Petroleum and Mining Department
Second Grade- Fall Semester

Statics- 3D of force (Lecture 3)

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Lecture Content:

- Cartesian Vectors
- Addition of Cartesian Vectors
- Position Vectors
- Force Vector Directed Along a Line
- Dot Product
- Solving problems

Learning Outcomes:

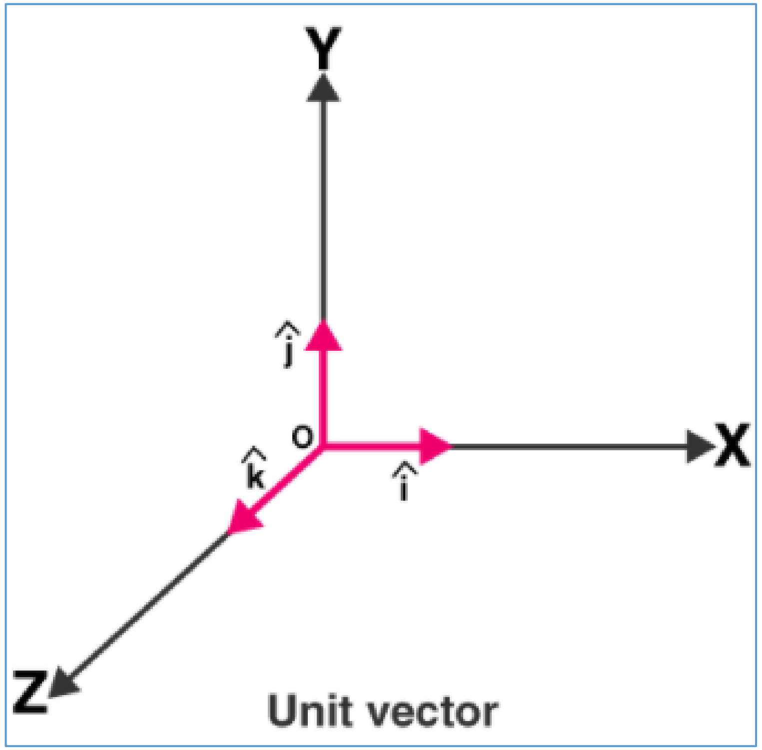
At the end of this lecture, you will be able to:

- Evaluate the system of force methods for three-dimensional system.
- Express the position vector and how to use it.
- Introduce the dot product in order to determine the angle between two vectors or the projection of one vector onto another.
- Solving problems using related equations

Unit Vector:

A vector is a quantity that has both magnitude, as well as direction. A vector that has a magnitude of **1** is a unit vector. It is also known as Direction Vector.

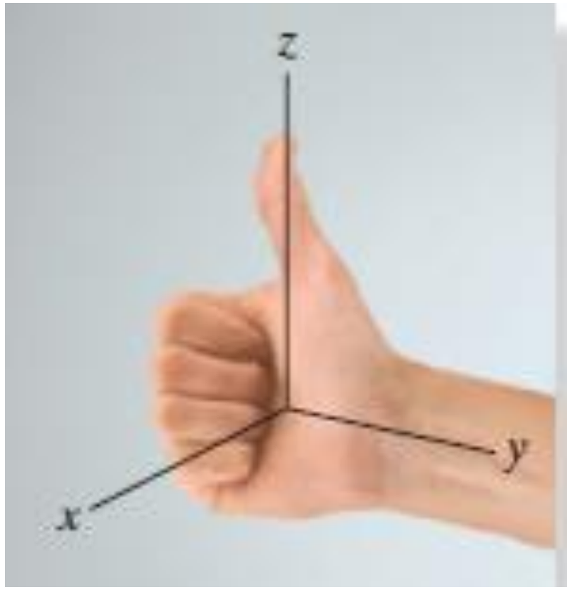
$$\text{Unit Vector} = \frac{\text{Vector}}{\text{Magnitude of Vector}}$$

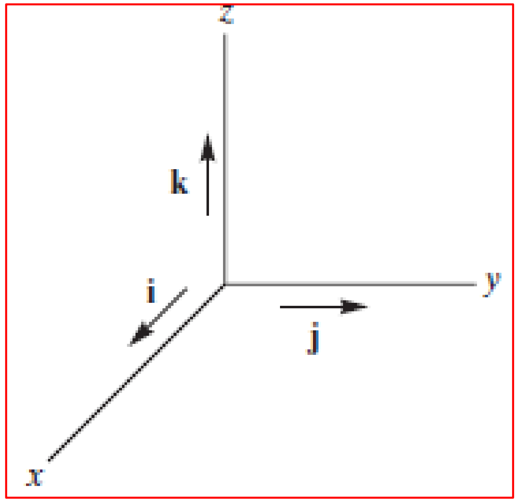


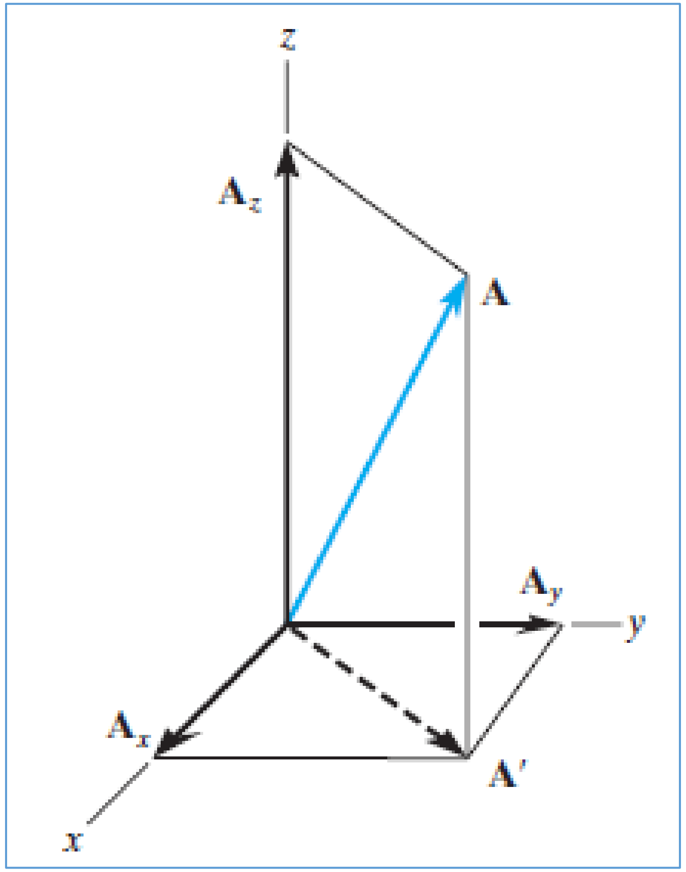
EXAMPLE// Find the unit vector $\vec{q}$ for the given vector, $-2\hat{\imath}+4\hat{\jmath}-4\hat{k}$. Show Unit vector component **q**.

$$\boldsymbol{q} = \frac{-2\hat{\imath}+4\hat{\jmath}-4\hat{k}}{\sqrt{(-2)^2+(4)^2+(-4)^2}} = \frac{-2\hat{\imath}+4\hat{\jmath}-4\hat{k}}{\sqrt{36}} = \frac{-2\hat{\imath}+4\hat{\jmath}-4\hat{k}}{6} = \frac{-2\hat{\imath}}{6} + \frac{4\hat{\jmath}}{6} - \frac{4\hat{k}}{6} = \frac{-1}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$$

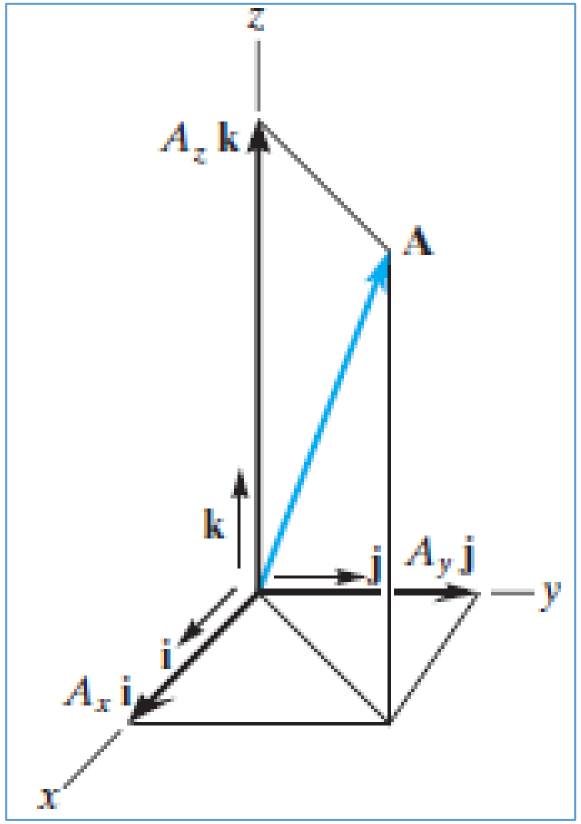
• Cartesian Vectors:



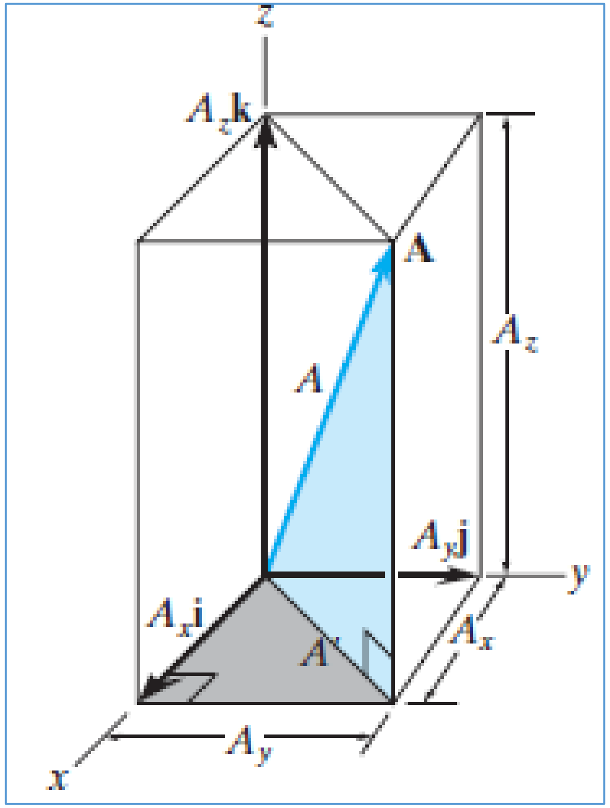




$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$$

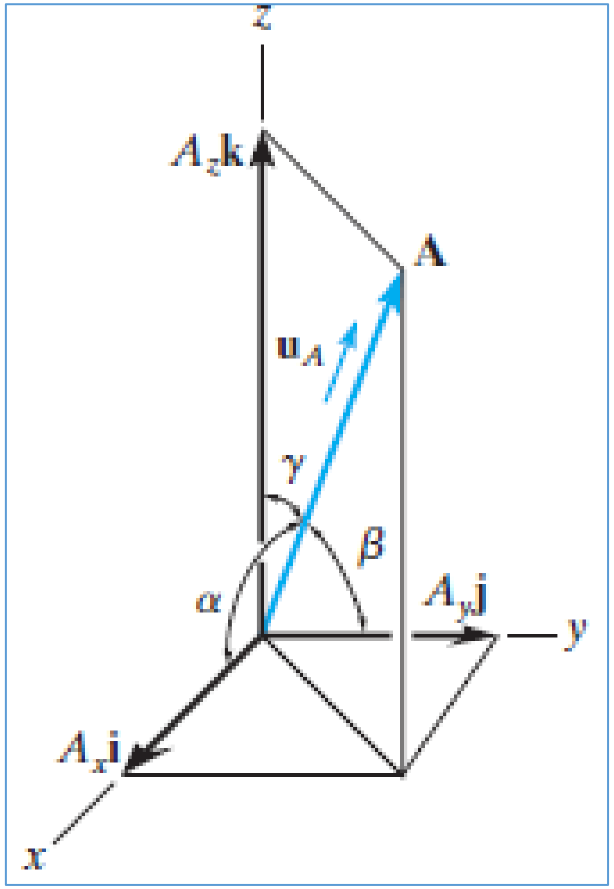


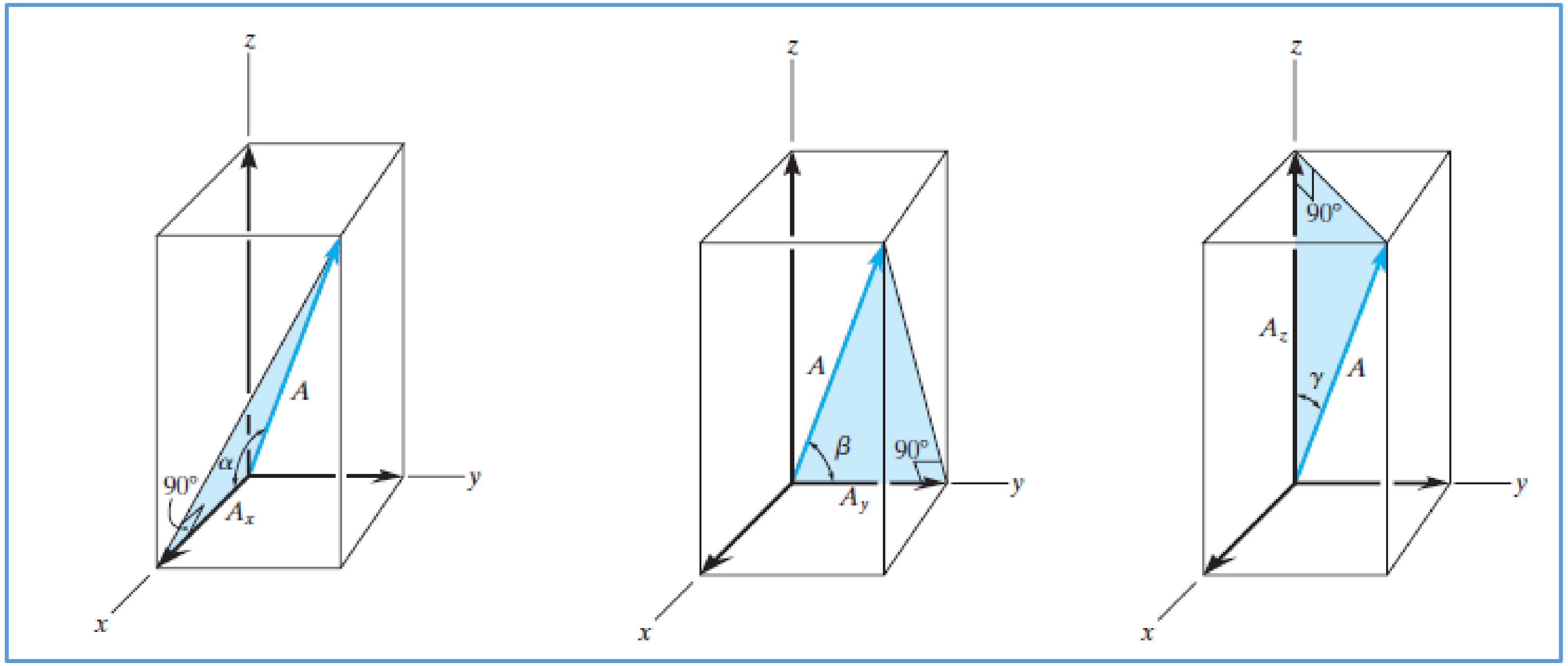
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$



$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

• Cartesian Vectors:





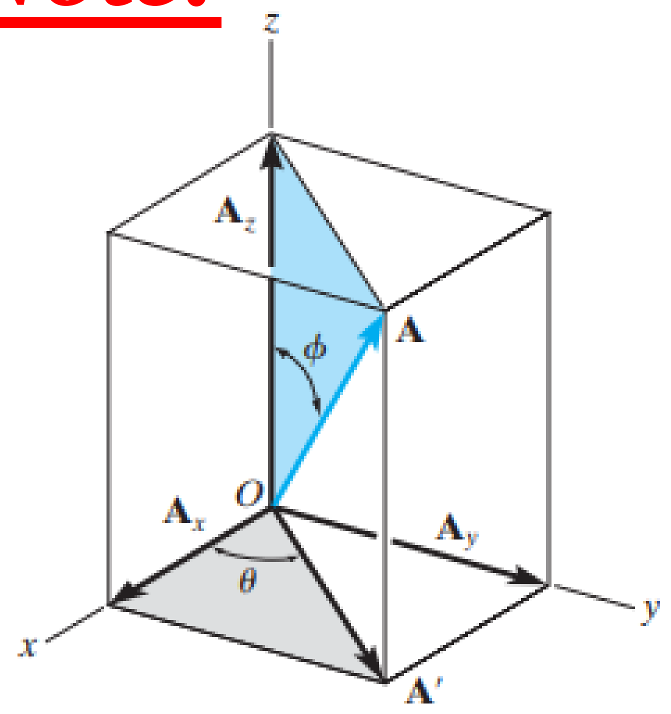
$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\begin{aligned}\mathbf{A} &= A\mathbf{u}_A \\ &= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k} \\ &= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\end{aligned}$$

Note:



Sometimes, the direction of **A** can be specified using two angles, θ and ϕ (phi), such as shown in Fig. . The components of **A** can then be determined by applying trigonometry first to the blue right triangle, which yields

$$A_z = A \cos \phi$$

and

$$A' = A \sin \phi$$

Now applying trigonometry to the gray shaded right triangle,

$$A_x = A' \cos \theta = A \sin \phi \cos \theta$$

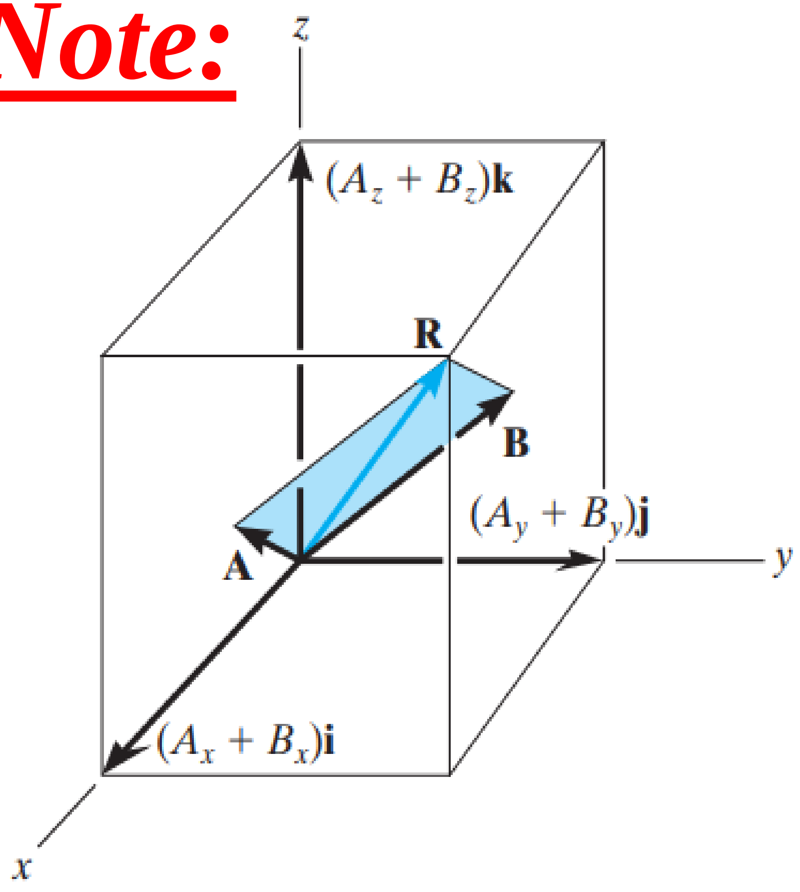
$$A_y = A' \sin \theta = A \sin \phi \sin \theta$$

Therefore **A** written in Cartesian vector form becomes

$$\mathbf{A} = A \sin \phi \cos \theta \, \mathbf{i} + A \sin \phi \sin \theta \, \mathbf{j} + A \cos \phi \, \mathbf{k}$$

You should not memorize this equation, rather it is important to understand how the components were determined using trigonometry.

Note:



The addition (or subtraction) of two or more vectors is greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, if $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, Fig. then the resultant vector, **R**, has components which are the scalar sums of the **i**, **j**, **k** components of **A** and **B**, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

Here ΣF_x, ΣF_y, and ΣF_z represent the algebraic sums of the respective x, y, z or **i**, **j**, **k** components of each force in the system.

Important Points

- Cartesian vector analysis is often used to solve problems in three dimensions.
- The positive directions of the *x, y, z* axes are defined by the Cartesian unit vectors **i**, **j**, **k**, respectively.
- The *magnitude* of a Cartesian vector is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.
- The *direction* of a Cartesian vector is specified using coordinate direction angles α, β, γ which the tail of the vector makes with the positive *x, y, z* axes, respectively. The components of the unit vector $\mathbf{u}_A = \mathbf{A}/A$ represent the direction cosines of α, β, γ. Only two of the angles α, β, γ have to be specified. The third angle is determined from the relationship $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
- Sometimes the direction of a vector is defined using the two angles θ and ϕ as in Fig. 2–28. In this case the vector components are obtained by vector resolution using trigonometry.
- To find the *resultant* of a concurrent force system, express each force as a Cartesian vector and add the **i**, **j**, **k** components of all the forces in the system.

Example 1: **Express the force F shown in figure as a Cartesian vector.**

Solution:

Since only two coordinate direction angles are specified, the third angle α must be determined from Eq.i.e.,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\alpha + \cos^2 60° + \cos^2 45° = 1$$

$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Hence, two possibilities exist, namely,

$$\alpha = \cos^{-1}(0.5) = 60° \quad \text{or} \quad \alpha = \cos^{-1}(-0.5) = 120°$$

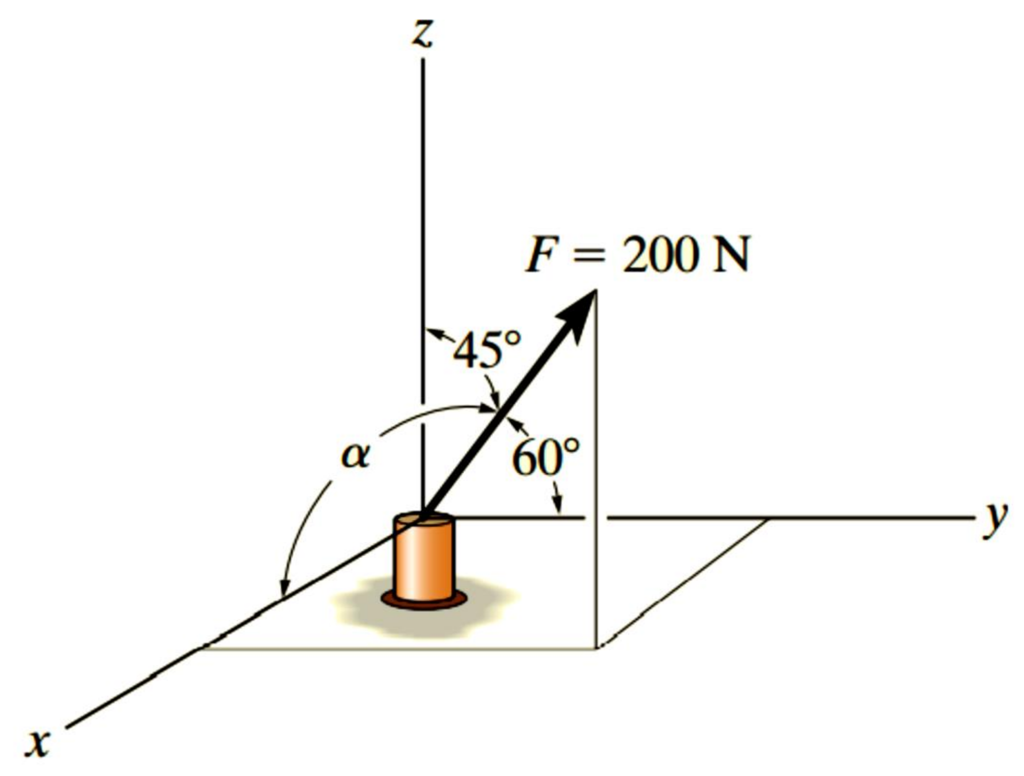
By inspection it is necessary that $\alpha = 60°$, since F_x must be in the $+x$ direction.

Using Eq. , with $F = 200$ N, we have

$$\mathbf{F} = F\cos\alpha\,\mathbf{i} + F\cos\beta\,\mathbf{j} + F\cos\gamma\,\mathbf{k}$$

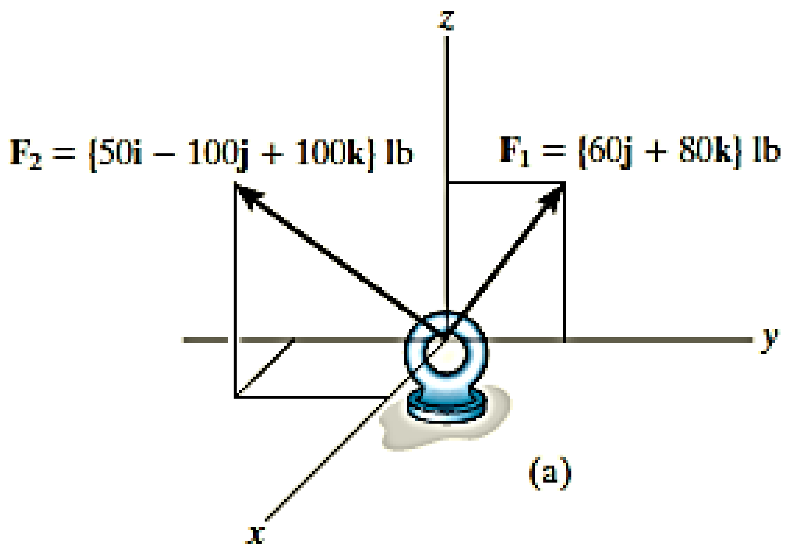
$$= (200\cos 60°\text{ N})\mathbf{i} + (200\cos 60°\text{ N})\mathbf{j} + (200\cos 45°\text{ N})\mathbf{k}$$

$$= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$



Example 2:

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Figure a.



(a)

SOLUTION

Since each force is represented in Cartesian vector form, the resultant force, shown in is

$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\}\text{ lb} + \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{ lb}$$
$$= \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\}\text{ lb}$$

The magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(50\text{ lb})^2 + (-40\text{ lb})^2 + (180\text{ lb})^2} = 191.0\text{ lb}$$
$$= 191\text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles α, β, γ are determined from the components of the unit vector acting in the direction of $\mathbf{F}_R$.

$$\mathbf{u}_{F_R} = \frac{\mathbf{F}_R}{F_R} = \frac{50}{191.0}\mathbf{i} - \frac{40}{191.0}\mathbf{j} + \frac{180}{191.0}\mathbf{k}$$

$$= 0.2617\mathbf{i} - 0.2094\mathbf{j} + 0.9422\mathbf{k}$$

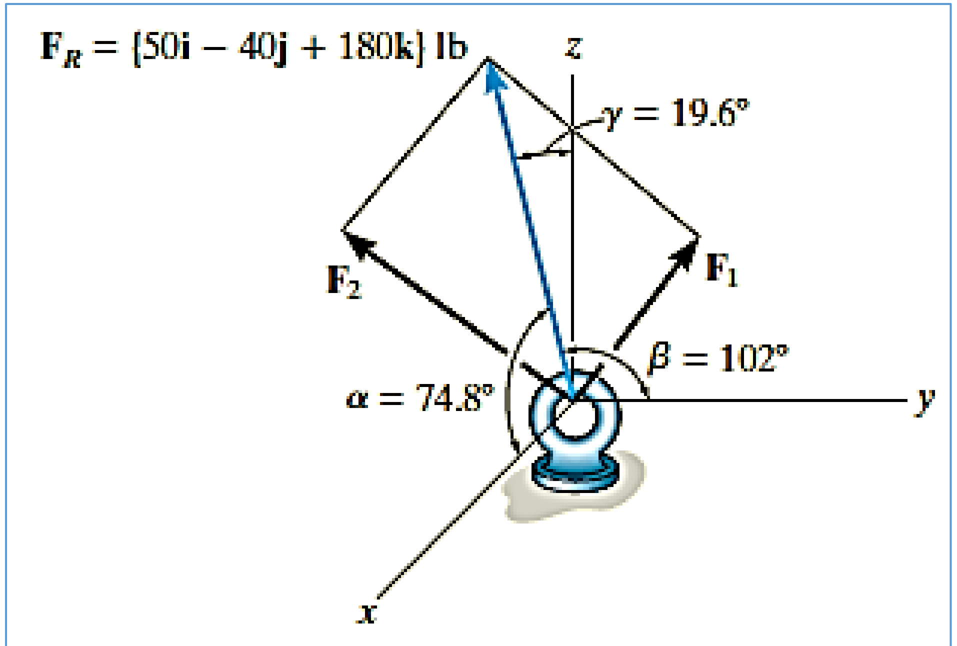
so that

$$\cos\alpha = 0.2617 \qquad \alpha = 74.8^\circ$$

$$\cos\beta = -0.2094 \qquad \beta = 102^\circ$$

$$\cos\gamma = 0.9422 \qquad \gamma = 19.6^\circ$$

These angles are shown in Fig. *b*.

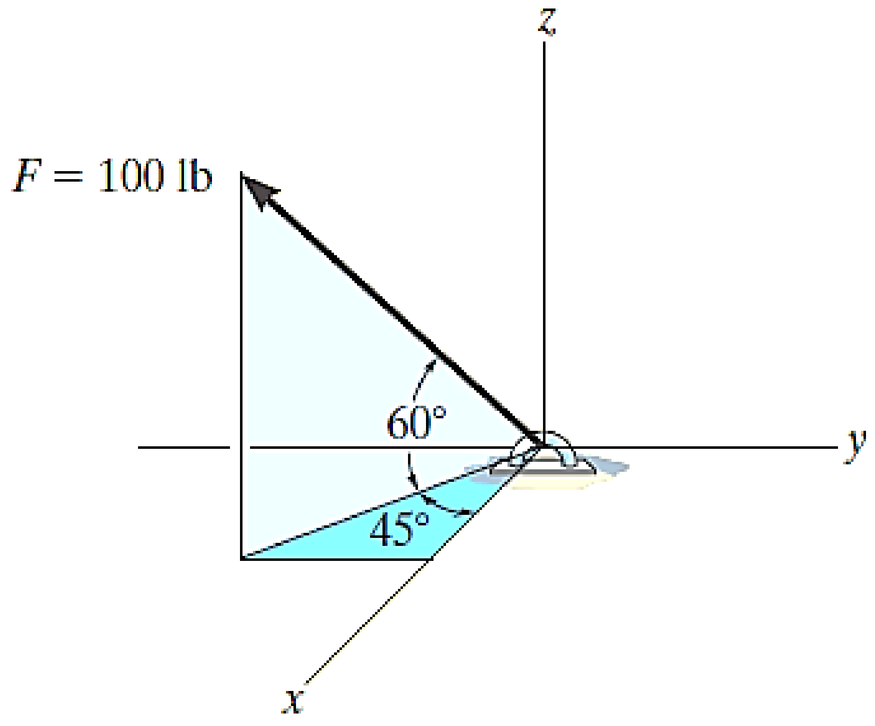


(b)

NOTE: In particular, notice that $\beta > 90^\circ$ since the **j** component of $\mathbf{u}_{F_R}$ is negative.

Example 3: Express the force **F** shown in Fig. *a* as a Cartesian vector and direction angles of the resultant force.

Solution:



(a)

$$F_z = 100 \sin 60° \text{ lb} = 86.6 \text{ lb}$$

$$F' = 100 \cos 60° \text{ lb} = 50 \text{ lb}$$

$$F_x = F' \cos 45° = 50 \cos 45° \text{ lb} = 35.4 \text{ lb}$$

$$F_y = F' \sin 45° = 50 \sin 45° \text{ lb} = 35.4 \text{ lb}$$

Realizing that $\mathbf{F}_y$ has a direction defined by $-\mathbf{j}$, we have

$$\mathbf{F} = \{35.4\mathbf{i} - 35.4\mathbf{j} + 86.6\mathbf{k}\} \text{ lb} \qquad \textit{Ans.}$$

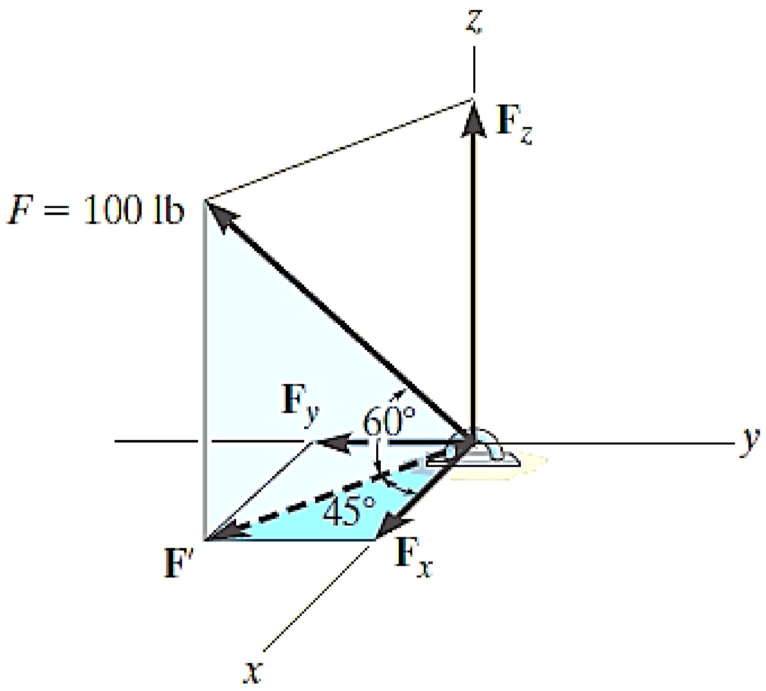
$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$
$$= \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = 100 \text{ lb}$$

If needed, the coordinate direction angles of **F** can be determined from the components of the unit vector acting in the direction of **F**. Hence,

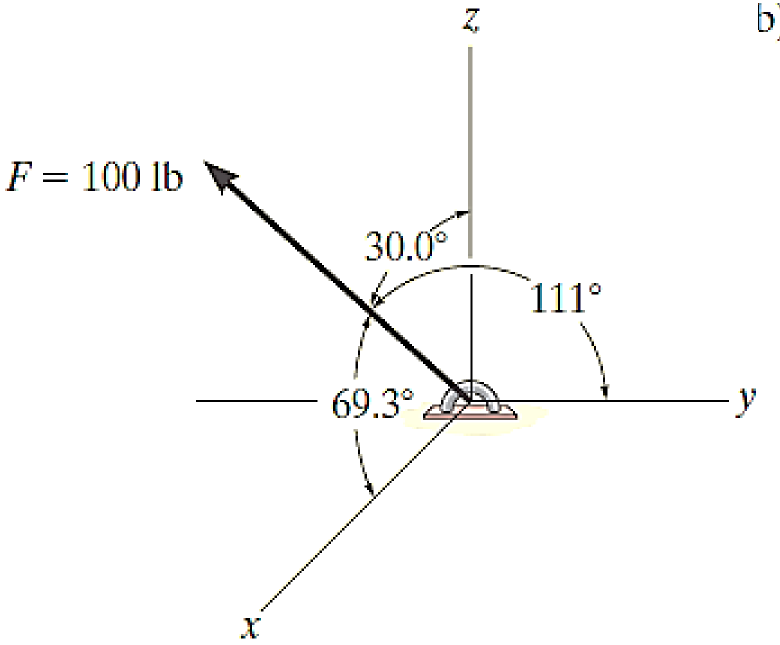
$$\mathbf{u} = \frac{\mathbf{F}}{F} = \frac{F_x}{F}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k}$$
$$= \frac{35.4}{100}\mathbf{i} - \frac{35.4}{100}\mathbf{j} + \frac{86.6}{100}\mathbf{k}$$
$$= 0.354\mathbf{i} - 0.354\mathbf{j} + 0.866\mathbf{k}$$

so that

$$\alpha = \cos^{-1}(0.354) = 69.3^\circ$$
$$\beta = \cos^{-1}(-0.354) = 111^\circ$$
$$\gamma = \cos^{-1}(0.866) = 30.0^\circ$$



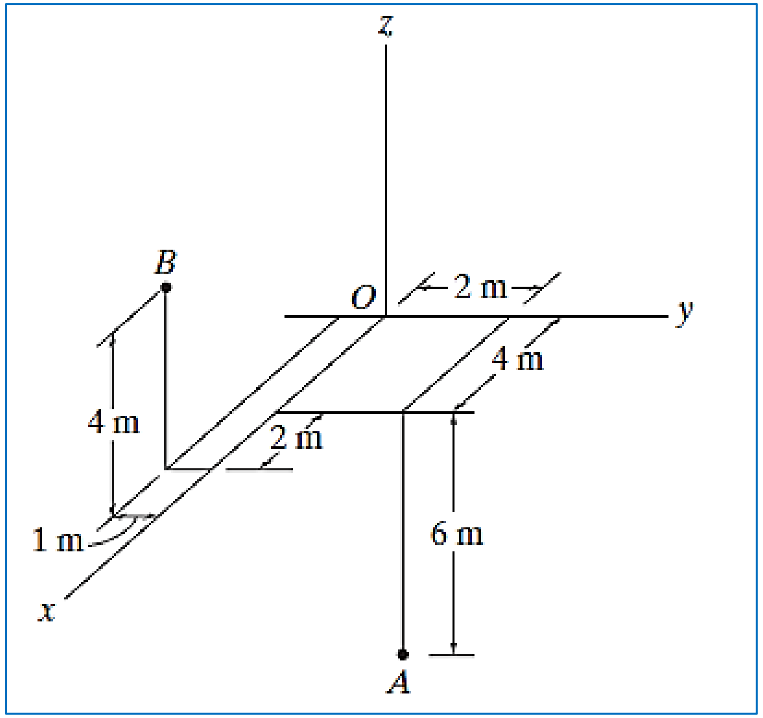
b)

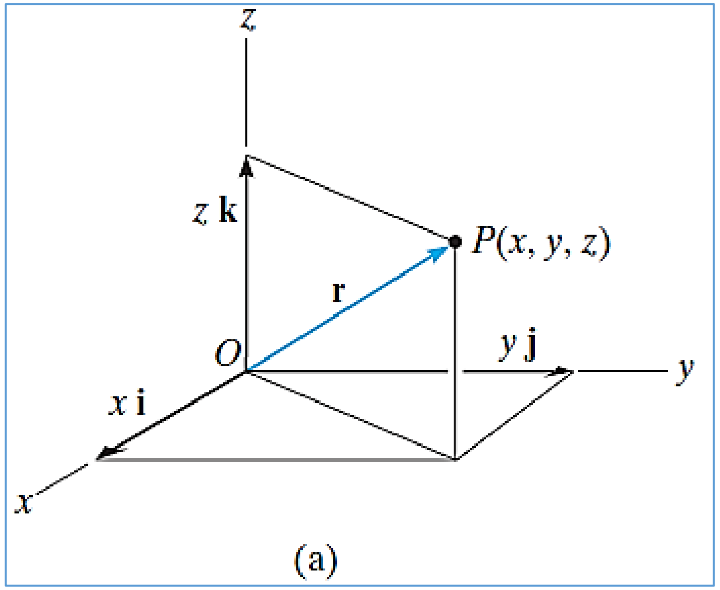


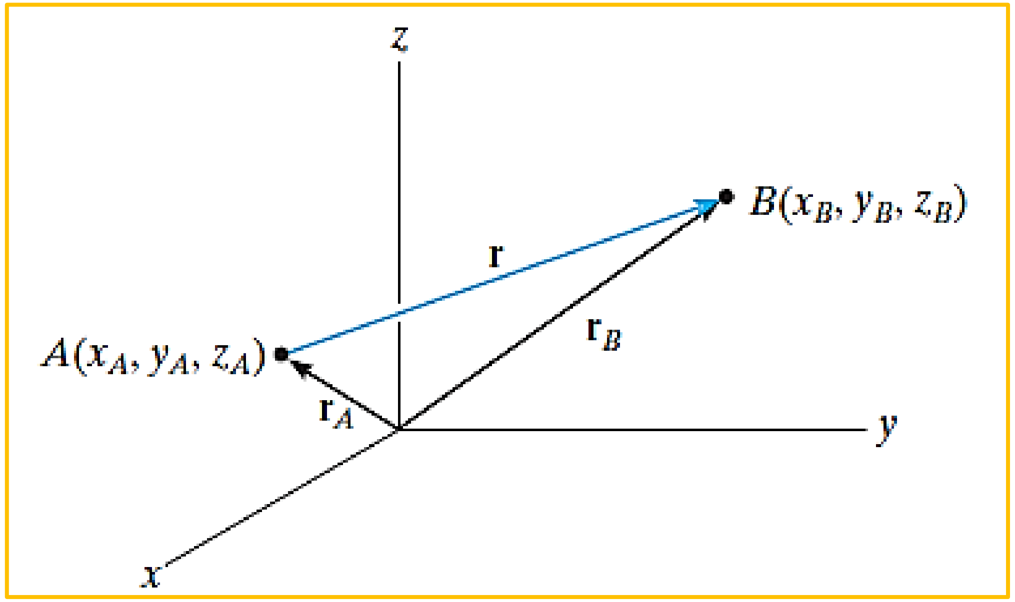
(c)

• ***Position Vectors (r):*** *(specify the forces by two points on the line of action)*

Defined as a fixed vector which locates a point in space relative to another point.







$$r = xi + yj + zk$$

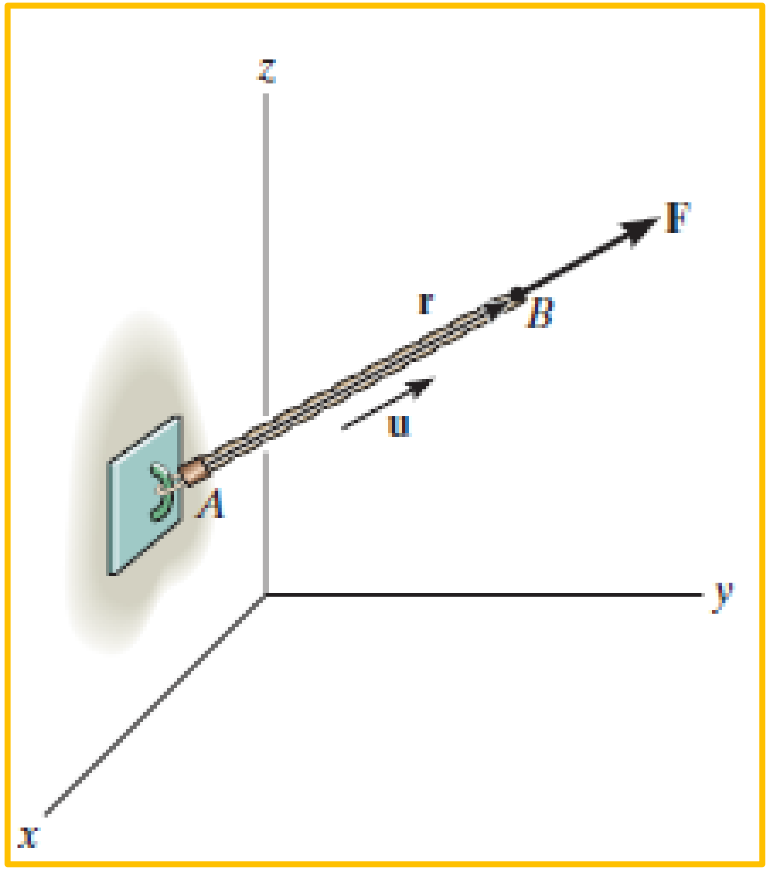
$$\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$$

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$

or

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

- ***Force Vector Directed Along a Line***



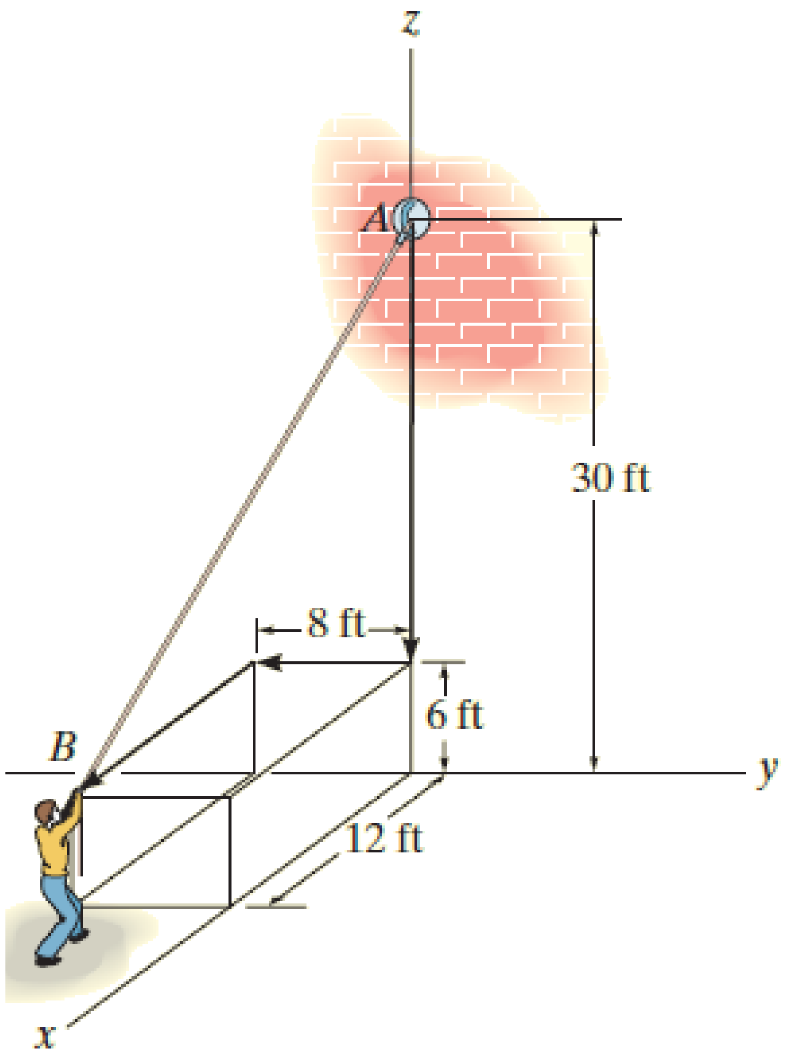
$$\boldsymbol{F} = F\mathbf{u} = F\frac{r_{AB}}{|r_{AB}|} = F\frac{(x_B - x_A)i + (y_B - y_A)j + (z_B - z_A)k}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

Important Points

- A position vector locates one point in space relative to another point.
- The easiest way to formulate the components of a position vector is to determine the distance and direction that must be traveled along the x, y, z directions—going from the tail to the head of the vector.
- A force $\mathbf{F}$ acting in the direction of a position vector $\mathbf{r}$ can be represented in Cartesian form if the unit vector $\mathbf{u}$ of the position vector is determined and it is multiplied by the magnitude of the force, i.e., $\mathbf{F} = F\mathbf{u} = F(\mathbf{r}/r)$.

Example 4//

The man shown in Figure pulls on the cord with a force of 70 lb. Represent this force acting on the support A as a Cartesian vector and determine its direction.



Solution//

$$\mathbf{r} = \{12\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}\}\ \text{ft}$$

The magnitude of **r**, which represents the *length* of cord AB, is

$$r = \sqrt{(12\ \text{ft})^2 + (-8\ \text{ft})^2 + (-24\ \text{ft})^2} = 28\ \text{ft}$$

Forming the unit vector that defines the direction and sense of both **r** and **F**, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}$$

Since **F** has a *magnitude* of 70 lb and a *direction* specified by **u**, then

$$\mathbf{F} = F\mathbf{u} = 70\ \text{lb}\left(\frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}\right)$$

$$= \{30\mathbf{i} - 20\mathbf{j} - 60\mathbf{k}\}\ \text{lb} \qquad \textit{Ans.}$$

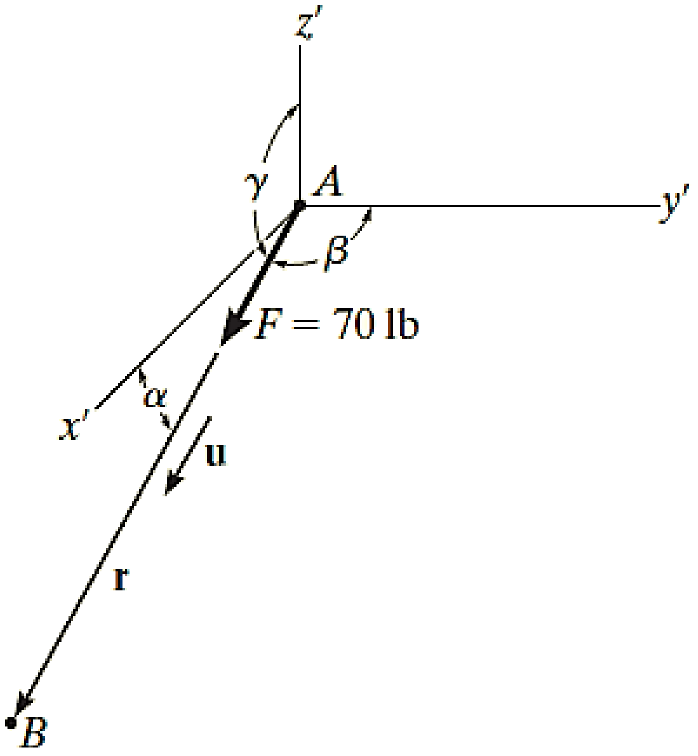
Solution cont.//

From the components of the unit vector:

$$\alpha = \cos^{-1}\left(\frac{12}{28}\right) = 64.6^\circ \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{-8}{28}\right) = 107^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{-24}{28}\right) = 149^\circ \qquad \textit{Ans.}$$

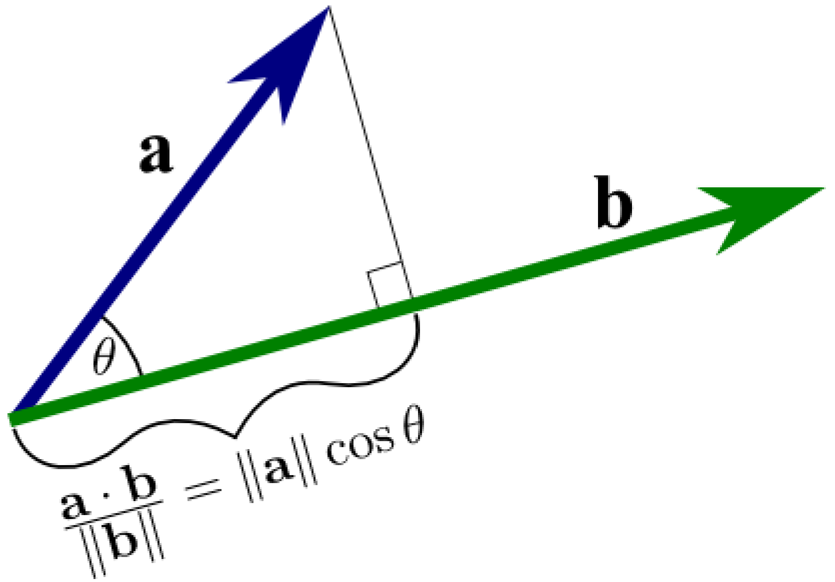


- **Dot Product** *(specify the forces by angle which oriented the line of action)*

- The dot product between two vectors **A** and **B** yields a scalar. If A and B are expressed in Cartesian vector form, then the dot product is the sum of the products of their x , y , and z components.
- The dot product is used to determine the angle between two vectors or the projection of a vector in a specified direction.

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

where $0° \le \theta \le 180°$



Laws of Operation.

1. Commutative law: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
2. Multiplication by a scalar: $a(\mathbf{A} \cdot \mathbf{B}) = (a\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a\mathbf{B})$
3. Distributive law: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})$

Cartesian Vector Formulation.

$$\mathbf{A}\cdot\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$
$$= A_xB_x(\mathbf{i}\cdot\mathbf{i}) + A_xB_y(\mathbf{i}\cdot\mathbf{j}) + A_xB_z(\mathbf{i}\cdot\mathbf{k})$$
$$+ A_yB_x(\mathbf{j}\cdot\mathbf{i}) + (A_yB_y(\mathbf{j}\cdot\mathbf{j}) + A_yB_z(\mathbf{j}\cdot\mathbf{k})$$
$$+ A_zB_x(\mathbf{k}\cdot\mathbf{i}) + A_zB_y(\mathbf{k}\cdot\mathbf{j}) + A_zB_z(\mathbf{k}\cdot\mathbf{k})$$

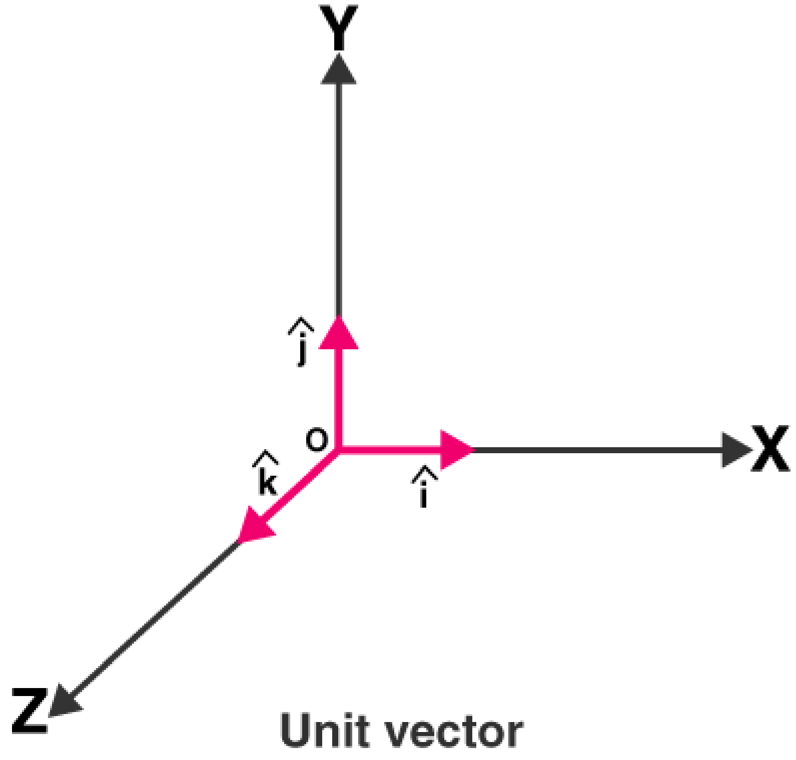
Carrying out the dot-product operations, the final result becomes

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z}$$

$$\mathbf{i}\cdot\mathbf{i} = (1)(1)\cos 0^\circ = 1$$
$$\mathbf{i}\cdot\mathbf{j} = (1)(1)\cos 90^\circ = 0.$$

$$\vec{i}\,.\,\vec{i} = 1$$
$$\vec{j}\cdot\vec{j} = 1$$
$$\vec{k}\,.\,\vec{k} = 1$$

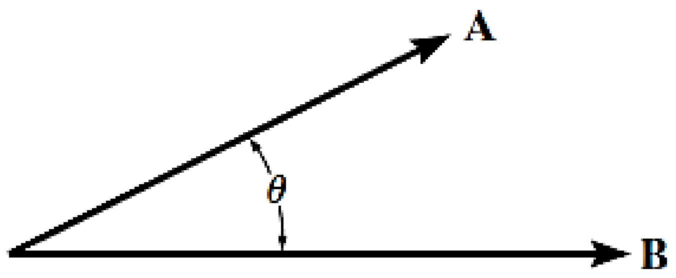
$$\vec{i}\,.\,\vec{j} = 0$$
$$\vec{j}\,.\,\vec{k} = 0$$
$$\vec{k}\,.\,\vec{i} = 0$$



Applications. The dot product has two important applications in mechanics.

- ***The angle formed between two vectors or intersecting lines.***

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) \quad 0^\circ \le \theta \le 180^\circ$$

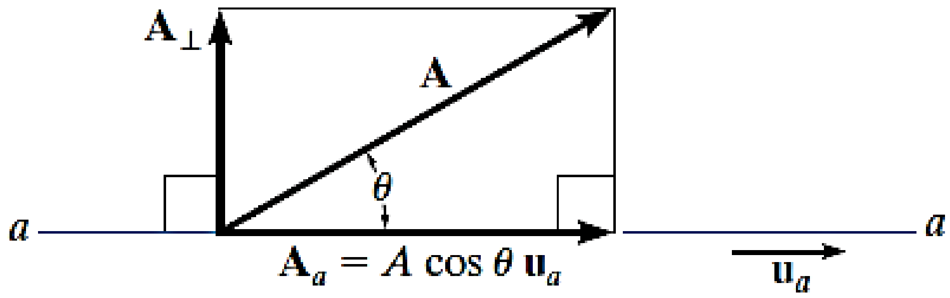


notice that if

$\mathbf{A}\cdot\mathbf{B} = 0, \theta = \cos^{-1} 0 = 90^\circ$ so that $\mathbf{A}$ will be *perpendicular* to $\mathbf{B}$.

- ***The components of a vector parallel and perpendicular to a line.***

$$A_a = A\cos\theta = \mathbf{A}\cdot\mathbf{u}_a$$



Since $\mathbf{A} = \mathbf{A}_a + \mathbf{A}_\perp$, then $\mathbf{A}_\perp = \mathbf{A} - \mathbf{A}_a$.

$$\theta = \cos^{-1}(\mathbf{A}\cdot\mathbf{u}_A / A)$$

$$A_\perp = A\sin\theta$$

Pythagorean's theorem

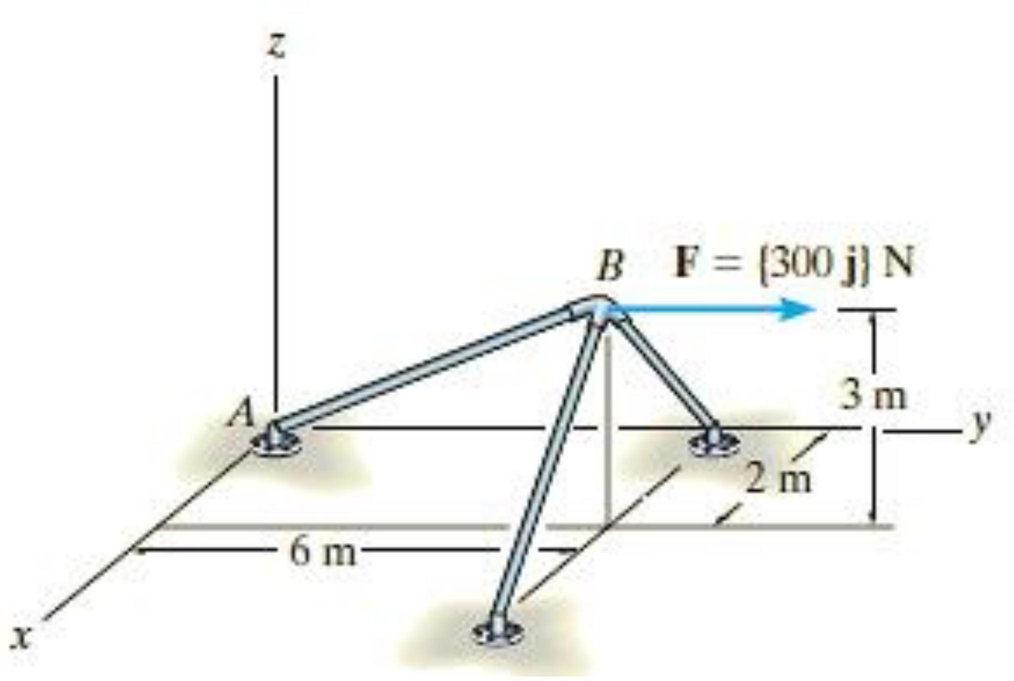
$$A_\perp = \sqrt{A^2 - A_a{}^2}$$

Important Points

- The dot product is used to determine the angle between two vectors or the projection of a vector in a specified direction.
- If vectors **A** and **B** are expressed in Cartesian vector form, the dot product is determined by multiplying the respective x, y, z scalar components and algebraically adding the results, i.e., $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$.
- From the definition of the dot product, the angle formed between the tails of vectors **A** and **B** is $\theta = \cos^{-1}(\mathbf{A} \cdot \mathbf{B}/AB)$.
- The magnitude of the projection of vector **A** along a line aa whose direction is specified by $\mathbf{u}_a$ is determined from the dot product $A_a = \mathbf{A} \cdot \mathbf{u}_a$.

Example 5:

The frame shown in Figure is subjected to a horizontal force F = {300 j}. Determine the magnitude of the components of this force parallel and perpendicular to member AB .



Solution:

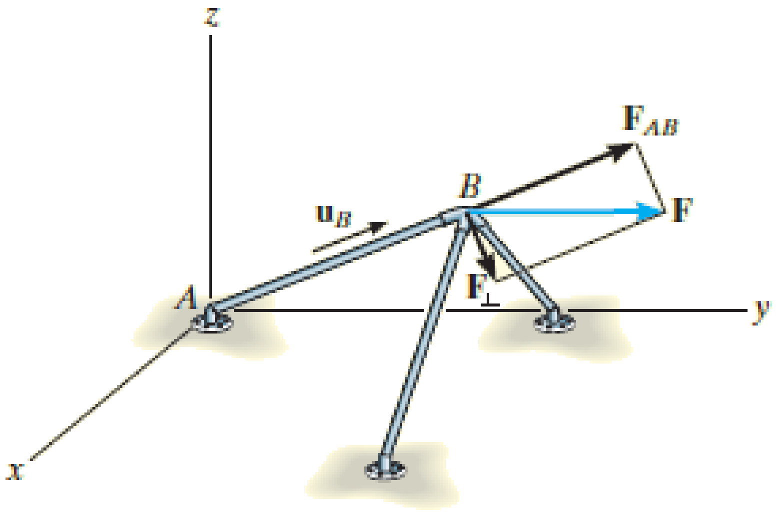
The magnitude of the component of **F** along AB is equal to the dot product of **F** and the unit vector $\mathbf{u}_B$, which defines the direction of AB,

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = 0.286\,\mathbf{i} + 0.857\,\mathbf{j} + 0.429\,\mathbf{k}$$

then

$$F_{AB} = F\cos\theta = \mathbf{F}\cdot\mathbf{u}_B = (300\mathbf{j})\cdot(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k})$$
$$= (0)(0.286) + (300)(0.857) + (0)(0.429)$$
$$= 257.1 \text{ N} \qquad \textit{Ans.}$$

Since the result is a positive scalar, $\mathbf{F}_{AB}$ has the same sense of direction as $\mathbf{u}_B$, Fig



Solution cont.:

Expressing $\mathbf{F}_{AB}$ in Cartesian vector form, we have

$$\mathbf{F}_{AB} = F_{AB}\mathbf{u}_B = (257.1\text{ N})(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\mathbf{k})$$
$$= \{73.5\mathbf{i} + 220\mathbf{j} + 110\mathbf{k}\}\text{ N} \qquad \textit{Ans.}$$

The perpendicular component, Fig. is therefore

$$\mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{AB} = 300\mathbf{j} - (73.5\mathbf{i} + 220\mathbf{j} + 110\mathbf{k})$$
$$= \{-73.5\mathbf{i} + 79.6\mathbf{j} - 110\mathbf{k}\}\text{ N}$$

Its magnitude can be determined either from this vector or by using the Pythagorean theorem, Fig.

$$F_{\perp} = \sqrt{F^2 - F_{AB}^2} = \sqrt{(300\text{ N})^2 - (257.1\text{ N})^2}$$
$$= 155\text{ N} \qquad \textit{Ans.}$$

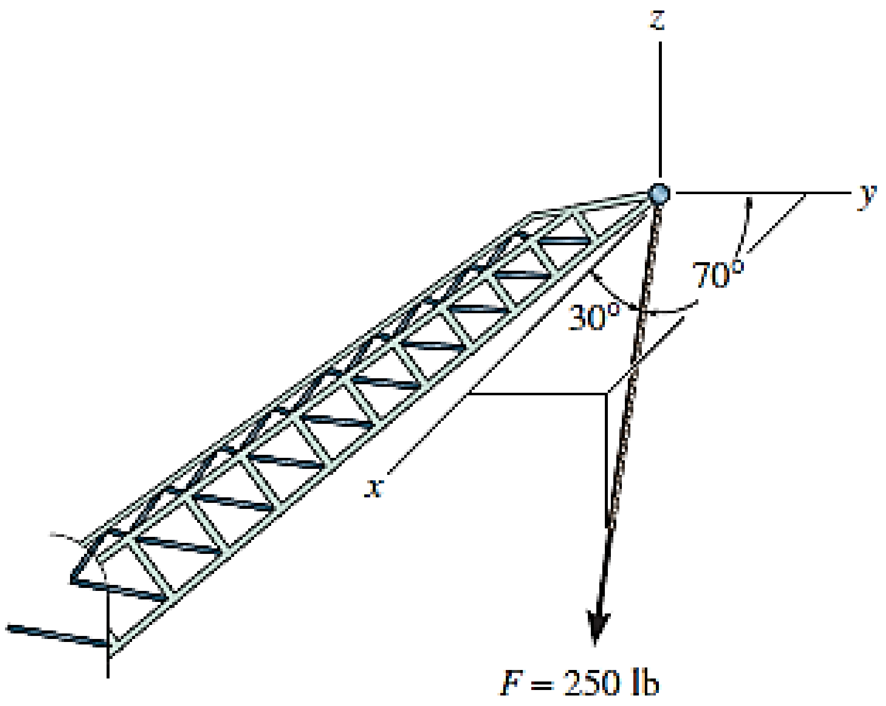
Next Lecture:

- Moment of a Force - Scalar Formation
- Cross Product
- Moment of a Force -Vector Formulation
- Solving Related examples

Assignment 1:

(solve this problems then submit your answer)

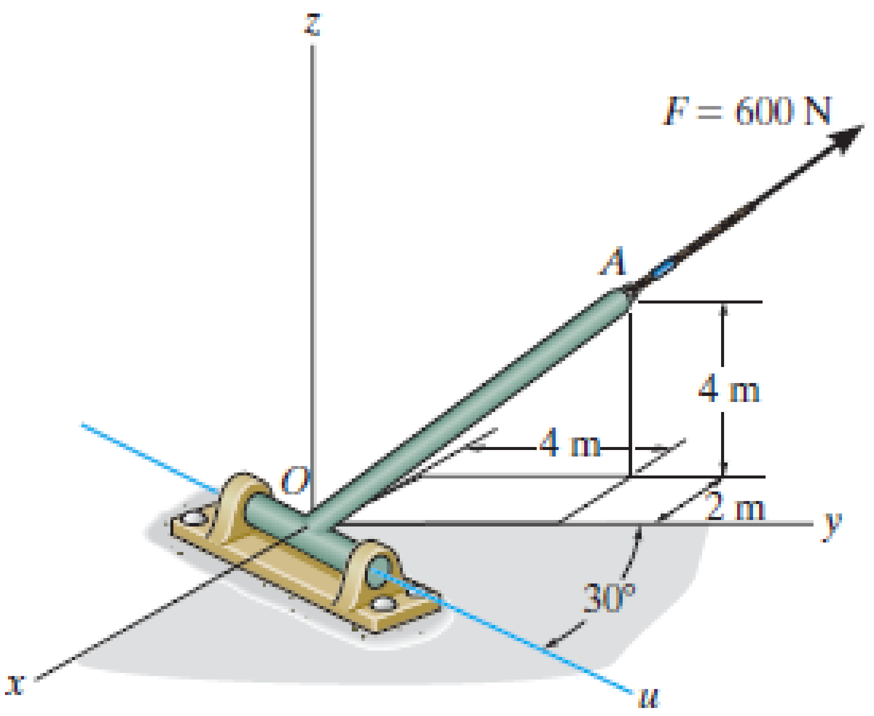
The cable at the end of the crane boom exerts a force of 250 lb on the boom as shown. Express **F** as a Cartesian vector.



Assignment 2:

(solve this problems then submit your answer)

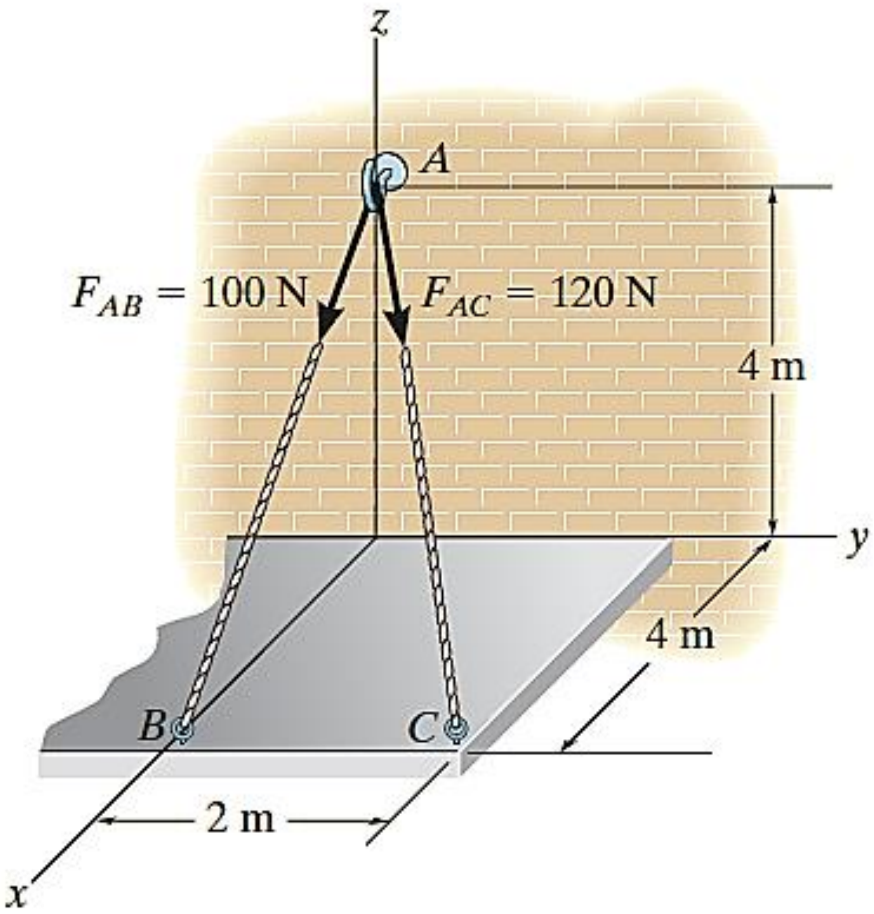
Determine the magnitude of the projection of force F = 600 N along the u axis.



Assignment 3:

(solve this problems then submit your answer)

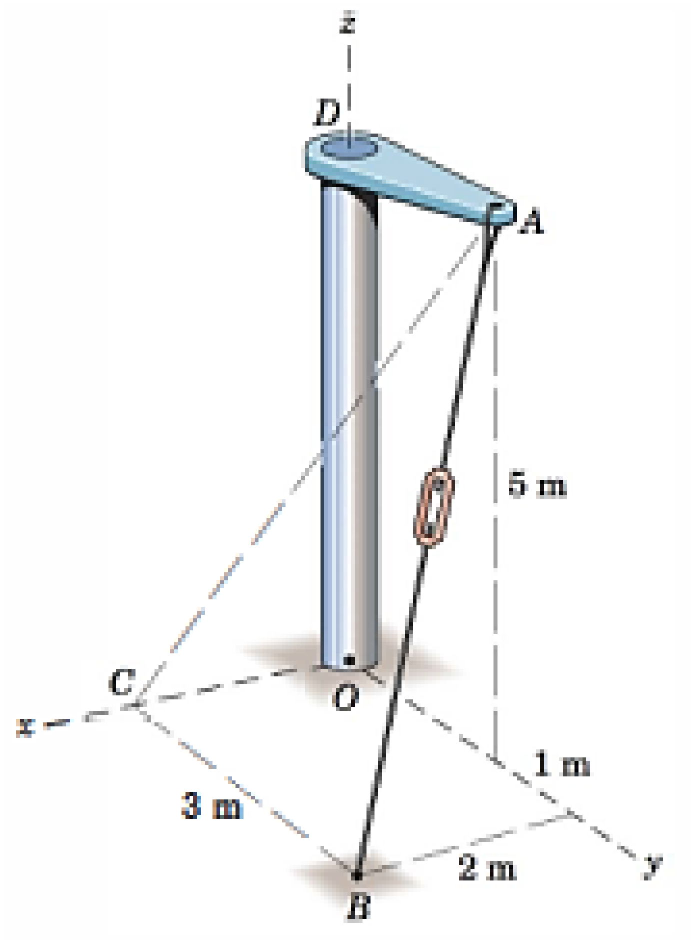
The roof is supported by cables as shown in the photo. If the cables exert forces $F_{AB} = 100\ N$ and $F_{AC} = 120\ N$ on the wall hook at A as shown in Fig., determine the resultant force acting at A . Express the result as a Cartesian vector.



Assignment 4:

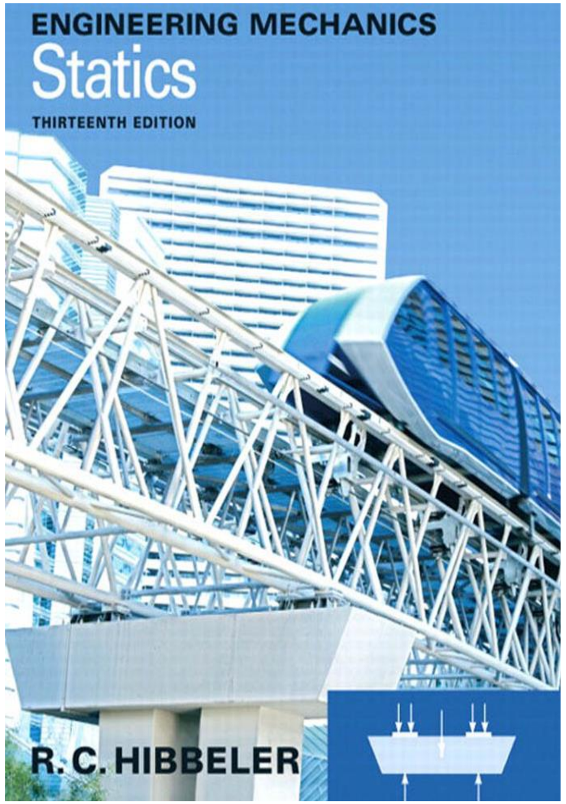
(solve this problems then submit your answer)

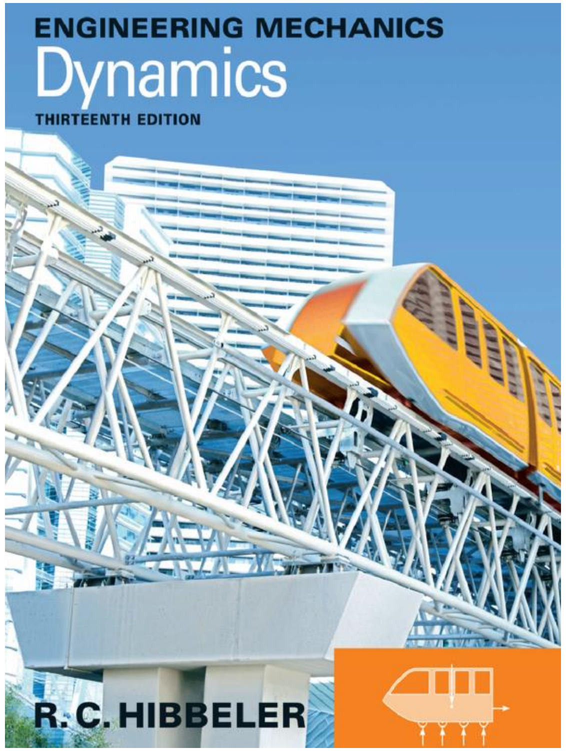
The turnbuckle is tightened until the tension in the cable AB equals $2.4\,kN$. Determine the vector expression for the tension T as a force acting on member AD. Also find the magnitude of the projection of T along the line AC



References:

Engineering Mechanics R.C. Hibbeler 13th edition (Statics and Dynamics).





The end of the lecture

Enjoy your time