q_1 + q$; $Q_2' = Q_2 + q$

For example, if the actual flows were:

$Q_i = 0.25\ m^3/s$; $Q_1 = 0.10\ m^3/s$; $Q_2 = -\ 0.15\ m^3/s$

in the directions shown, and the initial estimates were:

$Q_1 = 0.12\ m^3/s$; $Q_2 = -0.13\ m^3/s$

in the directions shown, then a correction of

$q = -0.02\ m^3/s$ is required.

Using the established sign convention

$Q_1' = Q_1 + q = 0.12 - 0.02 = 010\ m^3/s$

$Q_2' = Q_2 + q = -0.13 - 0.02 = -0.15\ m^3/s$

How can we determine the flow correction?

Substitute Hazen Williams Eq. into the balanced headloss equation

i.e. $\Sigma H = k_1(Q_1 + q)^{1.85} + k_2(Q_2 + q)^{1.85}$ note:

replace the exponent with n $k = (1/K)^{1.85}$

$\Sigma H = k_1(Q_1 + q)^n + k_2\ (Q_2 + q)^n$

Expand the term $(Q + q)^n$ by means of a Taylor series (see any mathematics handbook)

$$(Q + q)^n = Q^n + nQ^{n-1} q + \frac{n(n-1)}{2!} Q^{n-2} q^2 + \frac{n(n-1)(n-2)}{3!} Q^{n-3} q^3 + \cdots$$

$$= Q^n + nQ^n\left(\frac{q}{Q}\right) + \frac{n(n-1)\,Q^n}{2!}\left(\frac{q}{Q}\right)^2 + \frac{n(n-1)(n-2)Q^n}{3!}\left(\frac{q}{Q}\right)^3 + \cdots$$

as $\frac{q}{Q} \Rightarrow 0$ (i.e. estimates of Q improve) the third term and beyond becomes much smaller than the first two terms because $\frac{q}{Q}$ is raised to a power $\therefore$ neglect the third term and beyond

$$\sum H = k_1 Q_1^n + k_1 n Q_1^n \left(\frac{q}{Q_1}\right) + k_2 Q_2^n + k_2 n Q_2^n \left(\frac{q}{Q_2}\right) = 0$$

$$= k_1 Q_1^n + k_2 Q_2^n + q\, n \left[\frac{k_1 Q_1^n}{Q_1} + \frac{k_2 Q_2^n}{Q_2}\right] = 0$$

but $H = kQ^n$

$$\therefore \quad H_1 + H_2 + q\, n \left[\frac{H_1}{Q_1} + \frac{H_2}{Q_2}\right] = 0$$

solve for q

$$q = \frac{-[H_1 + H_2]}{n\left[\frac{H_1}{Q_1} + \frac{H_2}{Q_2}\right]} = \frac{-\ [H_1 + \ H_2]}{1.85\left[\frac{H_1}{Q_1} + \frac{H_2}{Q_2}\right]}$$

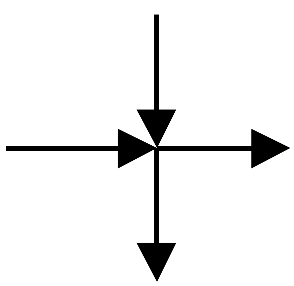
in general

$$q = \frac{-\ \Sigma H}{1.85\ \Sigma \frac{H}{Q}}$$

and more than one correction step is required because of the error associated with truncating the Taylor series term.

Method of Balancing Flows into Junctions

- require a sign convention to designate Q's



flows into a junction are +ve

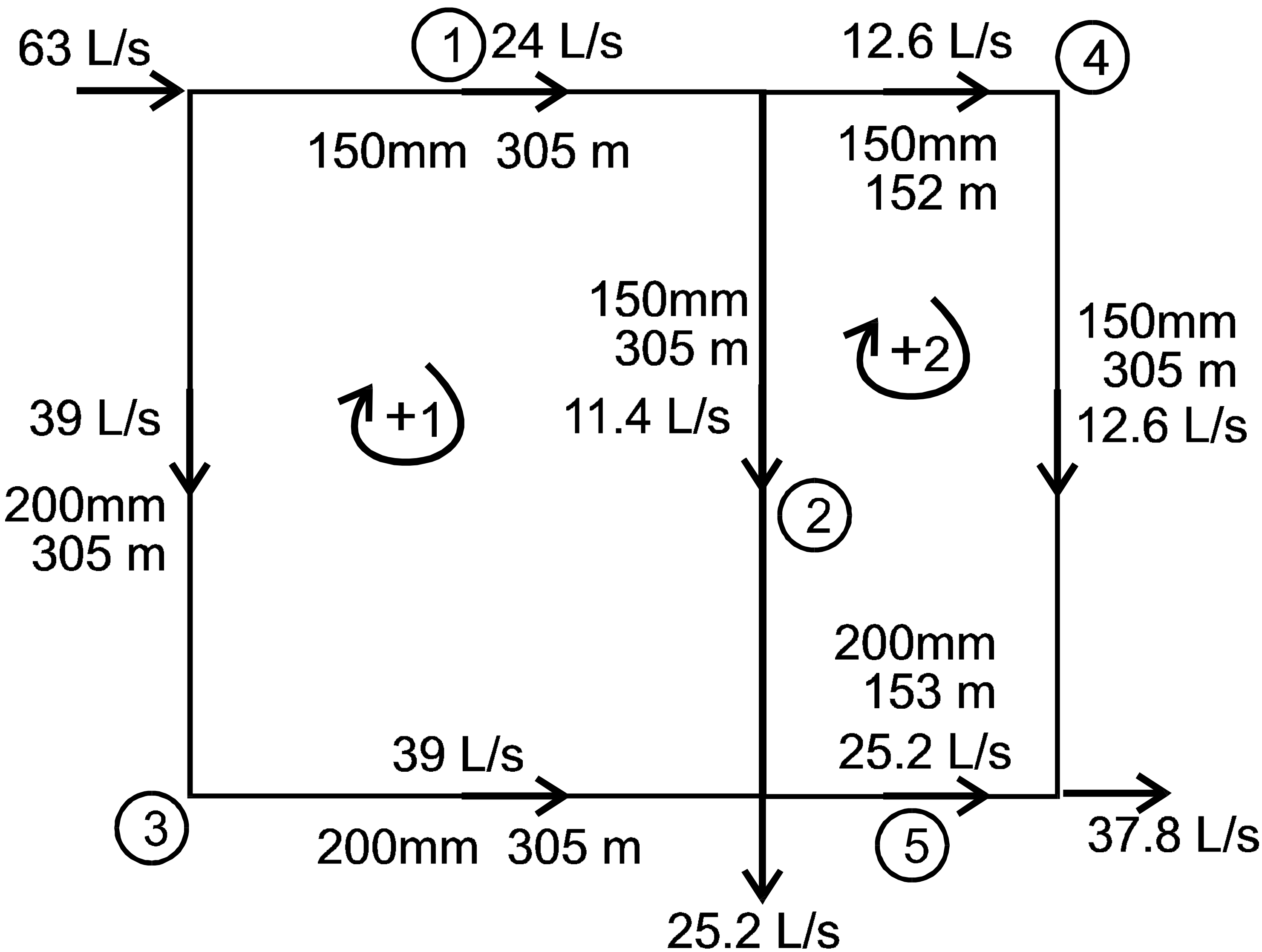
flows out of a junction are -ve

By a similar procedure to above:

$$\Delta h = \frac{-\Sigma Q}{0.54\ \Sigma \frac{Q}{H}} = \frac{-\ 1.85\ \Sigma Q}{\Sigma \frac{Q}{H}}$$

where Δh is the head correction at a node

Hardy Cross Analysis Example



C = 100 for all pipes

Hardy Cross Analysis Example

$$(1/K)^{1.85} = \frac{L}{[\beta C D^{2.63}]^{1.85}}, \beta = 278 \text{ for Q in L/s and D in m}$$

$$H = (1/K)^{1.85} Q^{1.85}$$

1st Iteration

Loop	Pipe	Dia (m)	L (m)	$(1/K)^{1.85}$	Q (L/s)	H (m)	H/Q (m/L/s)	Correction L/s	Q' L/s
1	1	0.150	305	0.0187	+24.0	+6.68	0.28	-0.24	+23.76
	2	0.150	305	0.0187	+11.4	+1.69	0.15	-0.24+0.57	+11.73
	3	0.200	610	0.0092	-39.0	-8.09	0.21	-0.24	-39.24
						+0.28	0.64		
2	2	0.150	305	0.0187	-11.4	-1.69	0.15	-0.57+0.24	-11.73
	4	0.150	457	0.0280	+12.6	+3.04	0.24	-0.57	+12.03
	5	0.200	153	0.0023	-25.2	-0.90	0.04	-0.57	-25.77
						+0.45	0.43		

$$q_{iter2loop1} = \frac{-\sum H}{n\sum H/Q} = \frac{-0.28}{1.85(0.64)} = -0.24 \quad ; \quad q_{iter1loop2} = \frac{-\sum H}{n\sum H/Q} = \frac{-0.45}{1.85(0.43)} = -0.57$$

2nd Iteration

Loop	Pipe	Dia (m)	L (m)	$(1/K)^{1.85}$	Q (L/s)	H (m)	H/Q (m/L/s)	Correction L/s	Q' L/s
1	1	0.150	305	0.0187	+23.76	+6.56	0.28	-0.15	+23.61
	2	0.150	305	0.0187	+11.73	+1.79	0.15	-0.15+0.09	+11.67
	3	0.200	610	0.0092	-39.24	-8.17	0.21	-0.15	-39.39
						+0.18	0.64		
2	2	0.150	305	0.0187	-11.73	-1.78	0.15	-0.09+0.15	-11.67
	4	0.150	457	0.0280	+12.03	+2.79	0.23	-0.09	+11.94
	5	0.200	153	0.0023	-25.77	-0.94	0.04	-0.09	-25.86
						+0.07	0.42		

$$q_{iter2loop1} = \frac{-\sum H}{n\sum H/Q} = \frac{-0.18}{1.85(0.64)} = -0.15 \quad ; \quad q_{iter2loop2} = \frac{-\sum H}{n\sum H/Q} = \frac{-0.07}{1.85(0.42)} = -0.09$$