‘Survey Adjustment’
Lecture-4
Mathematical Model

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Outline

- Mathematical Model
- Direct and indirect models
- Linear and nonlinear models
- Taylor series expansion
Mathematical Model

In surveying the measurements are rarely used directly as the required information often they are utilized in some other subsequent operations, to derive other quantities, such directions lengths, relative positions, areas and so on. The general relationships which relate the measurements to the other quantities of interest constitute what is known as “model”

Since almost all the relationships encountered in surveying computations are mathematical representations of underlying physical and geometrical conditions. The model is frequently called the “Mathematical model”.

A mathematical model is comprised of two parts

1. Functional model: describes the deterministic (physical geometric) relation between quantities.

2. Stochastic model: describes the non-deterministic (probabilistic) behavior of model quantities,

particular the observations,

   • Normal distribution
   • assessment
Types of mathematical models

There are various types of mathematical models. It can be classified as follow

A. Direct model (condition):

It is the mathematical model that express the relation between the observations and the results by using direct functions.

\[ X_{ux1} = g_{mx1}L_{nx1} \]

Where;

\( X: [X_1, X_2, \ldots, X_u] \) numbers of unknowns = \( u \)

\( g: [g_1, g_2, \ldots, X_m] \) numbers of functions = \( m \)

\( L: [L_1, L_2, \ldots, L_n] \) numbers of observations = \( n \)

\[ \Delta h_1 + \Delta h_2 + \Delta h_3 + \Delta h_4 + \Delta h_5 \neq 0 \]

So, this model is “direct” with respect to the parameters. And it has one equation per parameter \((u = m)\). The parameters are expressed directly as functions of the observations.
The direct mathematical model can be as follow

1. **Linear direct model**

The mathematical models are linear to the parameters (i.e. when the mathematical model are differentiated, they yield a vector of constant).

2. **Non-linear mathematical model**

   The models are non-linear to the parameters (i.e. when the mathematical models are differentiated, they yield a functions with degree less than the original ones). To solve this problem usually linearized model using a Taylor series expansion is adopted.

3. **Conditional direct mathematical model**

   A special case of the direct model. Where no parameters are expressed in the model. \( o = g_{m,1}(L_{n,1}) \)

   Such as estimating the internal angles of a triangle

**B. Indirect mathematical model (Parametric model)**

   It is the mathematical model that express the relation between the observations and the results in the term of another functions.

   \[ L_{n,1} = h_{m,1}(X_{u,1}) \]

   So, the model is “indirect” with respect to the parameters, and it has one equation per observation (n = m)
The indirect mathematical model can be as

A. Linear model

The following example can describe the linear indirect model.

The levelling network have the following observations, so, the model can be as follow.

\[ L = A \times X + C \]

Where: \( L \): vector of observations \((n = 5)\)
\( X \): vector of unknowns \((u = 3)\)
\( C \): vector of constants

\[
L = \begin{bmatrix}
\Delta h_1 \\
\Delta h_2 \\
\Delta h_3 \\
\Delta h_4 \\
\Delta h_5
\end{bmatrix},
X = \begin{bmatrix}
H_A \\
H_B \\
H_C
\end{bmatrix},
C = \begin{bmatrix}
H_{BM_1} \\
H_{BM_2}
\end{bmatrix}
\]

Thus, the redundancy \( = m - u = 5 - 3 = 2 \)

So, the mathematical model of this example became

\[
\begin{bmatrix}
\Delta h_1 \\
\Delta h_2 \\
\Delta h_3 \\
\Delta h_4 \\
\Delta h_5
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -10 & 0 \\
0 & -11 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
H_A \\
H_B \\
H_C
\end{bmatrix} + \begin{bmatrix}
-H_{BM_1} \\
0 \\
0 \\
-H_{BM_2} \\
0
\end{bmatrix}
\]

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B. Non-linear model

The mathematical method below can be described as a sample of non-linear indirect model

\[
\begin{bmatrix}
L_1 \\
L_2 \\
L_3
\end{bmatrix} = \begin{bmatrix}
h_1(x) \\
h_2(x) \\
h_3(x)
\end{bmatrix} = \begin{bmatrix}
(x_1 - x)^2 + (y_1 - y)^2 \\
(x_2 - x)^2 + (y_2 - y)^2 \\
(x_3 - x)^2 + (y_3 - y)^2
\end{bmatrix}^{1/2}
\]

Linearized of non-linear equations

In surveying, equations relating observations with other observations are rarely in a linear form, for example

The distance between two observed points A and B is defined as

\[
l_{AB} = [(E_A - E_B)^2 + (N_A - N_B)^2]^{1/2} = 0
\]

Such equation can not be normally capable of being expressed in term of a series of unknown multiplied by their own numerical coefficient. The solution of series of equations by matrix methods required that this solution for example. The following pair of equations.

\[
3X + 4Y = 24 , \quad 4X + 3Y = 25
\]

Expressed in the matrix form

\[
\begin{bmatrix}
3 & 4 \\
4 & 3
\end{bmatrix} \begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
24 \\
25
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
3 & 4 \\
4 & 3
\end{bmatrix}^{-1} \begin{bmatrix}
24 \\
25
\end{bmatrix}
\]
The same solution can not be done with the following non-linear equations

\[
(x - 3)^2 + (2y - 8)^2 = 5 \\
(x - 4)^2 + (y - 10)^2 = 49
\]

So, for the discussion of least squares has been based on the assumption that the condition equations are linear in the observation and parameters, because the direct use of non-linear equation is very complex and rarely done instead.

The equations are linearized using Taylor’s series expansion and solving the resulting linear equations, then iterating until the effect of the neglected higher order term is minimized.

Let \( y = f(x) \), be a function that is non-linear in \( x \). The Taylor series expansion is:

\[
y = f(x^\circ) + \frac{dy}{dx} \Delta x + \frac{1}{2!} \frac{d^2y}{dx^2} \Delta x^2 + \cdots \text{higher order term}
\]

In which \( x^\circ \) is the approximate value of the variable at which the function is evaluated. The first term of the right-hand side of above equation is the zero order term which is equal to the value of the function evaluated at \( x = x^\circ \),

The second term is the first order term, which contains the first derivative evaluated at \( x = x^\circ \), and so on
Thus, in order to have linear form. Only the zero and first order terms are used from the series expansion.

If, \( y = f(x_1, x_2) \) is a non-linear function of the two variables \( x_1, x_2 \) so the linearized form is

\[
y = f(x_{o_1}, x_{o_2}) + \frac{dy}{dx_1} x_{o_1} x_{o_2} \Delta x_1 + \frac{dy}{dx_2} x_{o_1} x_{o_2} \Delta x_2 = 0 \quad \text{(1)}
\]

Let, \( \frac{dy}{dx_1} x_{o_1}, x_{o_2} = j_1 \) and \( \frac{dy}{dx_2} x_{o_1}, x_{o_2} = j_2 \) then

The equation (1) can be written as follow

\[
y = f(x_{o_1}, x_{o_2}) + j_1 \Delta x_1 + j_2 \Delta x_2 = 0
\]

\[
G = f(x_1, x_2, x_3)
\]

\[
G = f(x_1, x_2, x_3) + j_1 \Delta x_1 + j_2 \Delta x_2 + j_3 \Delta x_3 = 0
\]

And in matrix form we have the following form

\[
y = y^* + j \Delta x = 0 \quad \text{... (3)}
\]

Where:

\[
y^* = f(x_{o_1}, x_{o_2})
\]

\[
J = [j_1 \ j_2]
\]

\[
\Delta x = [\Delta x_1 \ \Delta x_2]
\]

\[
y = y^* + [j_1 \ j_2] [\Delta x_1 \ \Delta x_2] = 0 \quad \text{or} \quad y - y^* = [j_1 \ j_2] [\Delta x_1 \ \Delta x_2] = 0
\]
The relation in equation (3) can be generalized to the case of m functions of y each in terms of some or all of n variables (x) i.e. \( y = f(x) \).

**Example-1:**

Suppose that, the following two non-linear function F(x, y) and G(x, y) expressed the relationship between observed values and unknowns x, y

\[
F(x, y) = x^2 + 3y = 115
\]

\[
G(x, y) = 5x + y^2 = 75
\]

mathematical models

And the approximate (observed values) of x and y are \( x^* = 9, y^* = 4 \)

**Solution:**

Using Taylor series expansion for linearization the two equations, we get

\[
f(x^*, y^*) + \frac{df}{dx} dx + \frac{df}{dy} dy = 115
\]

\[
G(x^*, y^*) + \frac{dG}{dx} dx + \frac{dG}{dy} dy = 75
\]

Linear form of two equations (linear mathematical model)

A/ first iteration: by substituting the values of \( x^*, y^* \) in the above equations, we get
These equations are now in linear form and containing only two unknowns (dx and dy). The solution of this pair of equations yields the following values of dx and dy
\[ \text{dx} = 1.04 \quad \text{and} \quad \text{dy} = 1.10 \]

B/ Second iteration: using correction from the first iteration new approximate values of \( x_0 \) and \( y_0 \) are computed as:
\[ x_0 = 9 + 1.04 = 10.04 \quad \text{and} \quad y_0 = 4 + 1.1 = 5.10 \]

Now, we will use the new approximate values to repeat the solution. Substituting a gain into equations above, we will get

\[ (2 * 10.04)dx + 3dy = 115 - [10.04^2 + 3(5.10)] \]
\[ 5dx + (2 * 5.10)dy = 75 - [5(10.04) + 5.10^2] \]

Solving these equations, gives that
\[ \text{dx} = -0.04 \quad \text{and} \quad \text{dy} = -0.099 = -0.10 \]

C/ Third iteration: :
\[ x_0 = 10.04 - 0.04 = 10.0 \quad \text{and} \quad y_0 = 5.10 - 0.1 = 5 \]

\[ (2 * 10)dx + 3dy = 115 - [10^2 + 3(5)] \]
\[ 20dx + 3dy = 115 - [100 + 15] \]
\[ 20dx + 3dy = 0 \quad \ldots (1) \]
\[ 5dx + (2 * 5)dy = 75 - [5(10) + 5^2] \]
5\(dx + (2 \cdot 5)dy = 75 - [5(10) + 5^2]\) 
5\(dx + 10dy = 75 - [50 + 25]\) 
5\(dx + 10dy = 0 \quad \text{(2)}\)

\[dy = \frac{-5dx}{10} \quad \text{substituting in equation one}\]

\[20dx + 3 \cdot \frac{-5dx}{10} = 0\]

\[20dx - 1.5dx = 0, \quad dx = 0\]

\[dy = \frac{-5(0)}{10} \quad dy = 0\]

So, \(x^* = x\) and \(y^* = y\)

**Example-2:**

Solve the equation of \(F(x) = x^3 - 7x = 6\). in the region of \(x = 3.5\)

**Example-3 (HW):**

Solve the following equation, using the approximate values of \(x = 5\) and \(y = 4\) and using the principle of iteration.

\[(x - 3)^2 + (2y - 8)^2 = 5\]

\[(x - 4)^2 + (y - 10)^2 = 49\]