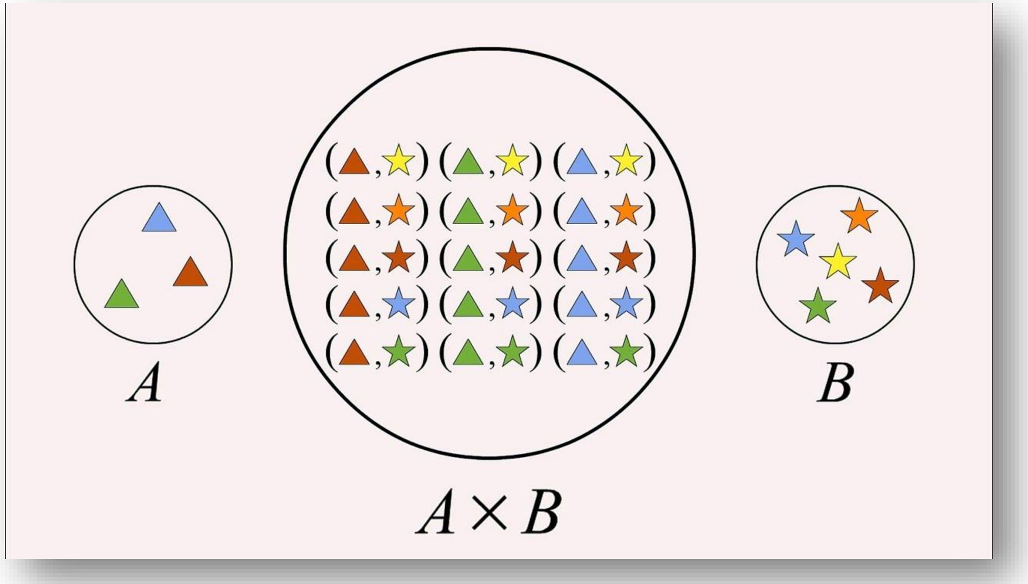
Lecture 3: Cartesian Products



Ms. Togzhan Nurtayeva

Course Code: IT 235/A
Semester 3
Week 5
Date: 02.11.2023

Objectives:

- Ordered Pairs
- Cartesian Products/Cross Products
- Cardinality
- Truth Sets
- Quantifiers

An **ordered pair** is a set of two numbers that identifies the location of a point on a coordinate plane.

- **An ordered pair (a, b) is a set:**

$$\{\{a\}, \{a, b\}\}$$

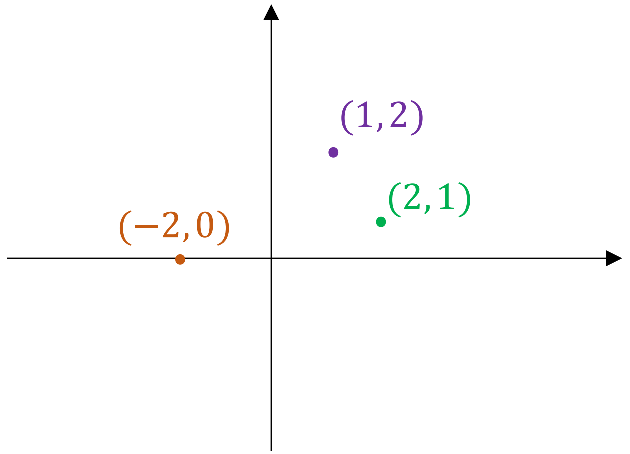
- **You've seen ordered pairs before as graph coordinates:**

$$(1, 2) = \{\{1\}, \{1, 2\}\}$$

$$\qquad\qquad\quad \neq \qquad\quad =$$

$$(2, 1) = \{\{2\}, \{2, 1\}\}$$

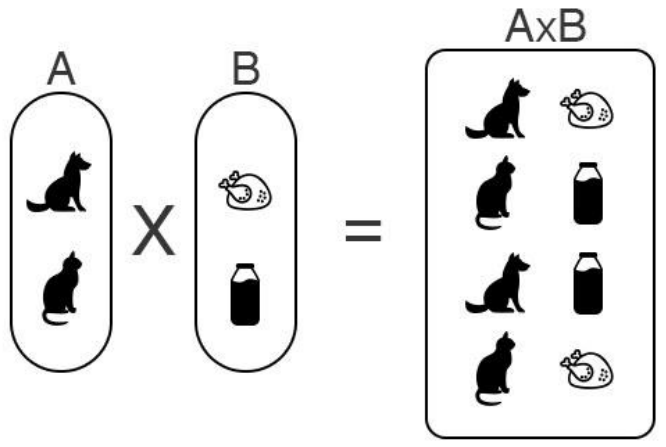
$$(-2, 0) = \{\{-2\}, \{-2, 0\}\}$$



<u>not the same</u> ordered pairs

- **The Cartesian Product, $A \times B$, is the set:**

$$\{(a, b) \mid a \in A \text{ and } b \in B\}$$



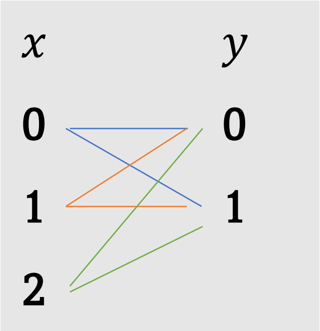
Cartesian Product of Two Sets.

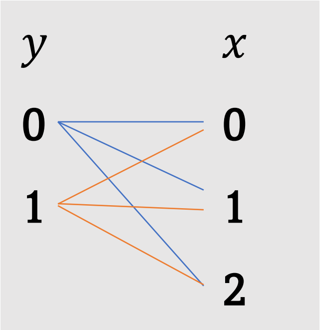
Also, it can be called Cross Product

- **Given $X = \{0,1,2\}$ and $Y = \{0,1\}$:**

$$X \times Y = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$$

$$Y \times X = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$





Cross join or Cartesian Join

Name
John
Paul
Sara

Department
IT
Marketing

CROSS JOIN

Name	Department
John	IT
Paul	IT
Sara	IT
John	Marketing
Paul	Marketing
Sara	Marketing

Usage in Database

id	des1	des2	id	des3	des4
100	desc11	desc12	101	desc41	desc42
100	desc11	desc12	103	desc51	desc52
100	desc11	desc12	105	desc61	desc62
101	desc21	desc22	101	desc41	desc42
101	desc21	desc22	103	desc51	desc52
101	desc21	desc22	105	desc61	desc62
102	desc31	desc32	101	desc41	desc42
102	desc31	desc32	103	desc51	desc52
102	desc31	desc32	105	desc61	desc62

Enrolled

student_id	course_name	credit_status
12345678	CS 105	ugrad
25252525	CS 111	ugrad
45678900	CS 460	grad
33566891	CS 105	non-credit
45678900	CS 510	grad

MajorsIn

student_id	dept_name
12345678	comp sci
45678900	mathematics
25252525	comp sci
45678900	english
66666666	the occult

Enrolled x MajorsIn

Enrolled. student_id	course_name	credit_status	MajorsIn. student_id	dept_name
12345678	CS 105	ugrad	12345678	comp sci
12345678	CS 105	ugrad	45678900	mathematics
12345678	CS 105	ugrad	25252525	comp sci
12345678	CS 105	ugrad	45678900	english
12345678	CS 105	ugrad	66666666	the occult
25252525	CS 111	ugrad	12345678	comp sci
25252525	CS 111	ugrad	45678900	mathematics
...	...	...	...	...

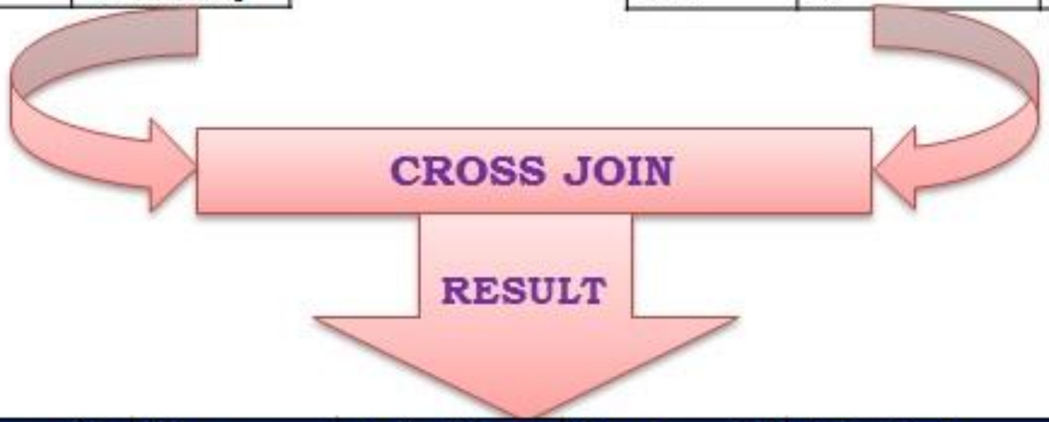
CROSS JOIN

Customers

CustomerId	Name
1	Shree
2	Kalpana
3	Basavaraj

Orders

OrderId	CustomerId	OrderDate
100	1	2014-01-29 23:56:57.700
200	4	2014-01-30 23:56:57.700
300	3	2014-01-31 23:56:57.700



CustomerId	Name	OrderId	CustomerId	OrderDate
1	Shree	100	1	2014-01-30 23:48:32.850
2	Kalpana	100	1	2014-01-30 23:48:32.850
3	Basavaraj	100	1	2014-01-30 23:48:32.850
1	Shree	200	4	2014-01-31 23:48:32.853
2	Kalpana	200	4	2014-01-31 23:48:32.853
3	Basavaraj	200	4	2014-01-31 23:48:32.853
1	Shree	300	3	2014-02-01 23:48:32.853
2	Kalpana	300	3	2014-02-01 23:48:32.853
3	Basavaraj	300	3	2014-02-01 23:48:32.853

What is the cardinality of $A \times B$?

If $|A| = m$ **and** $|B| = n$ **then**

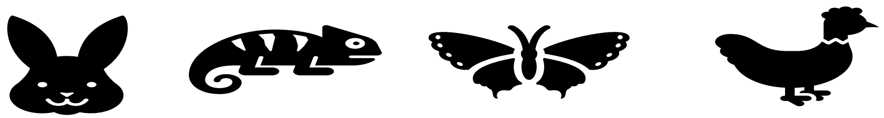
$$|A \times B| = m \cdot n$$

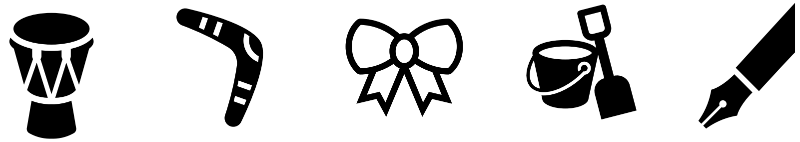
$|\boldsymbol{X}| = \mathbf{3}$

$|\boldsymbol{Y}| = \mathbf{2}$

$|\boldsymbol{X} \times \boldsymbol{Y}| = \mathbf{3 \cdot 2 = 6}$

Cardinality – the number of elements in a set.

$A = \{$  $\}$

$T = \{$  $\}$

$Cardinality = 4 \cdot 5 = 20$

Cartesian Products can generalize to n-tuples

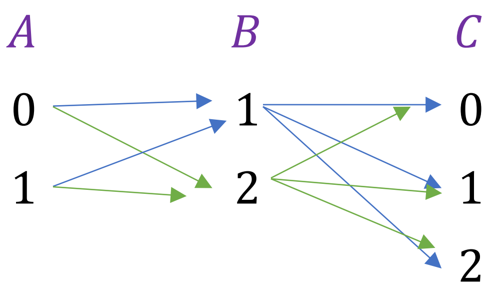
- **3-tuple**

$$A \times B \times C = \{(a, b, c) |\ a \in A, b \in B, c \in C\}$$

- **n-tuple**

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1, a_2, a_3, \ldots, a_n) |\ a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots, a_n \in A_n\}$$

❑ **What is the Cartesian Product,** $A \times B \times C$**, where** $A = \{0, 1\}, B = \{1, 2\}$**, and** $C = \{0, 1, 2\}$**?**



$$A \times B \times C = \{0,1,0\},\{0,1,1\},\{0,1,2\},\{0,2,0\},\{0,2,1\},\{0,2,2\},$$
$$\{1,1,0\},\{1,1,1\},\{1,1,2\},\{1,2,0\},\{1,2,1\},\{1,2,2\}$$

❑ **Find the Cartesian product of** $C \times A \times B$, A^2 **and** A^3.

Truth Sets

- **A truth set of P is the set of elements x in D such that $P(x)$ is true.**

a mathematical or logical set containing all the elements that make a given statement of relationships true when substituted in it the equation.

- **Notation:** $\{x \in D | P(x)\}$

the equation x + 7 = 10 has as its truth set the single number 3

EXAMPLE

$D = \mathbb{Z}$

$P(x): |x| = 3$ ⟹ $\{-3, 3\}$

absolute value

Quantifiers

universal

- ❑ $\forall x P(x)$: "For all x, x is P"

for all x, P(x).

Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," "for each," and "for any."

existential

- ❑ $\exists x P(x)$: "For some x, x is P"

there exists an x for which P(x).

Besides the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is."

Quantifier Negation

$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x);$$
$$\neg \forall x P(x)) \Leftrightarrow \exists x \neg P(x).$$

All students in my class have not taken a course in programming.

There is a student in my class who hasn't taken a course in programming.

We use this notation everywhere in mathematics

- For every real number n, there is a real number m such that $m^2 = n$.

$$\forall n \in \mathbb{R} \quad \exists m \in \mathbb{R} \quad : m^2 = n$$

- Given two rationales x and y, $\sqrt{xy}$ will also be a rational.

$$\forall x, y \in \mathbb{Q}, \ \sqrt{xy} \in \mathbb{Q}$$

$$\updownarrow$$

$$\forall x \in \mathbb{Q}, \forall y \in \mathbb{Q}$$

FYI

A **conjunction** implies that both statements are true, while **disjunction** implies that at least one statement is true. With a conjunction, statements are connected by the word "and" while with disjunction statements are connected by the word "or." The symbol for conjunction is ∧. The symbol for disjunction is ∨.

Truth Sets

1. Translate each of these quantifications into English and determine its truth value.
   - **a)** $\forall x \in \mathbf{R}\ (x^2 \neq -1)$
   - **b)** $\exists x \in \mathbf{Z}\ (x^2 = 2)$
   - **c)** $\forall x \in \mathbf{Z}\ (x^2 > 0)$
   - **d)** $\exists x \in \mathbf{R}\ (x^2 = x)$
2. Translate each of these quantifications into English and determine its truth value.
   - **a)** $\exists x \in \mathbf{R}\ (x^3 = -1)$
   - **b)** $\exists x \in \mathbf{Z}\ (x + 1 > x)$
   - **c)** $\forall x \in \mathbf{Z}\ (x - 1 \in \mathbf{Z})$
   - **d)** $\forall x \in \mathbf{Z}\ (x^2 \in \mathbf{Z})$
3. Find the truth set of each of these predicates where the domain is the set of integers.
   - **a)** $P(x)$: $x^2 < 3$
   - **b)** $Q(x)$: $x^2 > x$
   - **c)** $R(x)$: $2x + 1 = 0$
4. Find the truth set of each of these predicates where the domain is the set of integers.
   - **a)** $P(x)$: $x^3 \geq 1$
   - **b)** $Q(x)$: $x^2 = 2$
   - **c)** $R(x)$: $x < x^2$

1) Let P(x) denote the statement "$x \leq 4$." What are these truth values?

a) P(0)
b) P(4)
c) P(6)

2) Let P(x) be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$
b) $\forall x P(x)$

3) Suppose the domain of the propositional function P(x, y) consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\exists x\ P(x, 3)$
b) $\forall y\ P(1, y)$
c) $\exists y \neg P(2, y)$
d) $\forall x\ \neg P(x, 2)$

4) Let P(x) be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) P(0)
b) P(1)
c) P(−1)
d) $\exists x P(x)$
e) P(2)
f) $\forall x P(x)$

5) Let N(x) be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$
b) $\forall x N(x)$

6) Let Q(x) be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) Q(0)
b) $\exists x Q(x)$
c) Q(−1)
d) $\forall x Q(x)$
e) Q(1)

Cartesian Products

Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$.

b) $C \times B \times A$.

c) $C \times A \times B$.

d) $B \times B \times B$.

Find A^2 if

a) $A = \{0, 1, 3\}$.

b) $A = \{1, 2, a, b\}$.

Exercise

Let $A = \{a, b\}$ **and** $B = \{c, d\}$

$A \times B =$

$B^2 =$

$\emptyset \times A =$

Exercise

Let $A = \{a, b\}$ **and** $B = \{c, d\}$

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$

$$B^2 = \mathrm{B} \times B = \{(c, c), (c, d), (d, c), (d, d)\}$$

$$\emptyset \times A = \emptyset$$

$$|\emptyset \times A| = |\emptyset| \cdot |A| = 0 \cdot 2 = 0$$

Exercise

If $|B| = m$ and $|A| = n$ then find:

$|A \times B| =$

$|A^2| =$

$|B^{32} \times A^{19}| =$

Answers

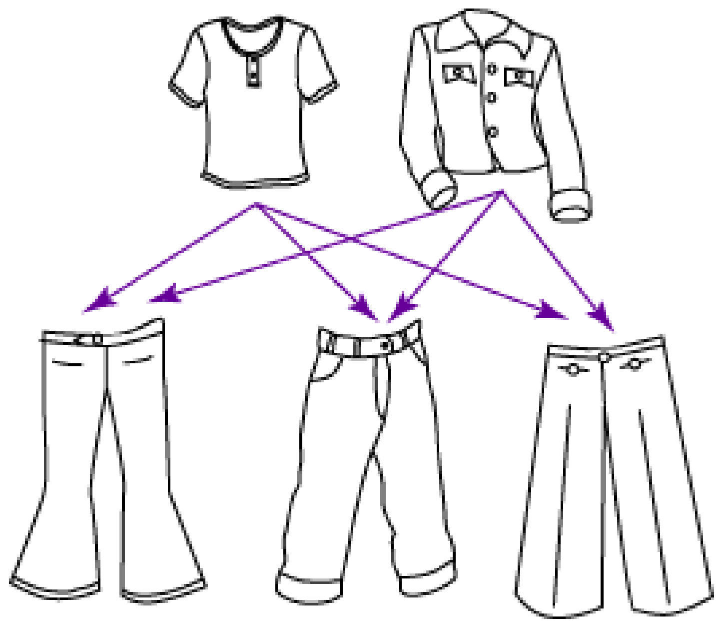
If $|B| = m$ **and** $|A| = n$ **then find:**

$|A \times B| = m \cdot n$

$|A^2| = n \cdot n = n^2$

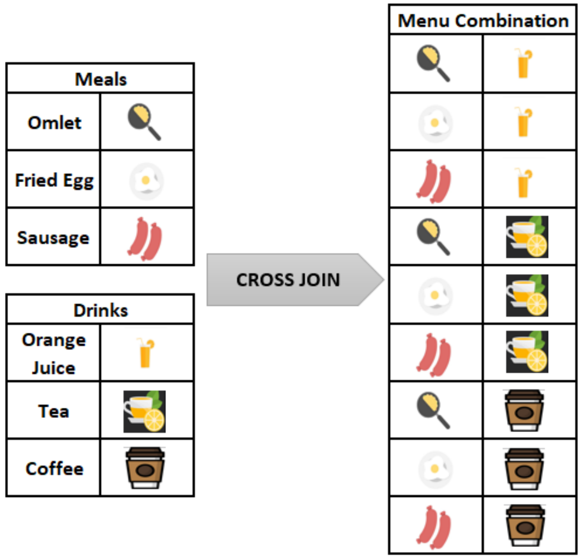
$|B^{32} \times A^{19}| = m^{32} n^{19}$

Where do we use Cartesian Products in real life?



a) The Spider hazard gets introduced. b) Later, it is part of a more challenging obstacle.

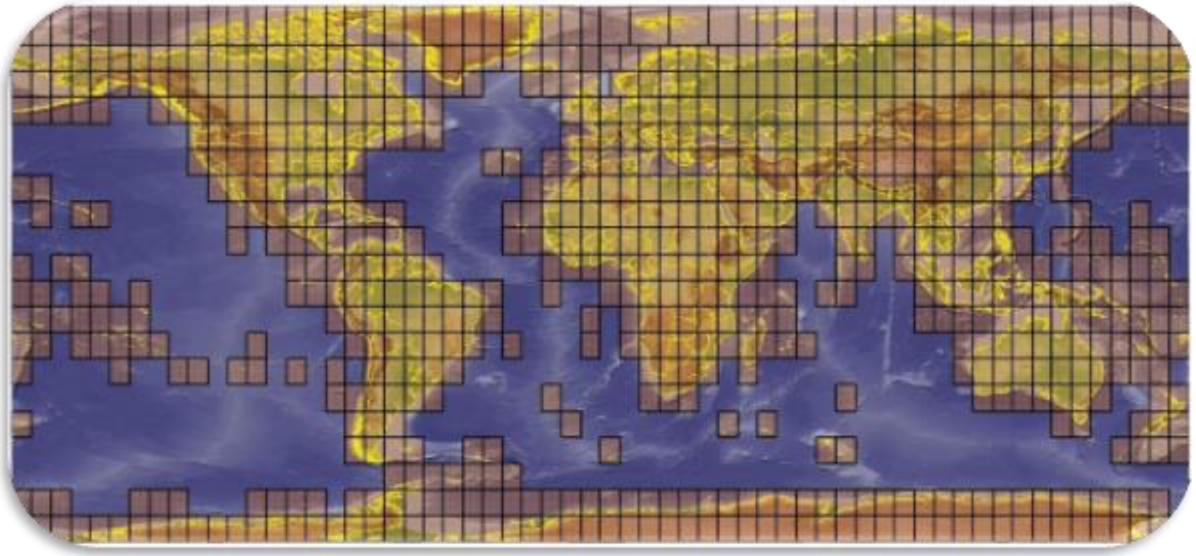
Mighty Jill Off foreshadowing a hazard to prepare the players for a later challenge in the game.

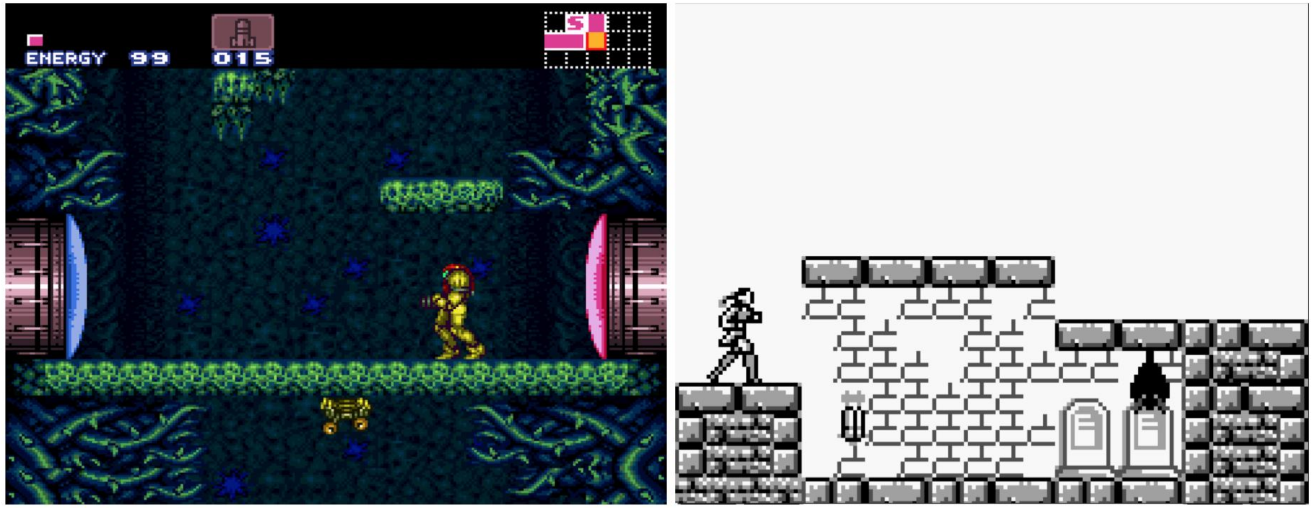


loops.py

```python
models = ["Toyota Corolla","Honda CRV","Chevy Cruze"]
colors = ["White","Grey","Red"]
capacities = ["1.4L","1.6L","1.8L"]

products = [[model,color,capacity] for model in models for color in colors for capacity in capacities]
print(products)
```





a) *Super Metroid* uses enemies to highlight the path the game wants them to follow.

b) Castlevania Legends uses the enemies to guide the player towards a trap.

An example of how *Super Metroid* and *Castlevania Legends* used enemies to guide players to specific areas.

Use Cartesian Product to create combinations from given sets for selling

$$C := \{\text{blue, red, yellow, green, brown}\}\,,$$
$$N := \{\text{round neck, V}-\text{neck, polo}-\text{neck}\}\,,$$
$$S := \{\text{XS, S, M, L, XL, XXL}\}$$

Make & Model : Toyota Corolla, Honda CRV, Chevy Cruze
Colors : White, Grey, Red
Engine Capacities : 1.4L, 1.6L, 1.8L