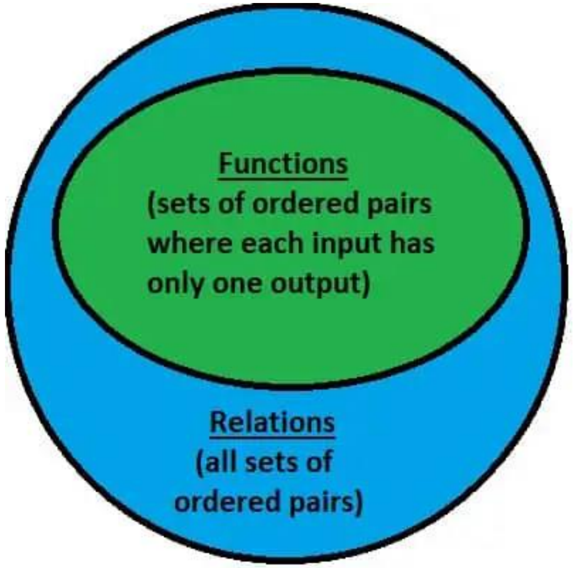
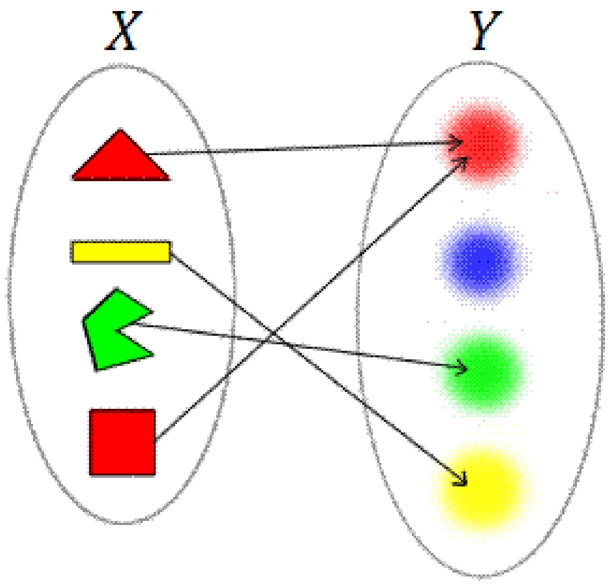
Lecture 4: Relations





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Course Code: IT 235/A
Semester 3
Week 5-6
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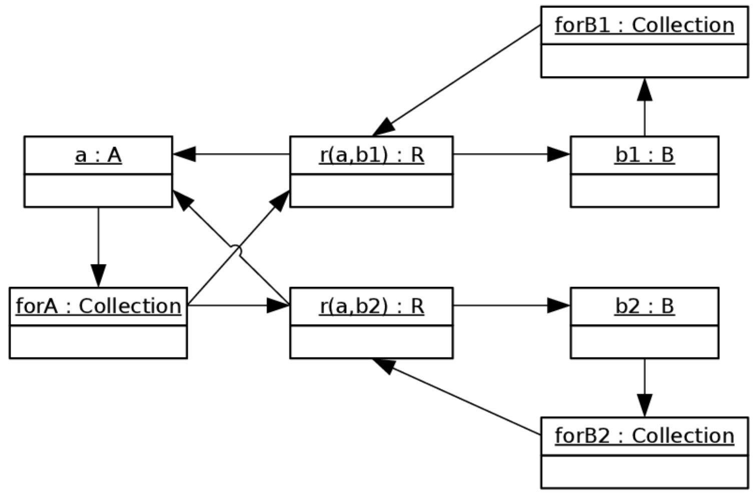
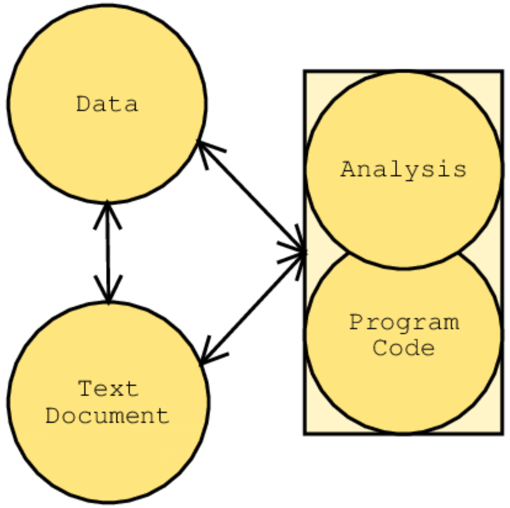
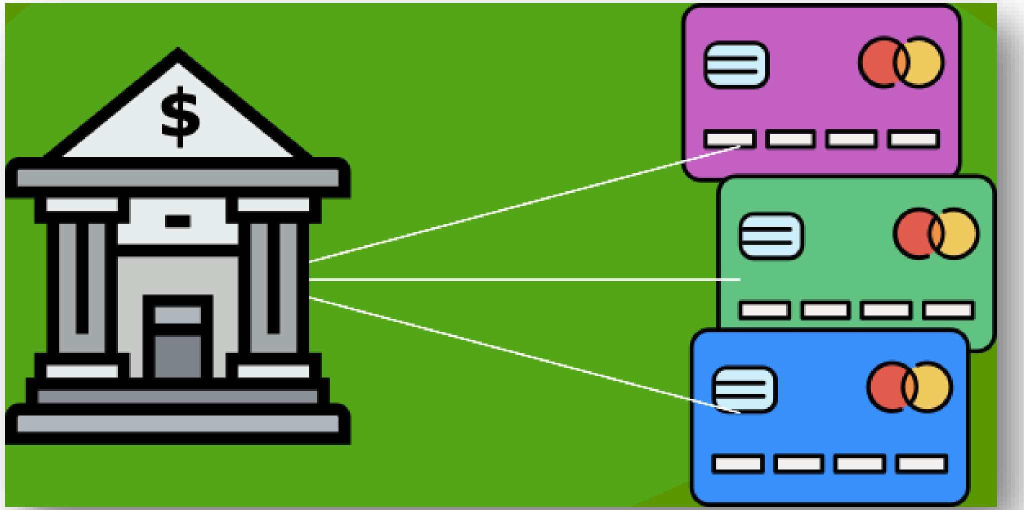
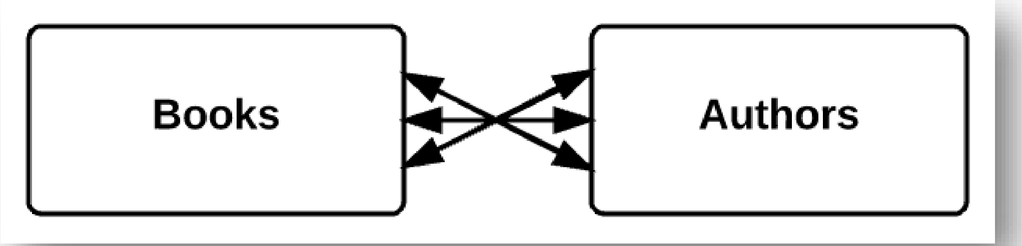
A relation is a mathematical tool for describing associations between elements of sets. Relations are widely used in computer science, especially in databases and scheduling applications. A relation can be defined across many items in many sets, but in this text, we will focus on binary relations, which represent an association between two items in one or two sets.



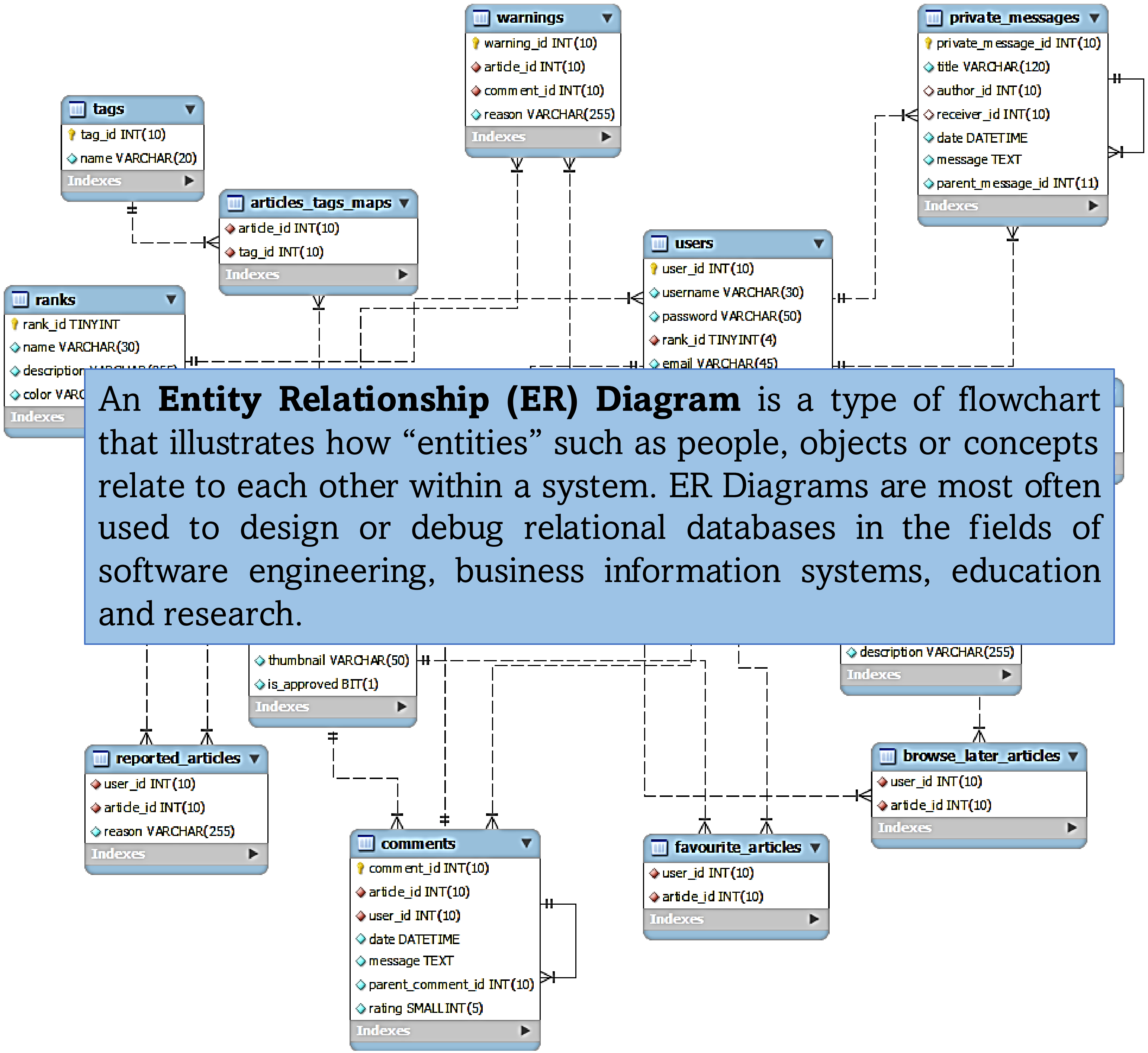
(c) Two-way, with attributes (as above, with ‘mutual friends’)



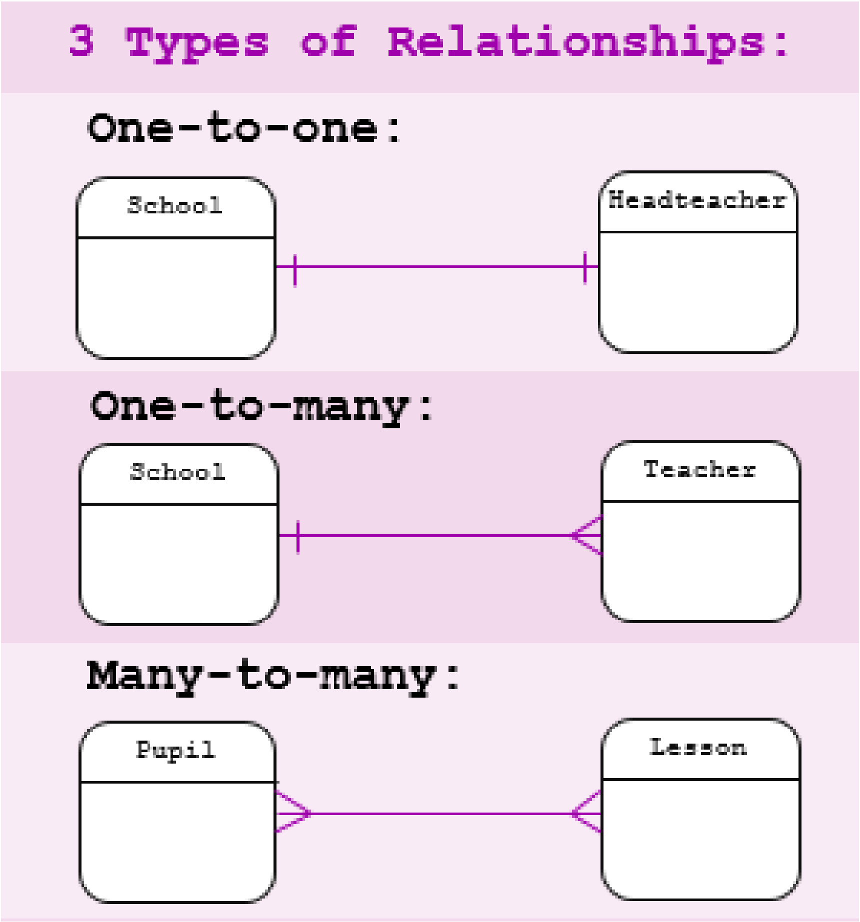


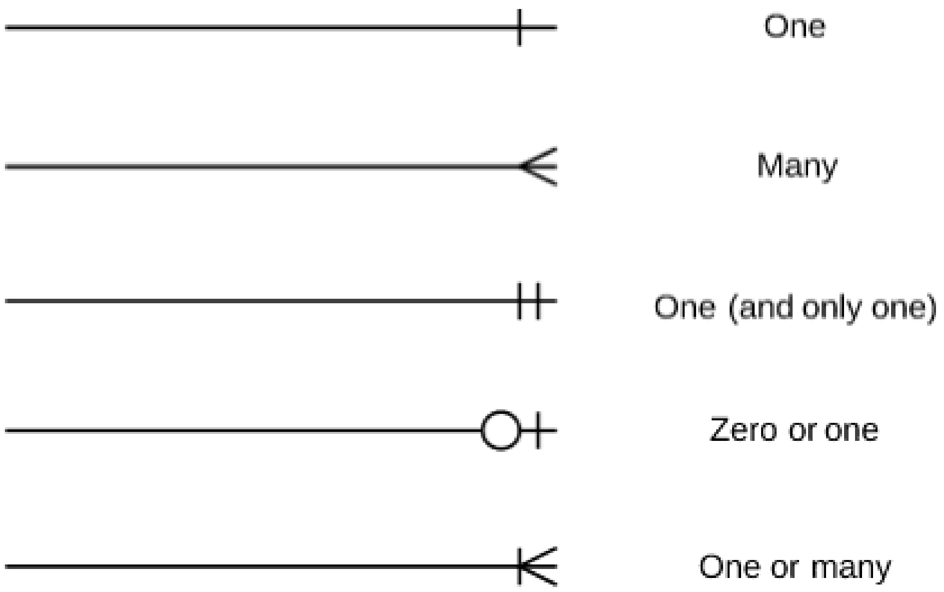

Books
Authors


ONE-TO-ONE RELATIONSHIP
PASSPORT
one passport
one person



An **Entity Relationship (ER) Diagram** is a type of flowchart that illustrates how “entities” such as people, objects or concepts relate to each other within a system. ER Diagrams are most often used to design or debug relational databases in the fields of software engineering, business information systems, education and research.





https://www.lucidchart.com/pages/er-diagrams

Definition 1:

Let A and B be sets. A binary relation from A to B is a subset of A × B.

When *(a, b)* belongs to *R*, *a* is said to be **related to** *b* by *R*.

$$a \; R \; b \qquad (a, b) \in R$$

When *a* is not related to *b*:

$$a \not R \, b \qquad (a, b) \notin R$$

Binary relations represent relationships between the elements of two sets.

EXAMPLE 1

Let A be the set of cities in the U.S.A., and let B be the set of the 50 states in the U.S.A.

Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b.

- ✓ Boulder, Colorado
- ✓ Bangor, Maine
- ✓ Ann Arbor, Michigan
- ✓ Middletown, New Jersey
- ✓ Cupertino, California
- ✓ Red Bank, New Jersey

are in R.

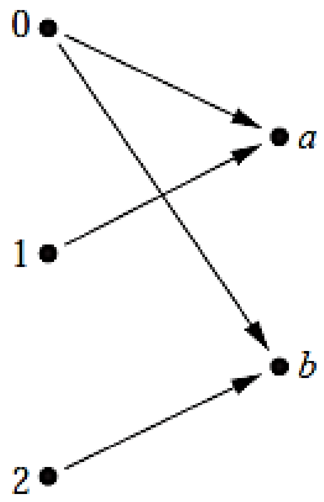
A relation can be used to express a **one-to-many** relationship between the elements of the sets A and B (as in Example 1), where an element of A may be related to more than one element of B. A function represents a relation where exactly one element of B is related to each element of A.

YOUR TURN

Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the region b (Kurdistan).

EXAMPLE 2

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.



Mapping representation

R	a	b
0	×	×
1	×	
2		×

Tabular representation

EXAMPLE 3

Let A be the set of students in your university, and let B be the set of courses. Let $\boldsymbol{R}$ be the relation that consists of those pairs (a, b), where a is a student enrolled in course b.

Ahmed and Sara are enrolled in Programming 1 course. Then the pairs (Ahmed, Programming 1) and (Sara, Programming 1) belong to R.

If Ahmed is also enrolled in Physics 1, then the pair (Ahmed, Physics 1) is also in R. However, if Sara is not enrolled in Biology, then the pair (Sara, Biology) is not in R.

Relations

- ❑ A relation R on a set X is a subset of $X \times X$.

If $(a, b) \in R$, we write xRy.

$\Rightarrow$ "x is related to y".

$\Rightarrow$ $R(x, y)$

$\Rightarrow$ Rxy (Predicate Logic)

xGy: x **is greater than** y. $(x, y) \in \mathbb{Z}$

- ❑ $(7, 5)$
- ❑ $(6, 2)$
- ❑ $(1, 9)$
- ❑ $(4.3, 2)$

xGy: x **is greater than** y. $(x, y) \in \mathbb{Z}$

❑ $(7, 5)$	$7G5$ or $G(7, 5)$	+
❑ $(6, 2)$	$6G2$	+
❑ $(1, 9)$	$1G9$	−
❑ $(4.3, 2)$	$G(4.3, 2)$	Don't make claim

not an $\mathbb{Z}$

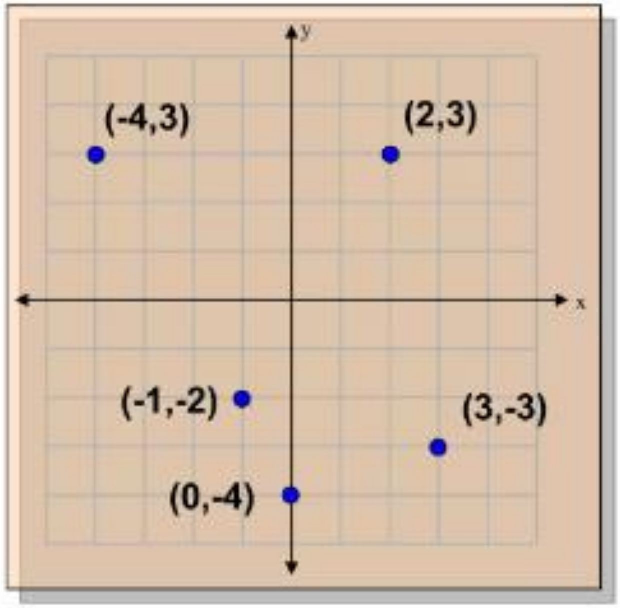
EXAMPLE

Given the relation:

$$\{(2,-6),(1,4),(2,4),(0,0),(1,-6),(3,0)\}$$

domain $:\{0,1,2,3\}$

range $:\{-6,0,4\}$



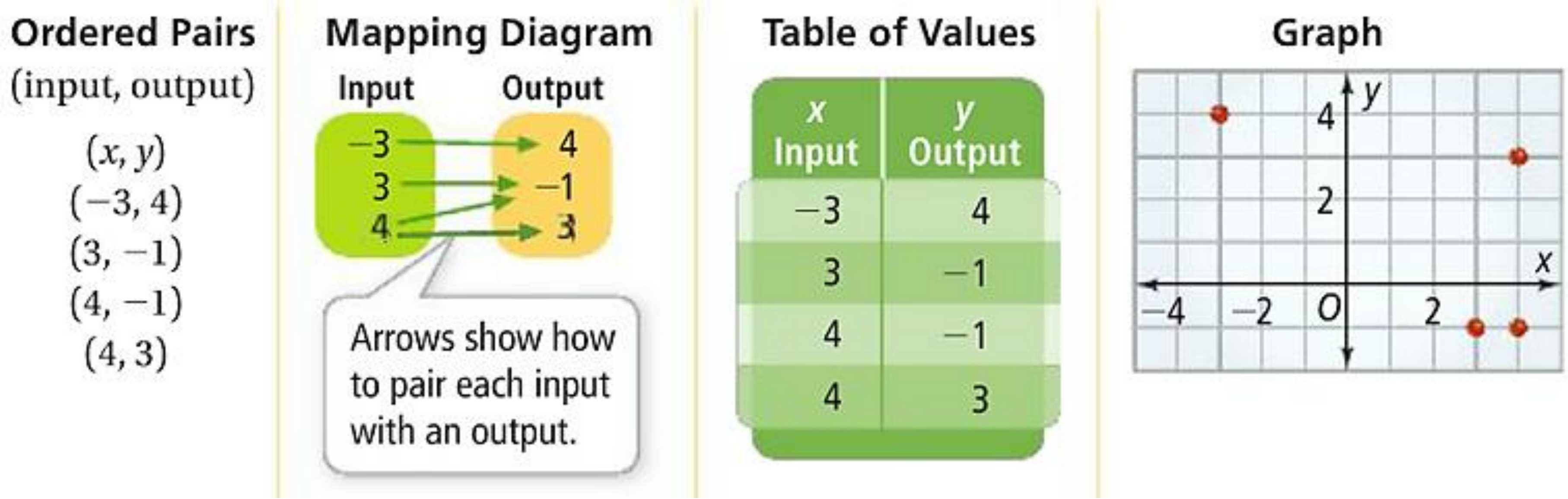
$$\{(-4,3),(-1,2),(0,-4),(2,3),(3,-3)\}$$

domain $:\{-4,-1,0,2,3\}$

range $:\{-4,-3,-2,3\}$

In Summary:

- A relation is a set of pairs of input and output values.
- There are four ways to represent relations:



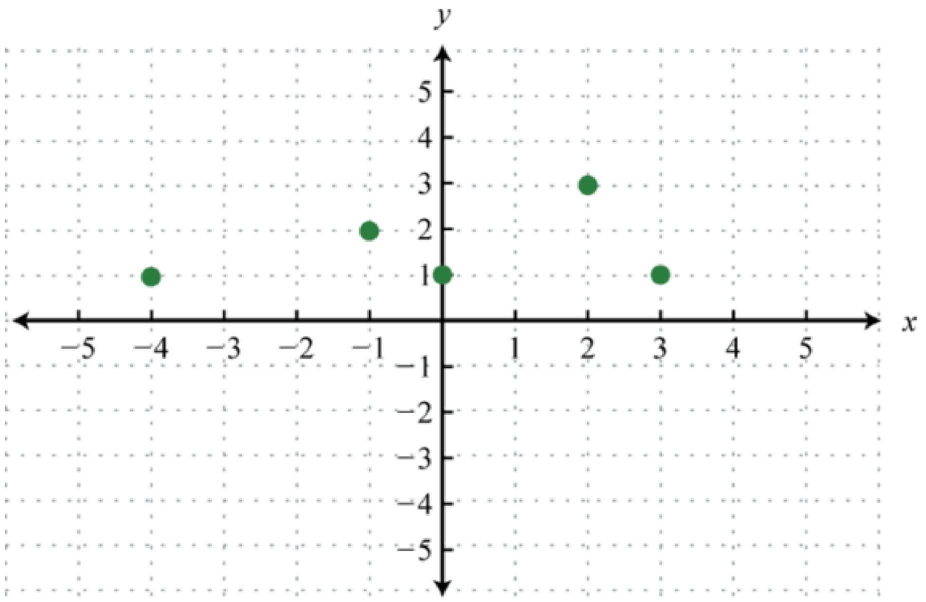
PRACTICE

Given the relation:

$$\{(2,-3),(4,6),(3,-1),(6,6),(2,3)\}$$

domain : $\{2,3,4,6\}$

range : $\{-3,-1,3,6\}$



domain : $\{-4,-1,0,2,3\}$

range : $\{1,2,3\}$

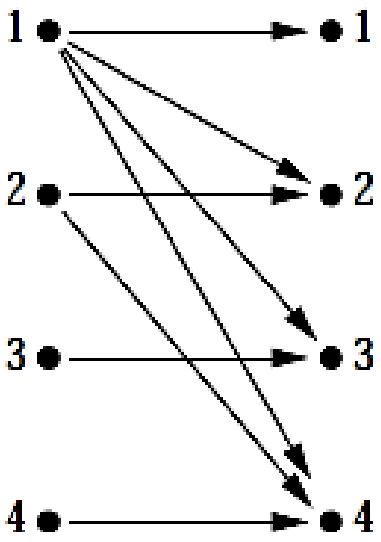
EXAMPLE 4

Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

a divides $b = a|b = a \cdot k = b$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

EXAMPLE 5

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \le b\}$,
$R_2 = \{(a, b) \mid a > b\}$,
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) \mid a = b\}$,
$R_5 = \{(a, b) \mid a = b + 1\}$,
$R_6 = \{(a, b) \mid a + b \le 3\}$.

Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

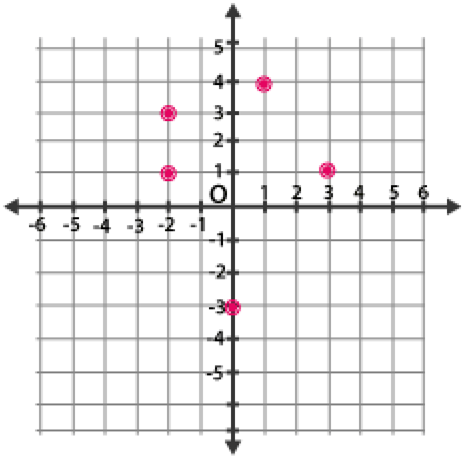
Solution: The pair $(1, 1)$ is in R_1, R_3, R_4, and R_6; $(1, 2)$ is in R_1 and R_6; $(2, 1)$ is in R_2, R_5, and R_6; $(1, -1)$ is in R_2, R_3, and R_6; and finally, $(2, 2)$ is in R_1, R_3, and R_4.

Relations as Functions

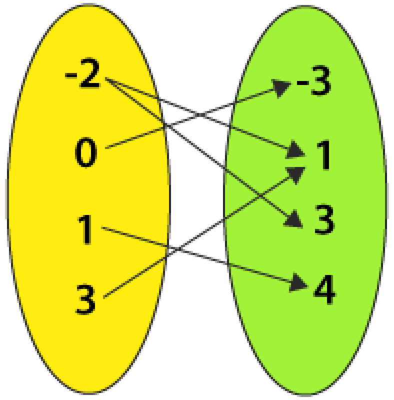
- ❑ Relations are a generalization of graphs of functions;
- ❑ The function f from A to B is the set of ordered pairs $(a, f(a))$ for $a \in A$

x	y
-2	1
-2	3
0	-3
1	4
3	1

Relation in table



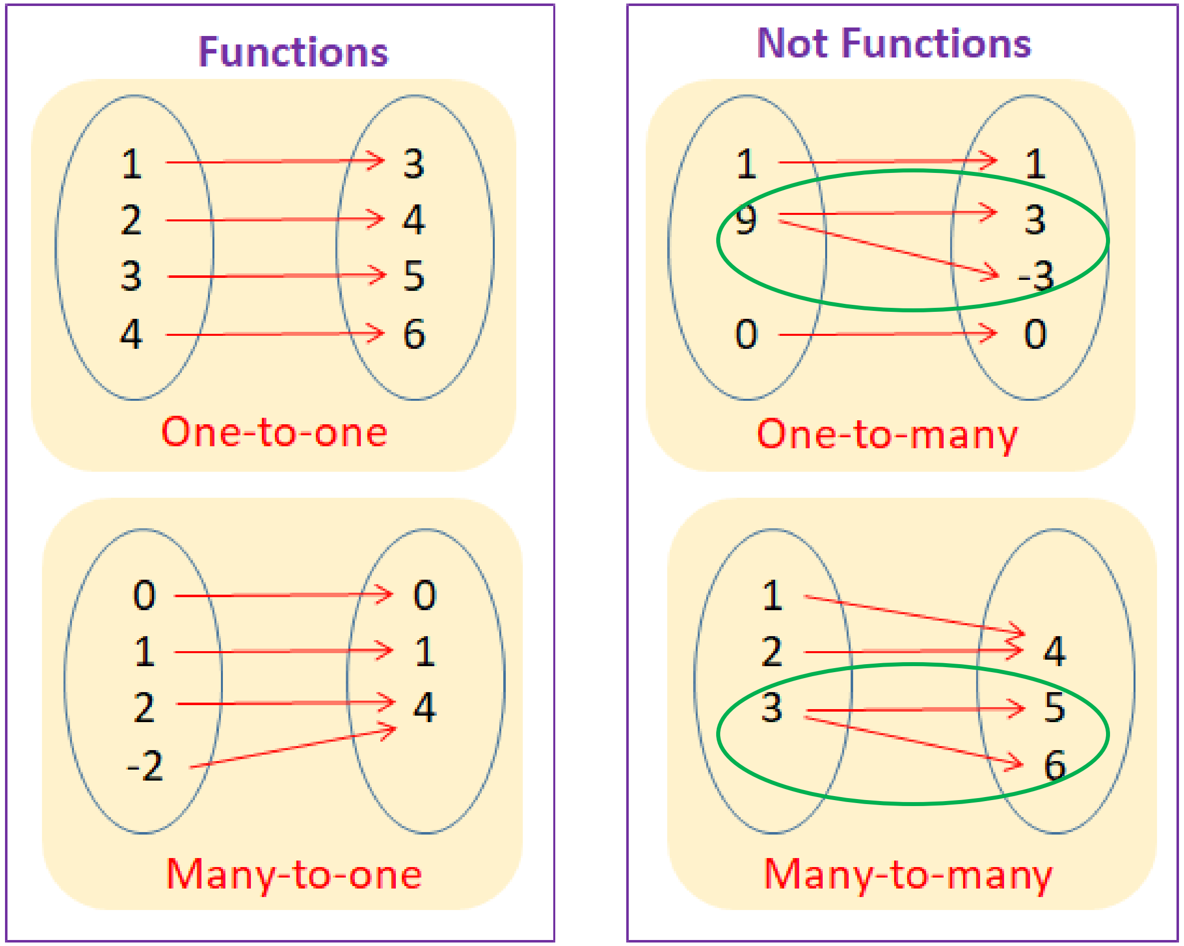
Relation in graph

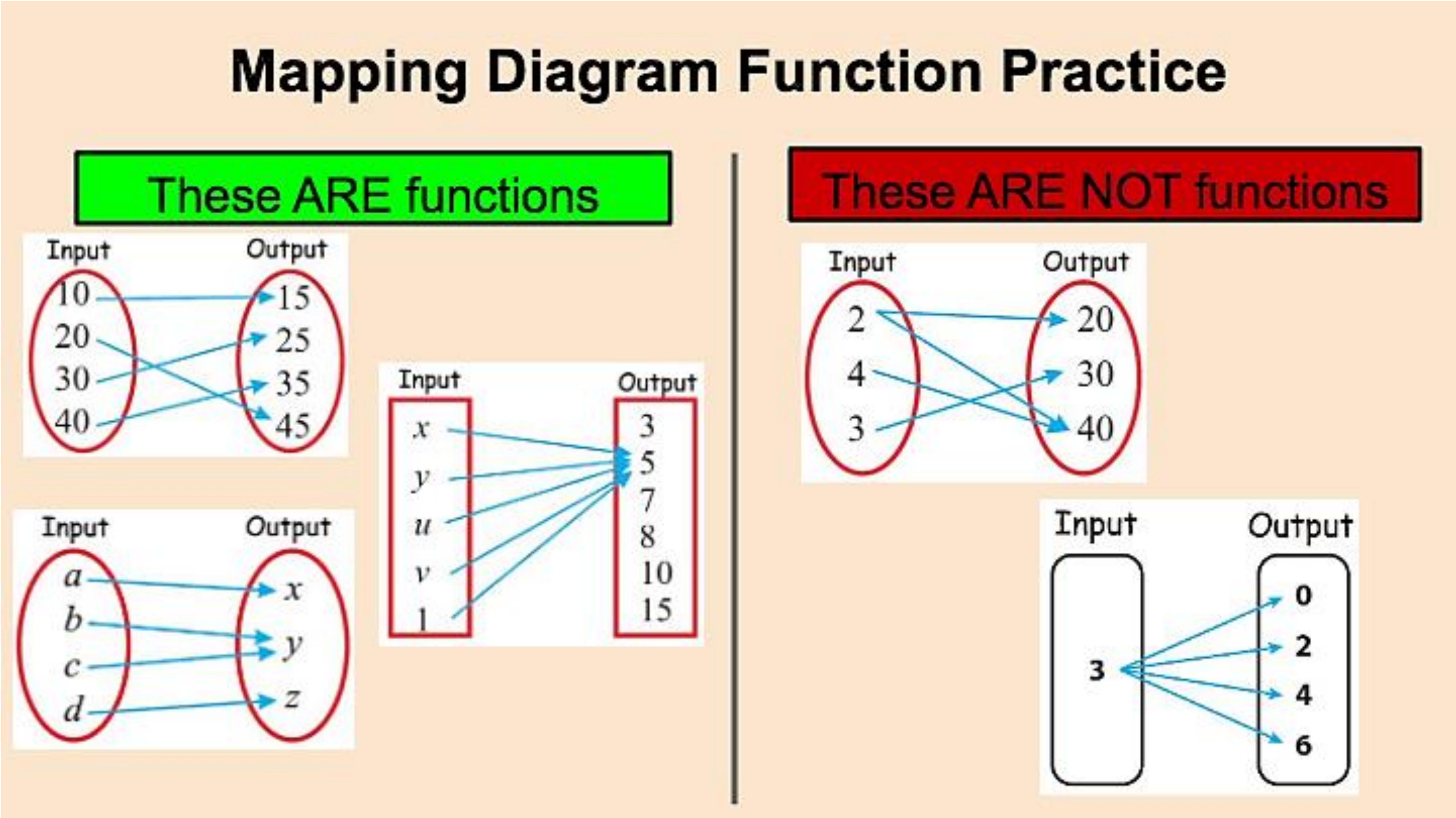


Relation in mapping diagram

Relations

A relation shows a relationship between two values.
A function is a relation where each input has only one output.

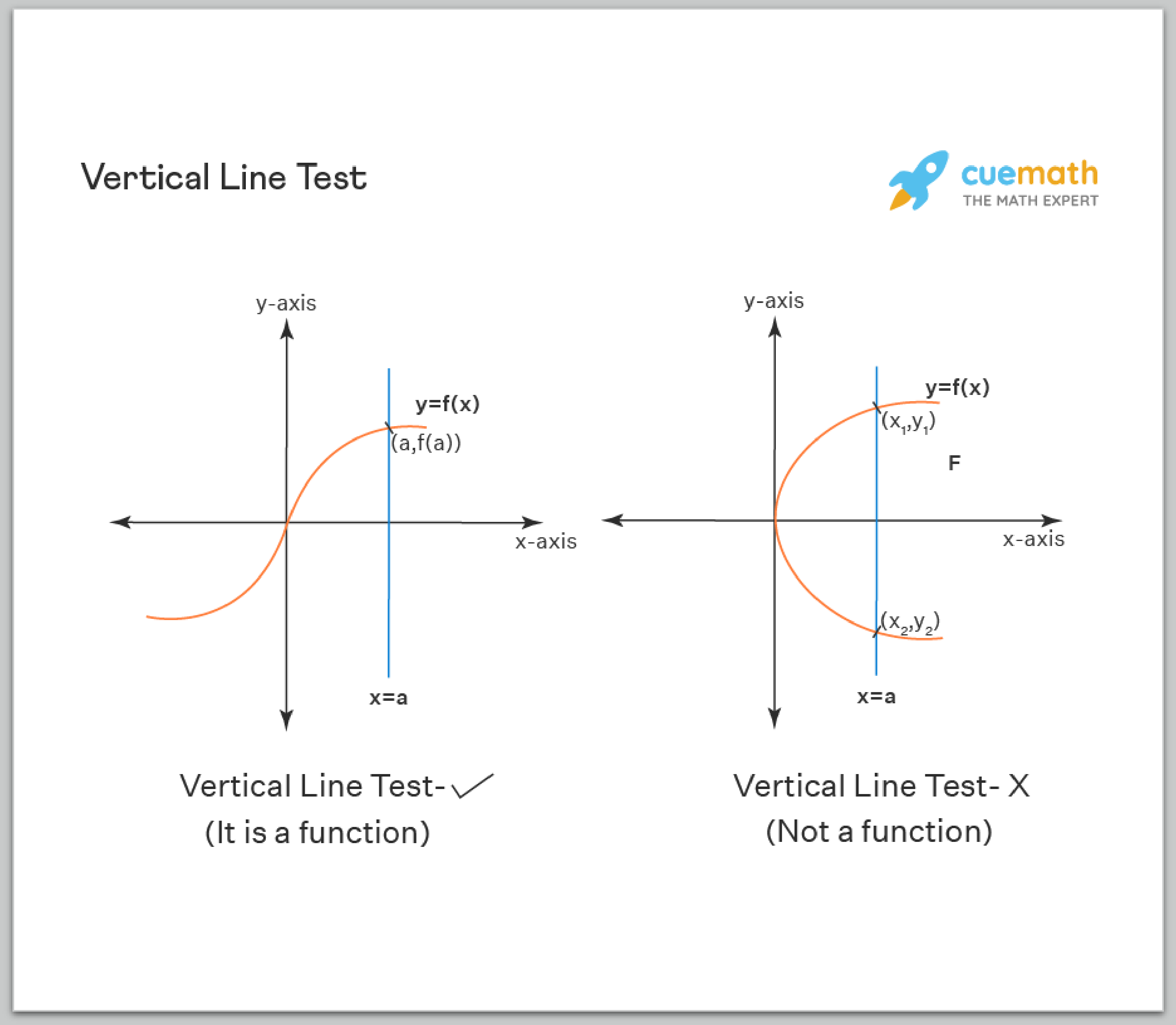



Mapping Diagram Function Practice
These ARE functions
Input
Output
10
20
30
40
15
25
35
45
Input
Output
a
b
c
d
x
y
z
Input
Output
x
y
u
v
1
3
5
7
8
10
15
These ARE NOT functions
Input
Output
2
4
3
20
30
40
Input
Output
3
0
2
4
6

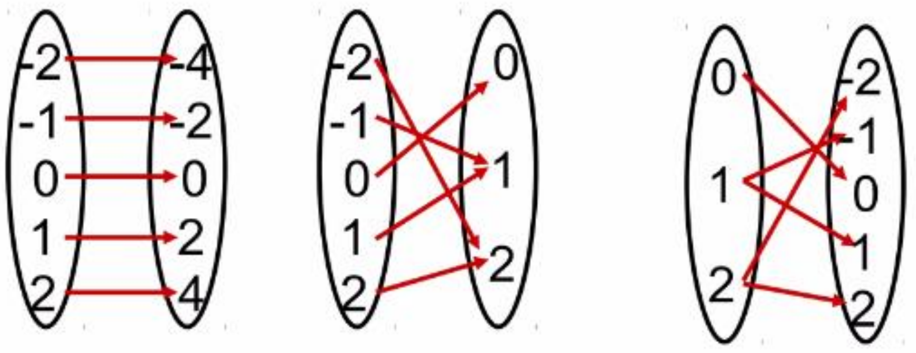
Define “Function” and “Not Function”:

Function	Not Function
(4,12)	(4,12)
(5,15)	(4,15)
(6,18)	(5,18)
(7,21)	(5,21)
(8,24)	(6,24)

Use **the vertical line test** to determine whether or not a graph represents a function. If a vertical line is moved across the graph and, at any time, touches the graph at only one point, then the graph is a function. If the vertical line touches the graph at more than one point, then the graph is not a function.

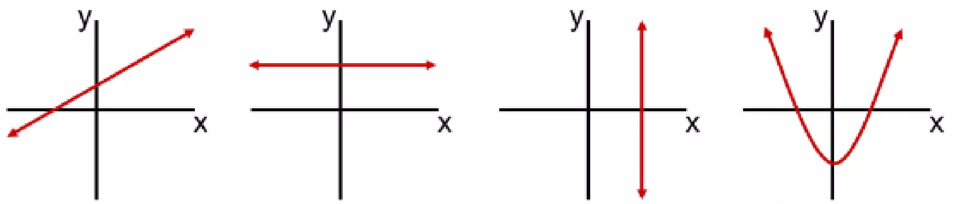


1. 2. 3.



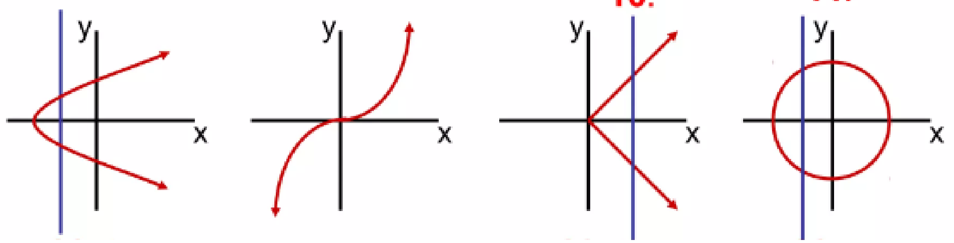
Function | Function | Not a Function

4. 5. 6. 7.

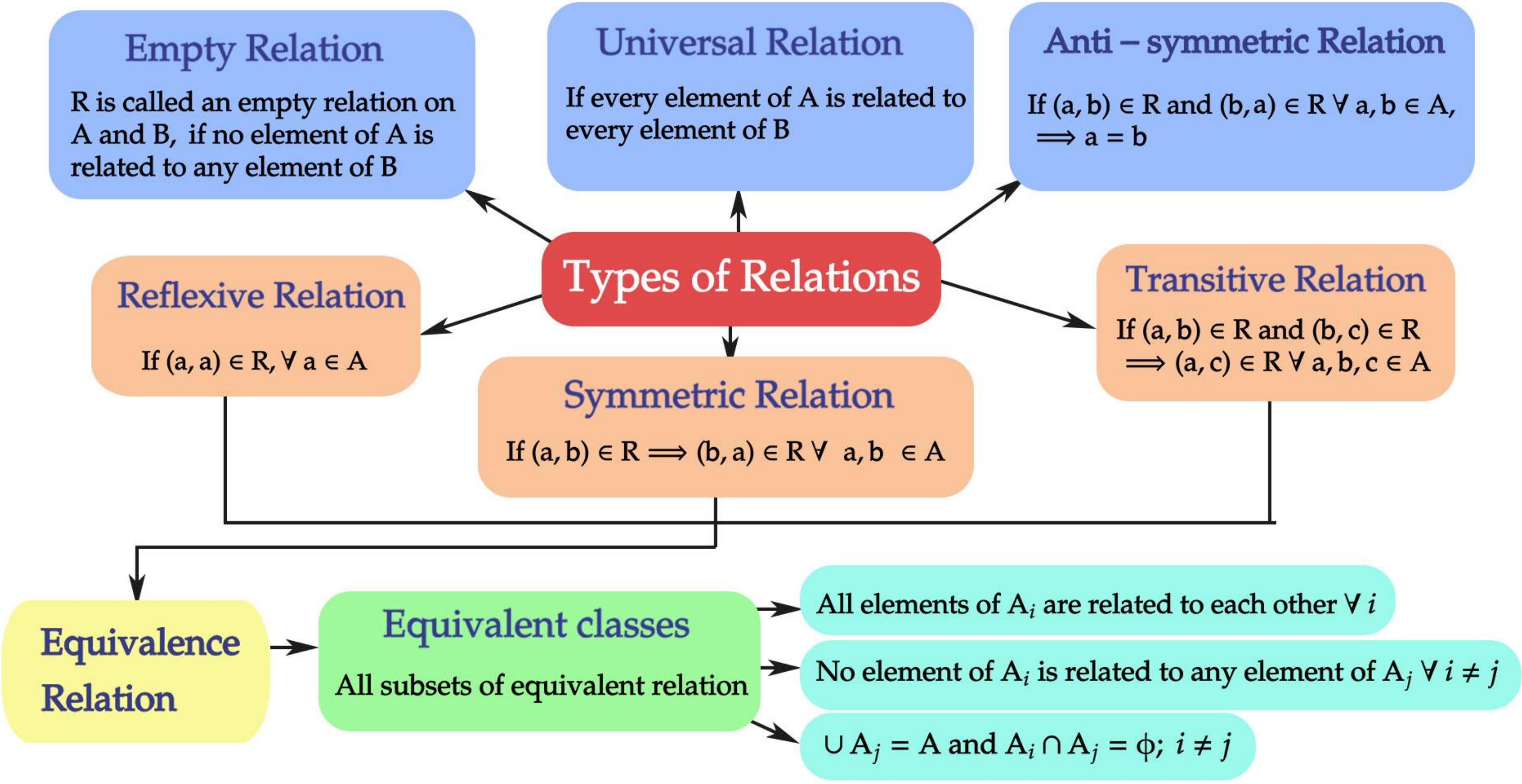


Function | Function | Not a Function | Function

8. 9. 10. 11.

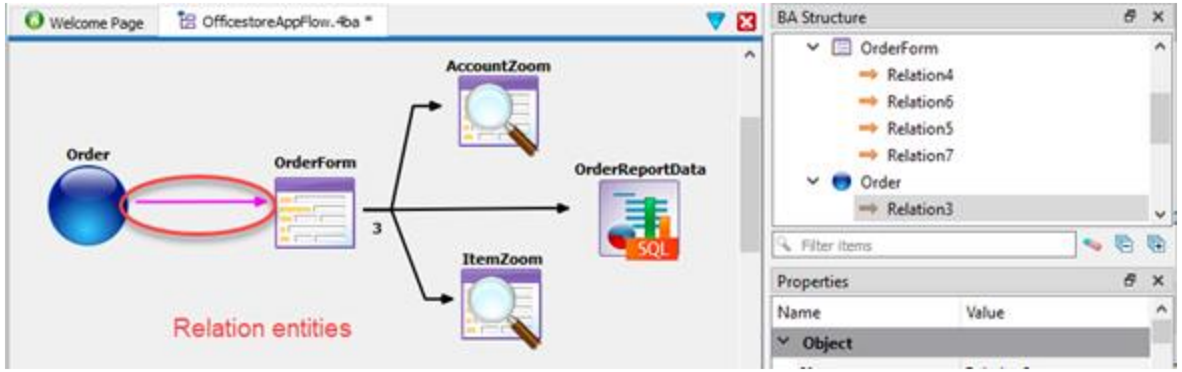


Not a Function | Function | Not a Function | Not a Function

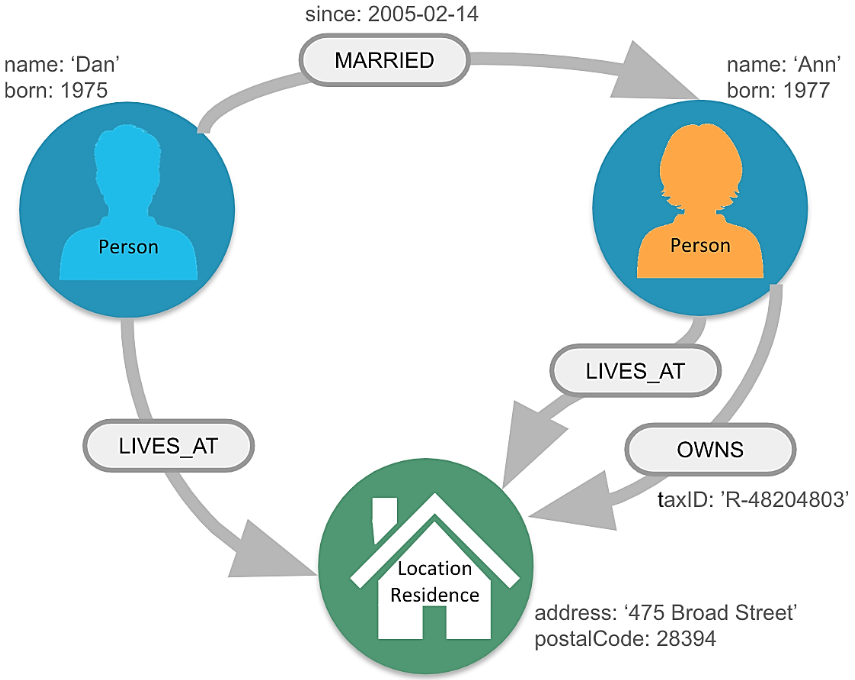


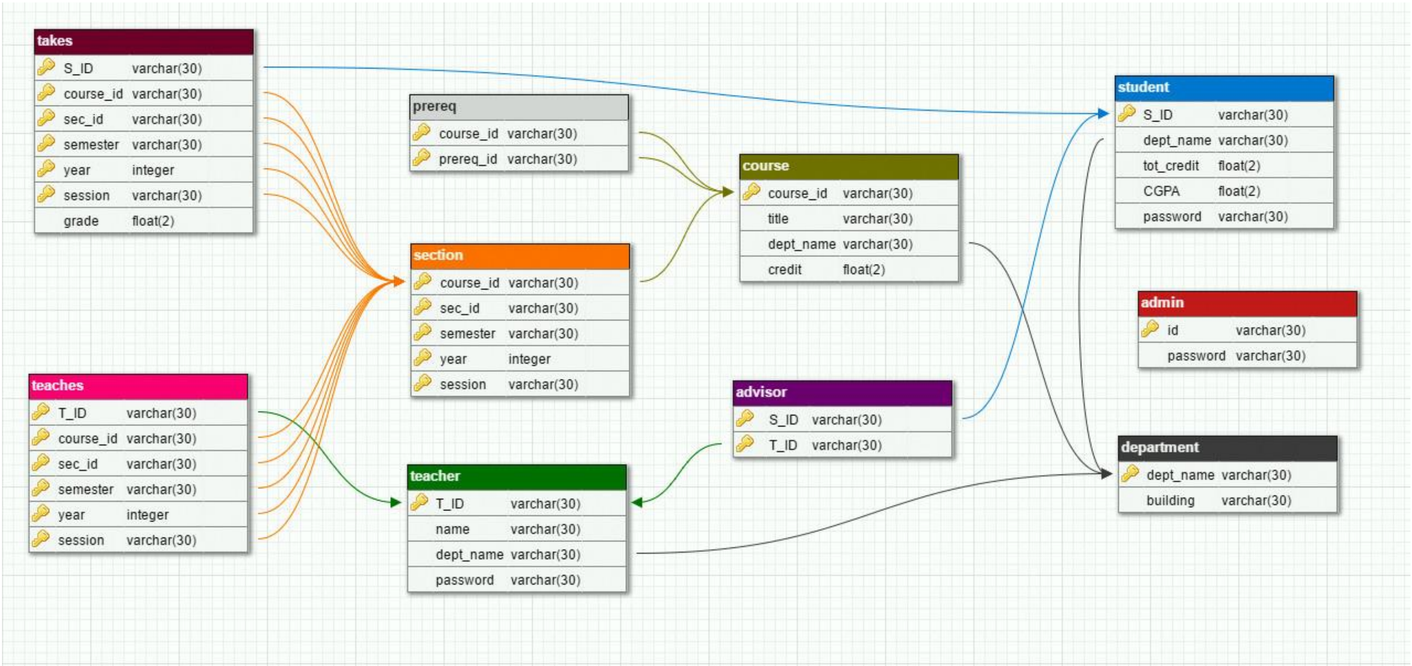
Empty Relation
R is called an empty relation on A and B, if no element of A is related to any element of B
Universal Relation
If every element of A is related to every element of B
Anti – symmetric Relation
If $(a, b) \in R$ and $(b, a) \in R\ \forall\ a, b \in A$, $\Longrightarrow a = b$
Types of Relations
Reflexive Relation
If $(a, a) \in R, \forall\ a \in A$
Transitive Relation
If $(a, b) \in R$ and $(b, c) \in R \Longrightarrow (a, c) \in R\ \forall\ a, b, c \in A$
Symmetric Relation
If $(a, b) \in R \Longrightarrow (b, a) \in R\ \forall\ a, b \in A$
Equivalence Relation
Equivalent classes
All subsets of equivalent relation
All elements of A_i are related to each other $\forall\ i$
No element of A_i is related to any element of $A_j\ \forall\ i \neq j$
$\cup A_j = A$ and $A_i \cap A_j = \phi;\ i \neq j$

Properties of Relations



- Reflexive
- Symmetric
- Antisymmetric
- Transitive





Advantages of relational databases:

- ✓ Simple Model
- ✓ Data Accuracy
- ✓ Easy to access Data
- ✓ Security
- ✓ Collaborate

A system used to maintain relational databases is a relational database management system **(RDBMS).** Many relational database systems are equipped with the option of using the **SQL (Structured Query Language) for querying and maintaining the database.**

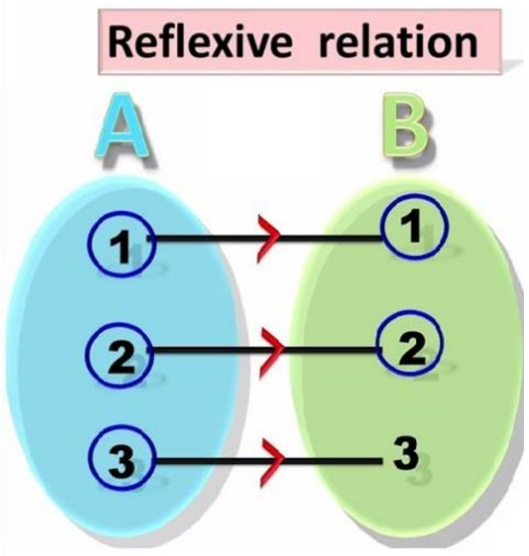
Properties of Relations: Reflexive

Properties of Relations

Reflexive

- A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

- We see that a relation on A is reflexive if every element of A is related to itself.



A relation R on a set A is said to be reflexive if $\forall\, a \in A, (a, a) \in R$.

$A = \{a, b, c\}$

- ✗ ○ $\emptyset$
- ✓ ○ $A \times A$
- ✓ ○ $\{(a,a),(b,b),(c,c)\}$
- ✗ ○ $\{(a,b),(b,a),(a,a),(b,b)\}$ **because there is no** (c, c)
- ✓ ○ $\{(a,a),(b,b),(c,c),(a,b),(b,c)\}$
- ✗ ○ $\{(a,b),(b,c),(a,c)\}$

$A \times A$

	a	b	c
a	aa	ab	ac
b	ba	bb	bc
c	ca	cb	cc

EXAMPLE 6

Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Which of these relations are reflexive?

Answer:

In reflexive relations there should be all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.

Therefore, only the relations R_3 and R_5 are reflexive.

Back to EXAMPLE 5

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$,
$R_2 = \{(a, b) \mid a > b\}$,
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) \mid a = b\}$,
$R_5 = \{(a, b) \mid a = b + 1\}$,
$R_6 = \{(a, b) \mid a + b \leq 3\}$.

Which of the relations are reflexive?

Answer: R_1 (because $a \leq a$ for every integer a), R_3, and R_4.

Properties of Relations: Symmetric

A relation R on a set A is said to be symmetric if $\forall\, a, b \in A, (a,b) \in R, (b,a) \in R$.

$$\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$$

$A = \{1, 2, 3\}$

- ✓ ○ $\emptyset$ — It is a possible subset of the cartesian product; As there is no condition for $\emptyset$, then it is symmetric.
- ✓ ○ $A \times A$
- ✓ ○ $\{(1,1), (2,2), (3,3)\}$
- ✓ ○ $\{(2,1), (1,2), (1,1)\}$
- ✕ ○ $\{(3,1), (1,3), (2,3)\}$
- ✕ ○ $\{(1,2), (2,3), (1,3)\}$

$A \times A$

	1	2	3
1	11	12	13
2	21	22	23
3	31	32	33

Properties of Relations: Antisymmetric

Given a relation R on a set A we say that R is *antisymmetric* if and only if:
$\forall(a, b) \in R$ where $a \neq b$ we must have $(b, a) \notin R$.

(3, 1) (7, 3) (1, 7) $\notin R_1$

$R_1 = \{(1,3), (3,7), (7,1)\}$ ✓ antisymmetric

(7, 1) (3, 7) $\notin R_1$

$R_2 = \{(1,7), (3,3), (7,3)\}$ ✓ antisymmetric

(1, 3) (3, 1) (7, 1)

$R_3 = \{(3,1), (1,3), (1,7)\}$ ✕ not antisymmetric

A relation R on a set A is said to be antisymmetric $\forall\, a, b \in A, (a, b) \in R, (b, a) \in R \quad a = b$.

If you have (a, b) then you can never have (b, a), but $a = b$.

$A = \{1, 2, 3\}$

- ✓ ○ $\emptyset$ — It is a possible subset of the cartesian product; As there is no condition for $\emptyset$, then it is both symmetric and antisymmetric.
- ✗ ○ $A \times A$ — In antisymmetric, if you have (1, 3) then you cannot have (3, 1).
- ✓ ○ $\{(1,1), (2,2), (3,3)\}$
- ✓ ○ $\{(2,1), (2,3), (1,1)\}$
- ✗ ○ $\{(2,3), (3,2), (2,2), (3,3)\}$
- ✓ ○ $\{(1,1), (2,2), (2,3), (1,3)\}$

	1	2	3
1	~~11~~	12	13
2	21	~~22~~	23
3	31	32	~~33~~

$A \times A$

Antisymmetric – means we cannot accept the values of the one side symmetric pair but diagonal.

Back to EXAMPLE 5

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$,
$R_2 = \{(a, b) \mid a > b\}$,
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) \mid a = b\}$,
$R_5 = \{(a, b) \mid a = b + 1\}$,
$R_6 = \{(a, b) \mid a + b \leq 3\}$.

Which of the relations are symmetric and which are antisymmetric?

Answer:

The relations R_3, R_4, and R_6 are symmetric.

R_3 is symmetric, for if $a = b$ or $a = -b$, then $b = a$ or $b = -a$.
R_4 is symmetric because $a = b$ implies that $b = a$.
R_6 is symmetric because $a + b \leq 3$ implies that $b + a \leq 3$.

The relations R_1, R_2, R_4, and R_5 are antisymmetric.

R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b$.

R_2 is antisymmetric because it is impossible that $a > b$ and $b > a$.

R_4 is antisymmetric, because two elements are related with respect to R_4 if and only if they are equal.

R_5 is antisymmetric because it is impossible that $a = b + 1$ and $b = a + 1$.

Back to EXAMPLE 6

Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Which of the relations are symmetric and which are antisymmetric?

Answer:

The relations R_2 and R_3 are symmetric.

R_4, R_5, and R_6 are all antisymmetric.

Properties of Relations: Transitive

A relation R on a set A is said to be transitive if $\forall\ a, b \in A, (a, b) \in R, (b, c) \in R\ then\ (a, c) \in R$.

$A = \{1, 2, 3\}$

✓ ○ $\emptyset$ — It is a possible subset of the cartesian product; As there is no condition for $\emptyset$, then it is symmetric, antisymmetric, and transitive.

✓ ○ $A \times A$

✓ ○ $\{(1, 1), (2, 2), (3, 3)\}$

✓ ○ $\{(2, 3), (1, 2), (1, 3)\}$

✓ ○ $\{(1, 2), (1, 3)\}$

✓ ○ $\{(2, 3)\}$

(for the two sets above:) there is no requirement to check transitivity, so, the set automatically considered transitive.

✕ ○ $\{(1, 2), (2, 1)\}$ — $(1,1)\ and\ (2,2)$ also should belong to R.

$A \times A$

	1	2	3
1	11	12	13
2	21	22	23
3	31	32	33

Detailed information about TRANSITIVE RELATION, when there are more than 3 pairs:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$

Start with the first pair, (1,1). Here a=1, b=1.
Are their pairs that have their 1st elements as b? Yes
Only one pair, (1, 2). Here b=1, c=2
Is (a, c) a member of R_1? Yes. ∴ R_1 can be transitive.

Now move to the next pair, (1, 2). Here a=1, b=2.
Are their pairs that have their 1st elements as b? Yes
Two pairs, (2, 1), (2, 2). Here b=2, c=1 and c=2
Is (a, c) a member of R_1? Yes for both c=1 and c=2. ∴ R_1 may be transitive.

Now move to the next pair, (2, 1). Here a=2, b=1.
Are their pairs that have their 1st elements as b? Yes
Two pairs, (1, 1), (1, 2). Here b=1, c=1 and c=2
Is (a, c) a member of R_1? Yes for both c=1 and c=2. ∴ R_1 may be transitive.

Now move to the next pair, (2, 2). Here a=2, b=2.
Are their pairs that have their 1st elements as b? Yes
Two pairs, (2, 1), (2, 2). Here b=2, c=1 and c=2
Is (a, c) a member of R_1? Yes for both c=1 and c=2. ∴ R_1 may be transitive.

Now move to the next pair, (3, 4). Here a=3, b=4.
Are their pairs that have their 1st elements as b? Yes
Two pairs, (4, 1), (4, 4). Here b=4, c=1 and c=4
Is (a, c) a member of R_1? No for c=1. ∴ R_1 is not transitive.

Finished! No need to do any further checking.

Another Method to check for Transitivity

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

(a, b)	Pairs with b as their 1 st element	c	$(a, c) \in R_4$?
(2, 1)	-	-	-
(3, 1)	-	-	-
(3, 2)	(2, 1)	1	Yes
(4, 1)	-	-	-
(4, 2)	(2, 1)	1	Yes
(4, 3)	(3, 1)	1	Yes
	(3, 2)	2	Yes

There are no "**No**" answers in the last column. Therefore, the relation R_4 is transitive.

Conclusion

- ○ ∅ - **not reflexive**, **but** **symmetric, antisymmetric, transitive**

- ❑ A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

= is reflexive (2=2)

- ❑ That is, a relation is symmetric if and only if a is related to b implies that b is related to a.

= is symmetric (x=2 implies 2=x)

- ❑ A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.

≤ is antisymmetric (x≤y and y≤x implies x=y)

- ❑ A relation R on a set A is said to be transitive if $\forall\, a, b \in A, (a, b) \in R, (b, c) \in R\ then\ (a, c) \in R$.

< is transitive (2<3 and 3<5 implies 2<5)

✓ X = {a, b, c, d} and R = {(a, a),(a, b),(a, c),(d, b),(b, b),(c, d)}

Only antisymmetric

✓ Let A = {0, 1, 2, 3} and R a relation over A:

R = {(0, 0),(0, 1),(0, 3),(1, 1),(1, 0),(2, 3),(3, 3)}

Only transitive

List Properties.

✓ If R and S are reflexive, then R ∩ S is so. Explain why.

✓ If R and S are symmetric, then R ∩ S is so. Explain why.

Prove.

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$

✗

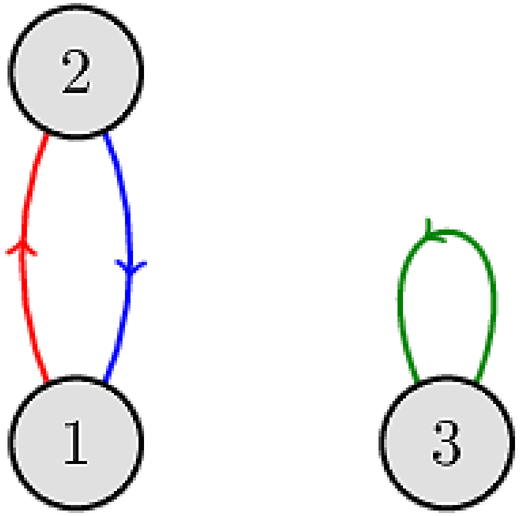
$$R = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$

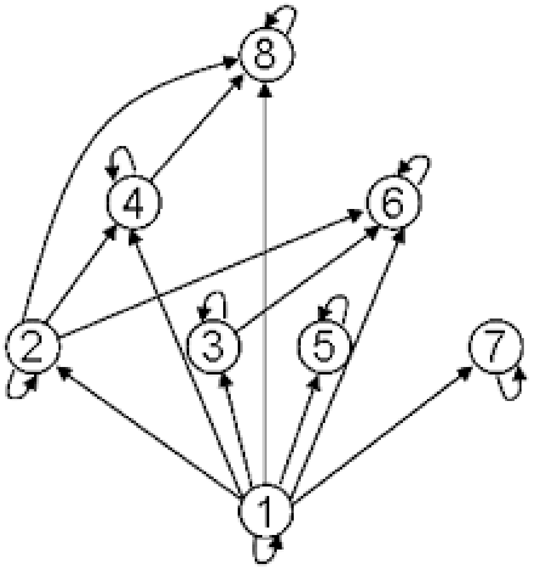
✓

Transitive?

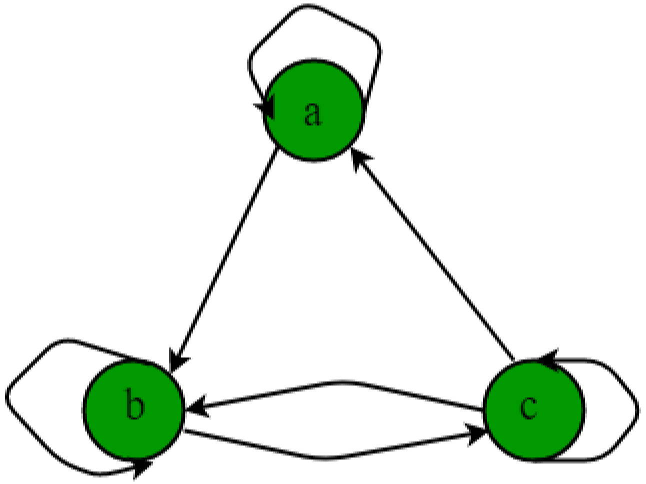
For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$





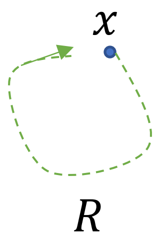
Directed Graphs or Digraphs



Reflexive

For any given item in the relation is going to be related to itself

$\forall x \quad xRx$

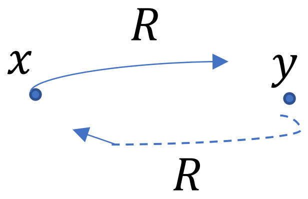


$= \quad 5 \in X$

$(5,5) \in R \Longrightarrow 5 = 5$

Symmetric

$\forall x \forall y \qquad xRy \rightarrow yRx$

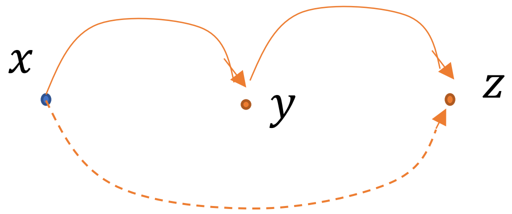


$\neq \qquad 10 \neq 11 \rightarrow 11 \neq 10$

$xRy \rightarrow yRx$

Transitive

$\forall x \forall y \forall z \qquad xRy \land yRz \rightarrow xRz$



$3 < 5, 5 < 7 \quad \Rightarrow 3 < 7$

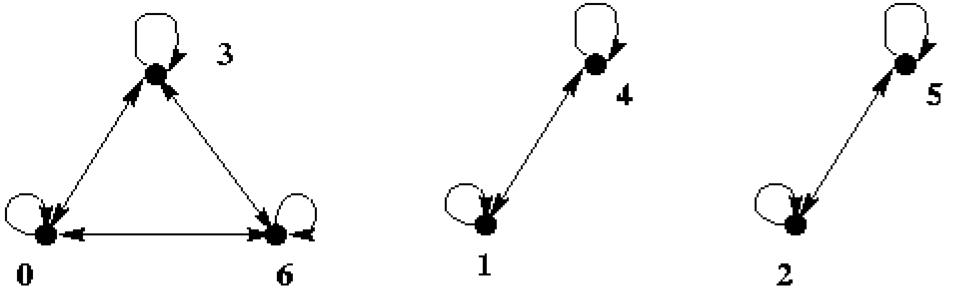
Equivalence Relation

- A binary relation ***R*** on a set ***A*** is an **equivalence relation** if and only if

  (1) ***R*** is reflexive → $a, \quad aRa$

  (2) ***R*** is symmetric, and→ $a, b \quad aRb \rightarrow bRa$

  (3) ***R*** is transitive. → $aRb \wedge bRc \rightarrow aRc$



Answer the Question: Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

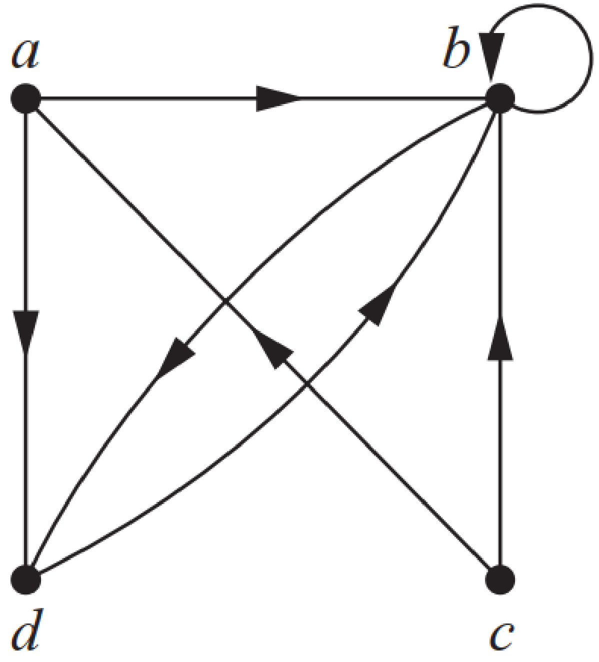
(a) $\{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0)\}$

(b) $\{(0,0), (1,1), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

(c) $\{(0,0), (0,1), (1,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$

(d) $\{(0,0), (0,1), (1,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

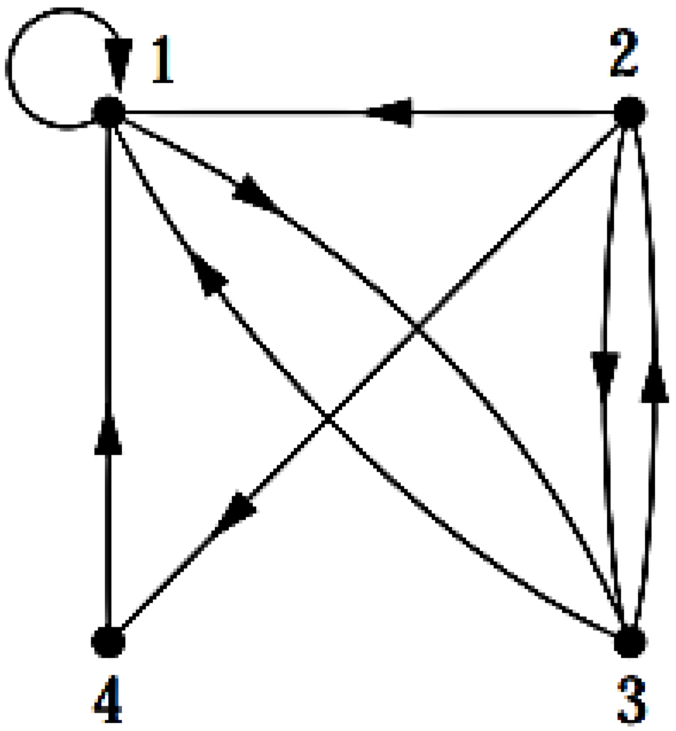
The directed graph with vertices a, b, c, and d, and edges:



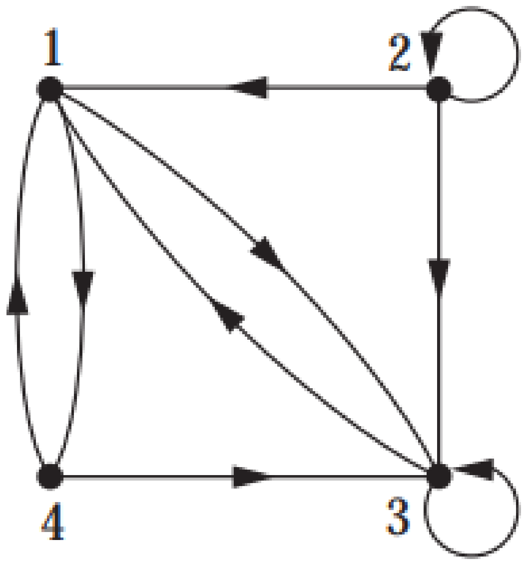
R = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}

The directed graph of the relation on the set {1, 2, 3, 4}:

$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$

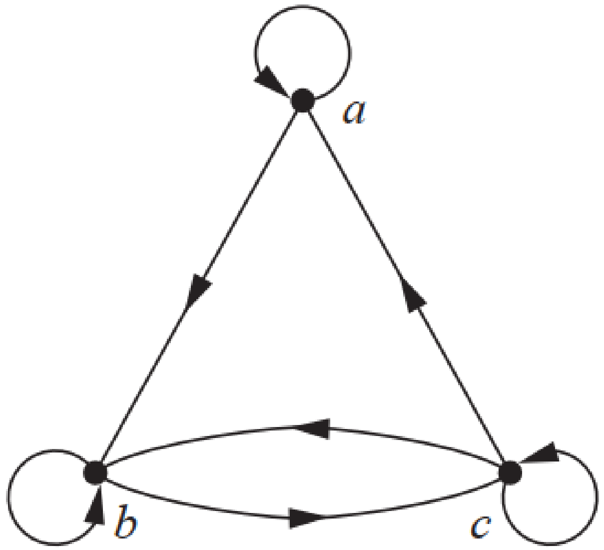


What are the ordered pairs in the relation R represented by the directed graph below?

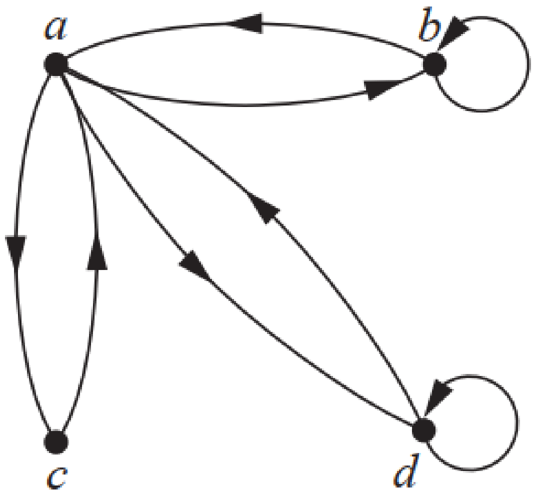


$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$

Determine whether the relations for the directed graphs shown below are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of R



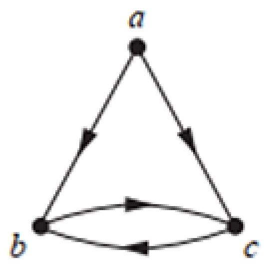
(b) Directed graph of S

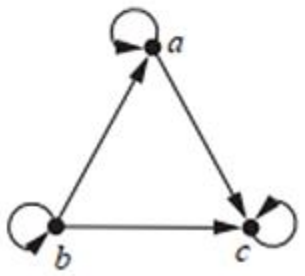
R is reflexive, neither symmetric nor antisymmetric and not transitive.

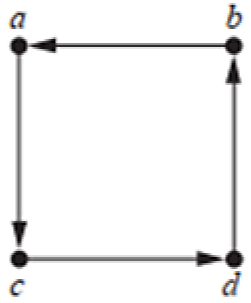
S is not reflexive, symmetric and not antisymmetric, not transitive.

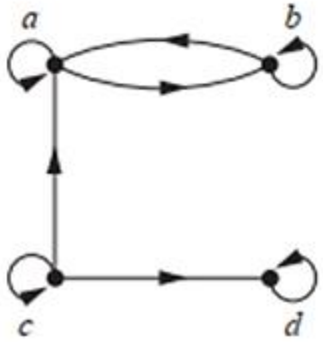
- Draw the directed graph that represents the relation
  {*(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)*}.

- List the ordered pairs in the relations represented by the directed graphs.









Determine their types.

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

EXAMPLE 17

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\},$$

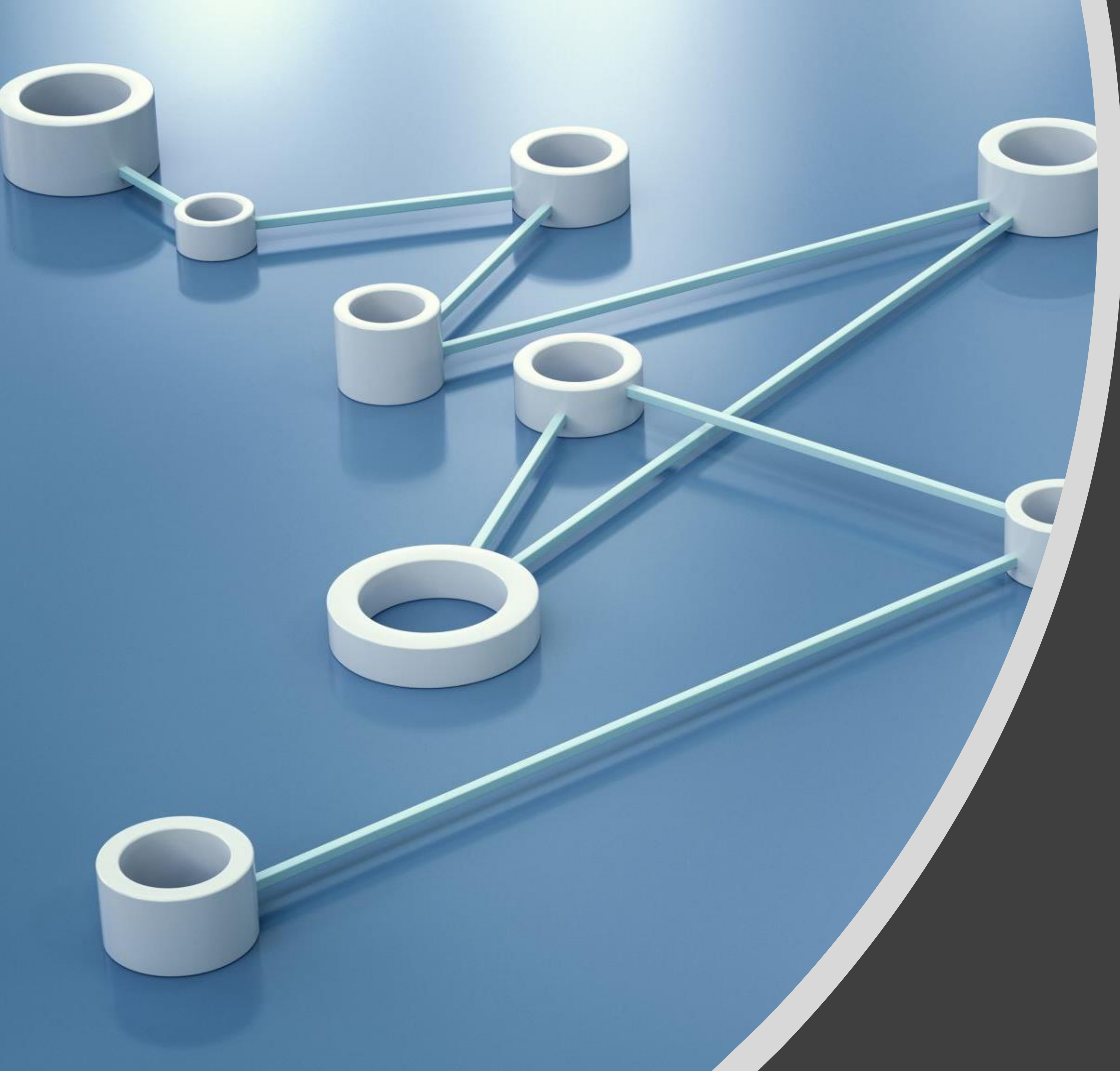
$$R_1 \cap R_2 = \{(1,1)\},$$

$$R_1 - R_2 = \{(2,2), (3,3)\},$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}.$$

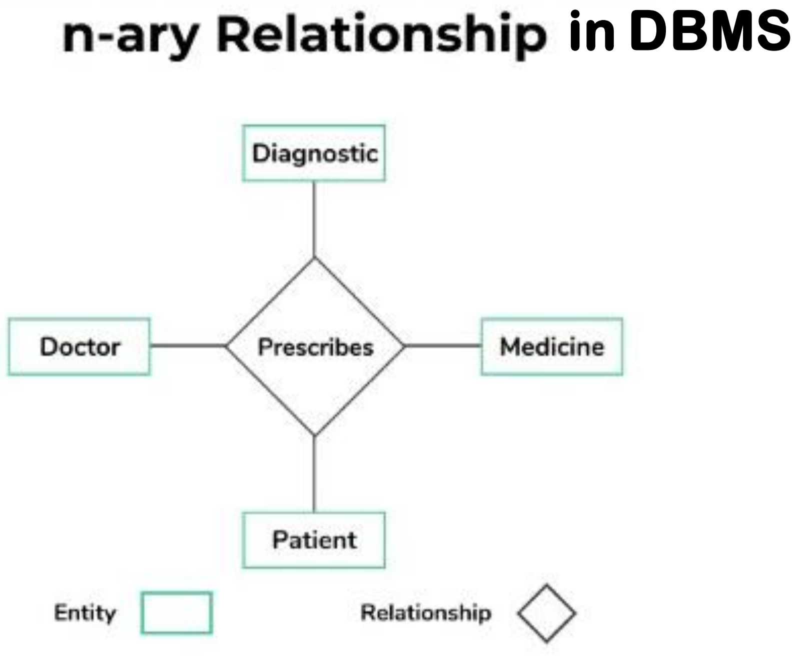
Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$.
Find:

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.



Databases and *n*-ary Relations

Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$ The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is called its degree.



- Let R be the relation on N × N × N (set of natural numbers) consisting of triples (a, b, c), where a, b, and c are integers with a < b < c. Then (1, 2, 3), (0, 2, 3), (3, 4, 5), … ∈ R, but (2, 4, 3) ∉ R.

The relation has degree 3

The domains of the relation are the set of natural numbers

- Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where $\boldsymbol{A}$ is the airline, $\boldsymbol{N}$ is the flight number, $\boldsymbol{S}$ is the starting point, $\boldsymbol{D}$ is the destination, and $\boldsymbol{T}$ is the departure time.

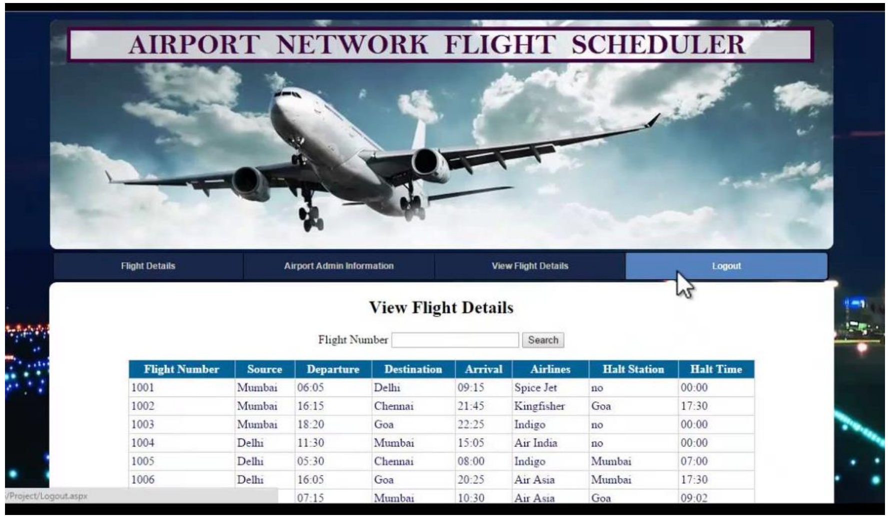


TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Student records are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*).
A sample database of six such records is:
(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49)
(Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90)
(Stevens, 786576, Psychology, 2.99)

- A database consists of **records**, which are *n*-tuples, made up of **fields**.

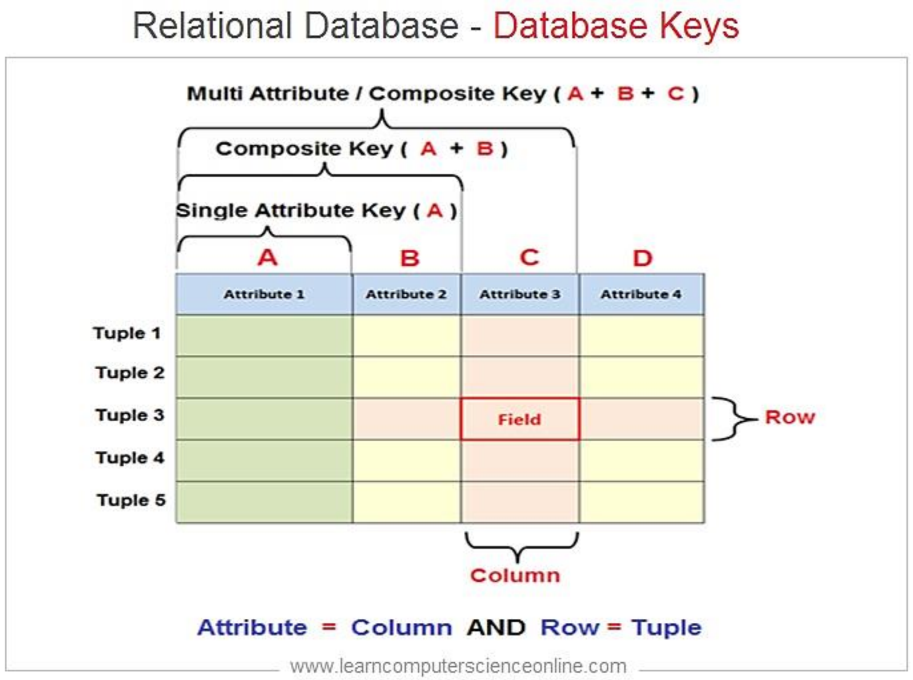
- Relations used to represent databases are also called **tables**, because these relations are often displayed as tables.

- Each column of the table corresponds to an *attribute* of the database. In Table 1, the attributes of this database are Student Name, ID Number, Major, and GPA.

- A domain of an *n*-ary relation is called a **primary key.**

- In Table 1, there is only one 4-tuple in this table for each student name, the domain of student names is a primary key. Similarly, the ID numbers in this table are unique, so the domain of ID numbers is also a primary key. However, the domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study. The domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA.

- The Cartesian product of these domains is called a **composite key**.



Primary Key

Stu_Id	Stu_Name	Stu_Age
101	Steve	23
102	John	24
103	Robert	28
104	Steve	29
105	Carl	29

Unique values

BeginnersBook.com

A composite key in SQL can be defined as **a combination of multiple columns**, and these columns are used to identify all the rows that are involved uniquely.

Composite Key

Composite Key (Name, Age)

Roll_No	Name	Age	Phone
1	Arya	21	7491901521
2	Bran	19	8491901000
3	John	24	9291018403
4	Max	24	7903084562

AuthorId	FirstName	LastName	Gender
1	Mark	Dunn	Male
2	Sara	Longhorn	Female
3	Jessica	Dale	Female
4	Steve	Wicht	Male

Composite Primary Key

Junction Table

AuthorId	BookId
1	3
1	2
3	4

BookId	BookTitle	Price	Published
1	Learn SQL	10	Yes
2	Learn C#	20	Yes
3	Learn CSS	15	Yes
4	Learn HTML	20	No

Note: Any key such as super key, primary key, candidate key, etc. can be called composite key if it has more than one attributes.

The 3-tuples in a 3-ary relation represent the following attributes of a student database: student ID number, name, phone number.

a) Is student ID number likely to be a primary key?
b) Is name likely to be a primary key?
c) Is phone number likely to be a primary key?

Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the ***selection operator*** (S_c) maps the n-ary relation $\boldsymbol{R}$ to the n-ary relation of all n-tuples from $\boldsymbol{R}$ that satisfy the condition $\boldsymbol{C}$.

To find the records of computer science majors in the n-ary relation R shown in Table 1, we use the operator S_{C_1}, where C_1 is the condition Major = "Computer Science."

The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49).

Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator S_{C_2}, where C_2 is the condition GPA > 3.5.

The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90).

Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator S_{C_3}, where C_3 is the condition (Major= "Computer Science" ∧ GPA > 3.5).

The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88).

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

We can use Relational Algebra to fetch data from this Table(relation)

Select Name students with age less than 17

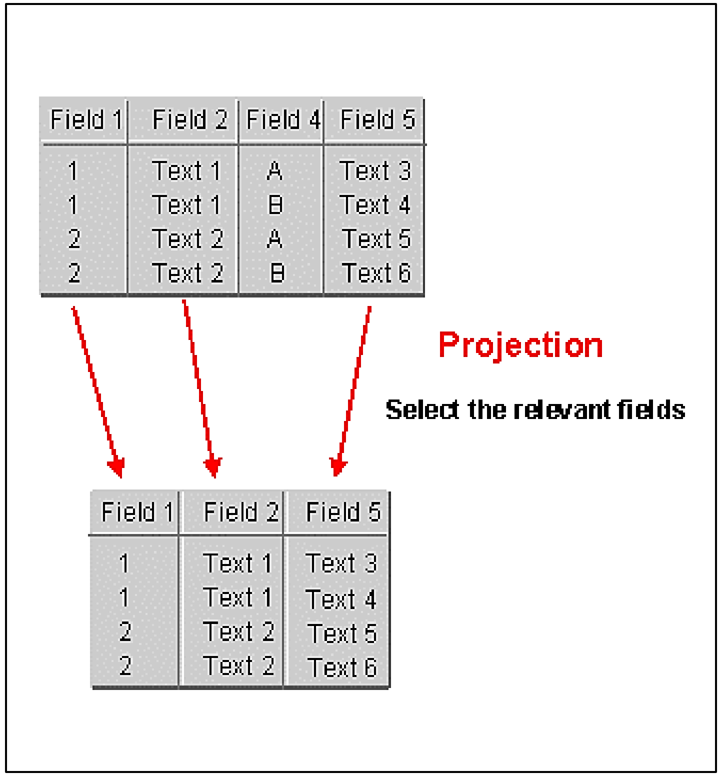
ID	Name	Age
1	Akon	17
2	Bkon	19
3	Ckon	15
4	Dkon	13

The output for query is also in form of a table(relation), with results in different columns

Projections are used to form new n-ary relations by deleting the same fields in every record of the relation.

The projection $P_{i_1 i_2 \ldots, i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the *n-tuple* $(a_1, a_2, \ldots, a_n)$ to the *m-tuple* $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

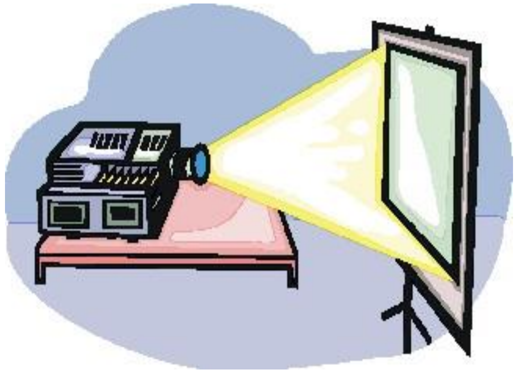
In other words, the projection $P_{i_1, i_2, \ldots, i_m}$ deletes $n - m$ of the components of an n-tuple, leaving the i_1th, i_2th, . . . , and i_mth components.



For instance:

- What results when the projection $P_{1,3}$ is applied to the 4-tuples $(2, 3, 0, 4)$, (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4)?

The projection $P_{1,3}$ sends these 4-tuples to $(2, 0)$, (Jane Doe, Geography), and (a_1, a_3), respectively.



What relation results when the projection $P_{1,4}$ is applied to the relation in Table 1?

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted.

TABLE 2 GPAs.

Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

What is the table obtained when the projection $P_{1,2}$ is applied to the relation in Table 3?

TABLE 3 Enrollments.

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

TABLE 4 Majors.

Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

The **join** operation is used to combine two tables into one when these tables share some identical fields.

TABLE 5 Teaching_assignments.		
Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.			
Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A M.
Zoology	335	A100	9:00 A M.
Zoology	412	A100	8:00 A M.

What relation results when the join operator J_2 is used to combine the relation displayed in Tables 5 and 6?

TABLE 7 Teaching_schedule.				
Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A M.
Cruz	Zoology	412	A100	8:00 A M.
Farber	Psychology	501	A100	3:00 P M.
Farber	Psychology	617	A110	11:00 A M.
Grammer	Physics	544	B505	4:00 P M.
Rosen	Computer Science	518	N521	2:00 P M.
Rosen	Mathematics	575	N502	3:00 P M.

SQL

```
SELECT Departure_time
FROM Flights
WHERE Destination='Detroit'
```

Output: 08:10, 08:47, 09:44

TABLE 8 Flights.

Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

```
SELECT Professor, Time
FROM Teaching_assignments, Class_schedule
WHERE Department='Mathematics'
```

Output: Rosen, 3:00 p.m.

TABLE 7 Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

TABLE 9 Part_needs.		
Supplier	*Part_number*	*Project*
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

TABLE 10 Parts_inventory.			
Part_number	*Project*	*Quantity*	*Color_code*
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

- What do you obtain when you apply the selection operator s_C, where C is the condition (Project = 2) ∧ (Quantity ≥ 50), to the database in Table 10?

- Construct the table obtained by applying the join operator J_2 to the relations in Tables 9 and 10.

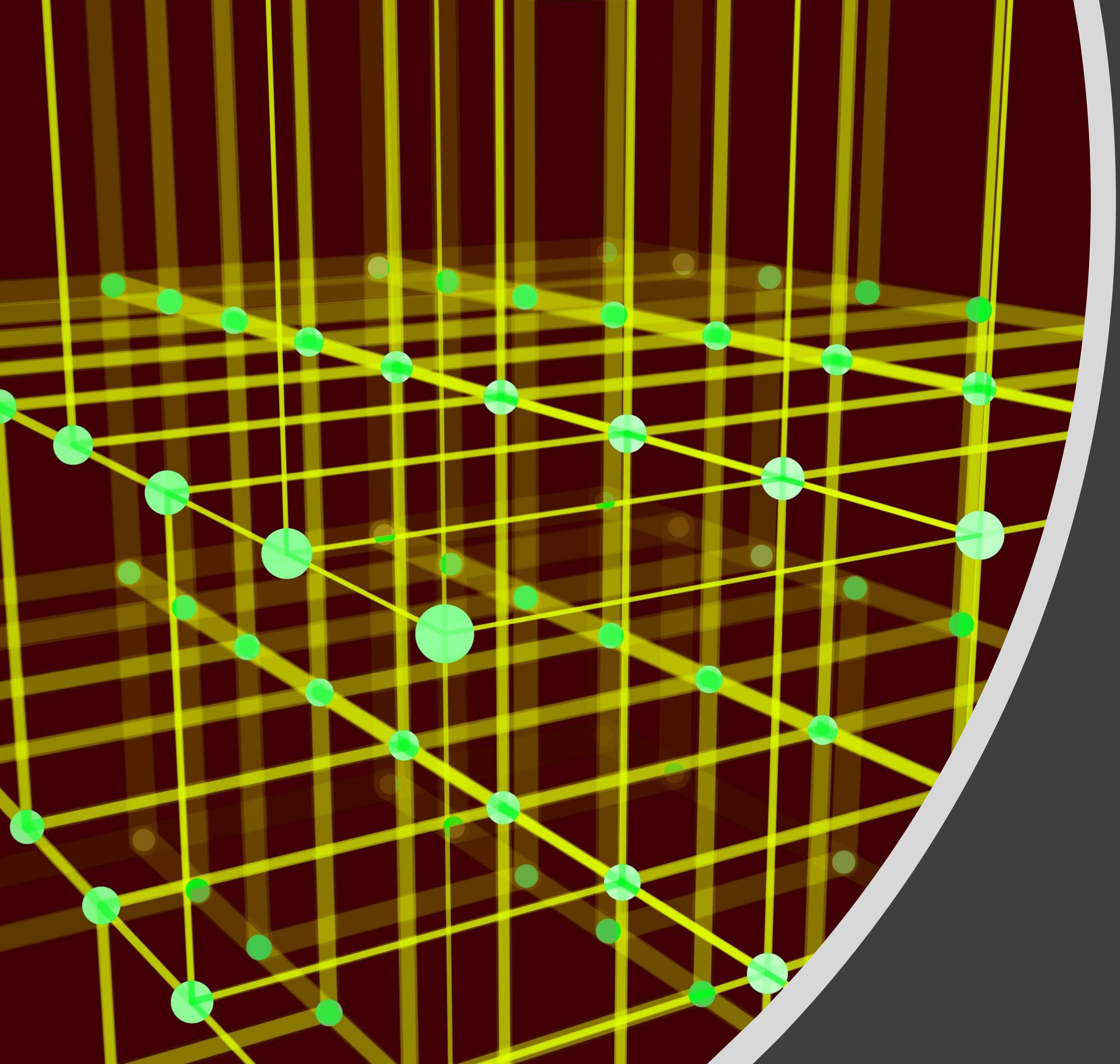
- What are the operations that correspond to the query expressed using this SQL statement?

a)

```
SELECT Supplier
FROM Part_needs
WHERE 1000 ≤ Part_number ≤ 5000
```

b)

```
SELECT Supplier, Project
FROM Part_needs, Parts_inventory
WHERE Quantity ≤ 10
```



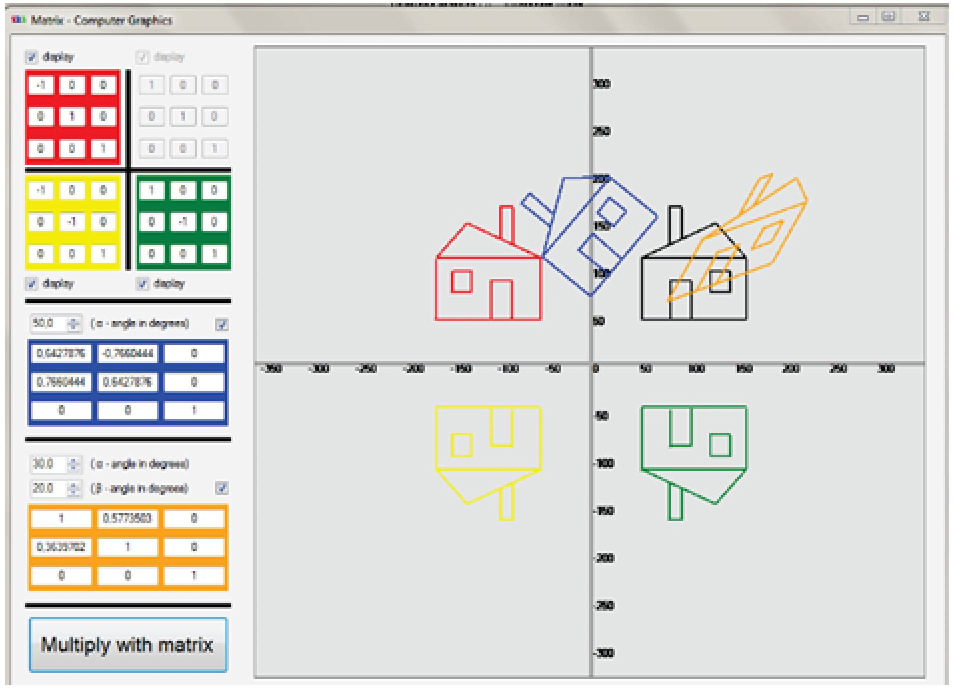
Representing Relations Using Matrices

$$m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R, \\ 0 \; if \; (a_i, b_j) \notin R. \end{cases}$$

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$



"Matrix - Computer Graphics" example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$

List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

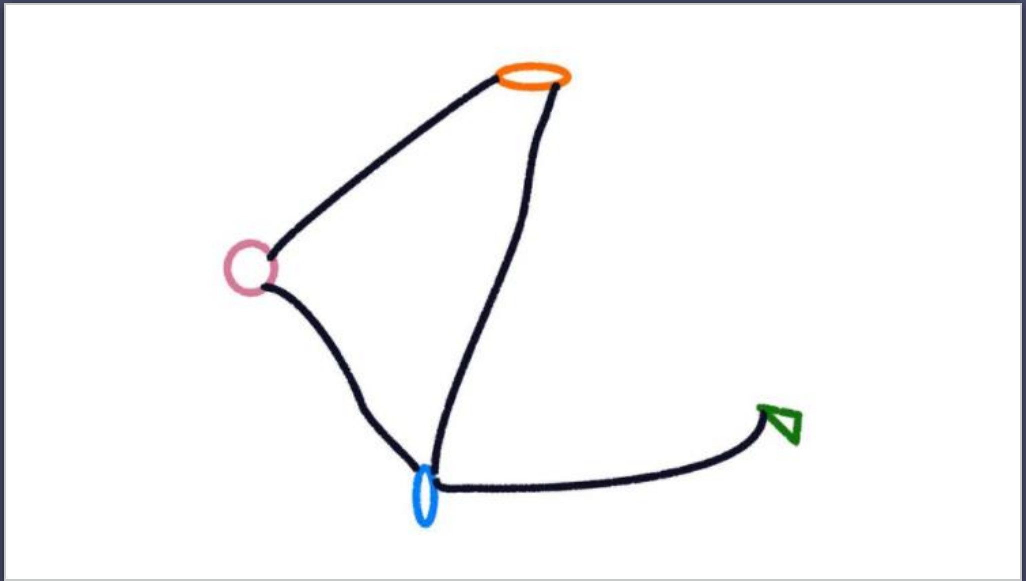
b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Determine their types.

Examples for matrices:

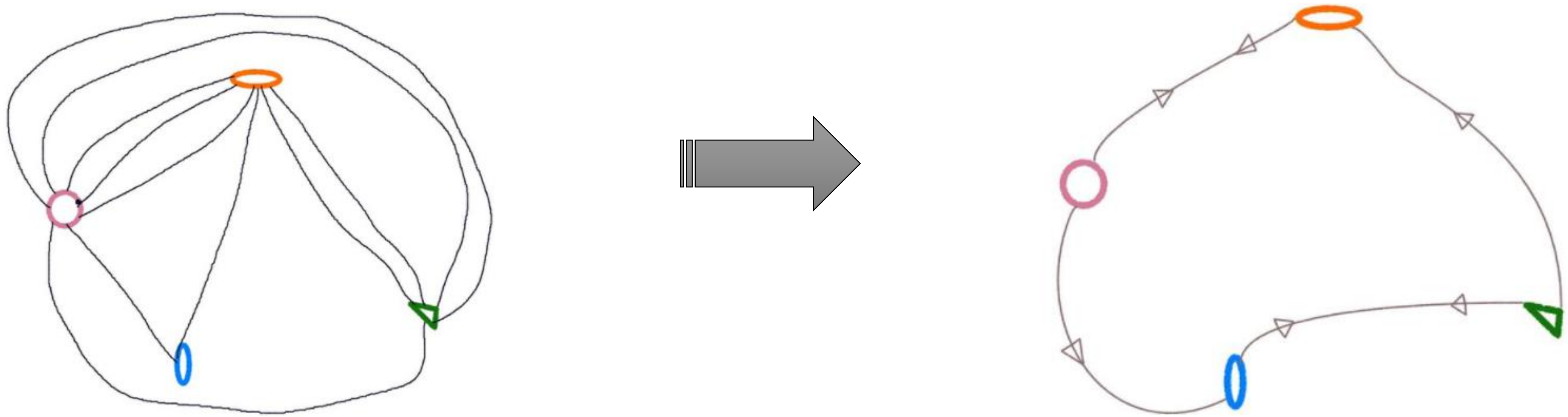
In true matrix style, we call the existence of a road link 1 and lack of a road link, a 0: Is there a link from Pink city to Blue city? yes. Pink to Pink? nonsensical, so 0. Pink to Green? 0. Green to Blue? 1. Giving us a matrix:



	P	B	O	G
P	0	1	1	0
B	1	0	1	1
O	1	1	0	0
G	0	1	0	0

We could also imagine that there are more than one roads going from one town to the other. For instance, see if you can write a matrix for this network:

The arrow from P to B means you can go to from P to B, but not from B to P. Two arrows on the same line mean the road is two-way:

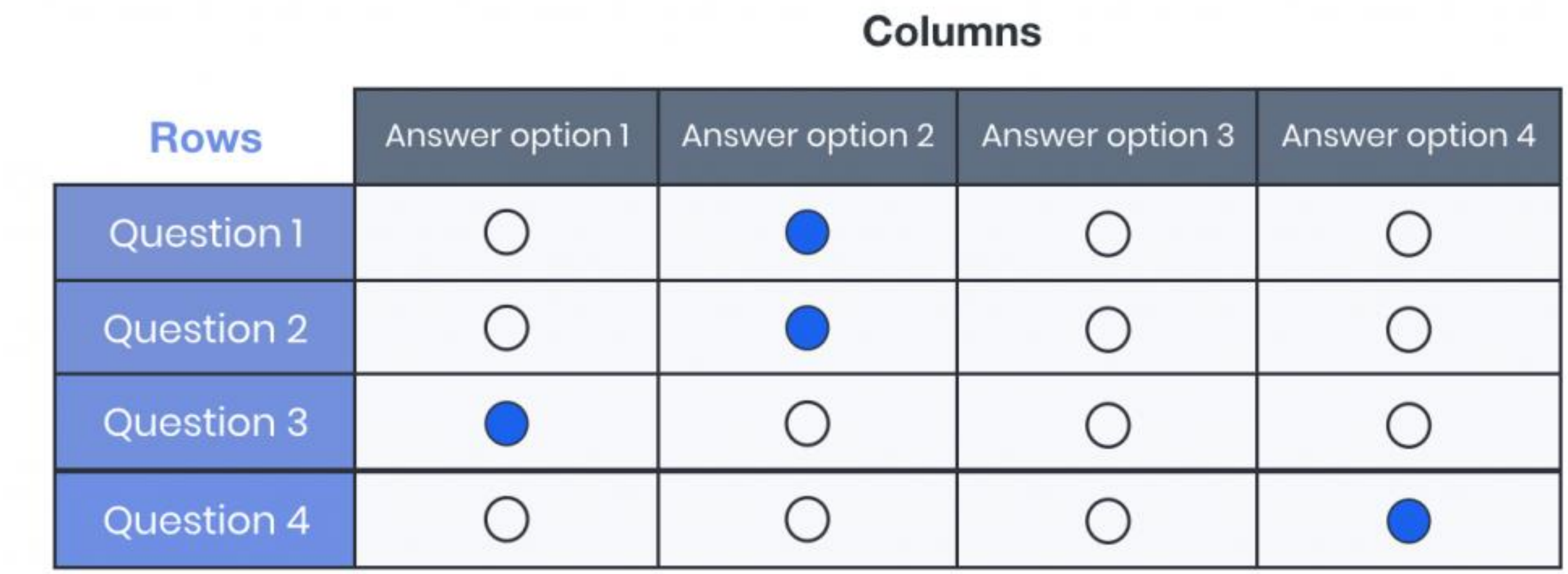


	P	B	O	G
P	0	1	1	0
B	1	0	0	1
O	1	0	0	0
G	0	1	1	0

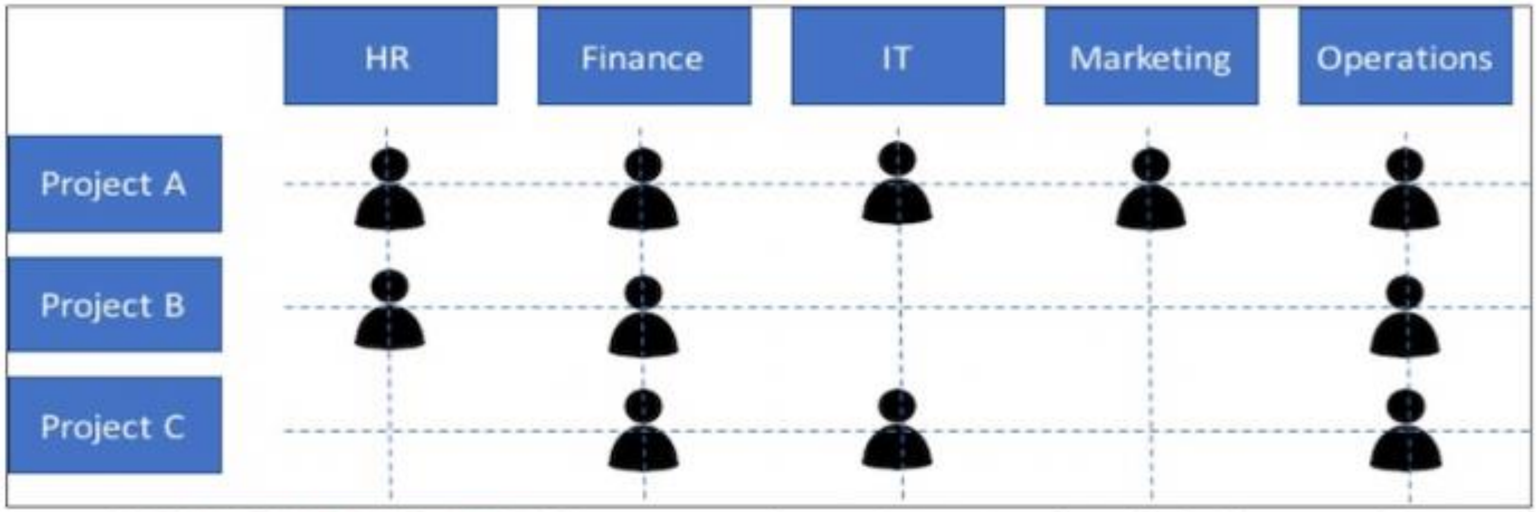
The **next important use** is that combining matrices can provide more information about the problem, than using each matrix alone. For instance, if we consider our first town routes representing train lines, (the golden-yellow matrix), and the city decided to add new bus lines (the sky-blue matrix) then combining these two we will have more information about how to move around these towns. Representing this new, more complex network as a matrix is as simple as multiplying the orange matrix to the blue one.

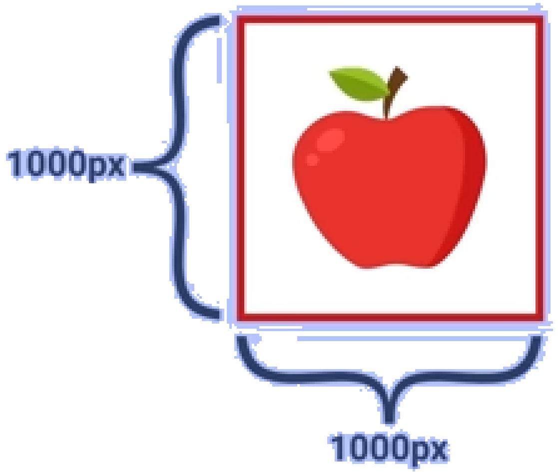
```
train = [0 1 1 0 ; 1 0 1 1 ; 1 1 0 0 ; 0 1 0 0]

train

>4×4 Array{Int64,2}:
 0  1  1  0
 1  0  1  1
 1  1  0  0
 0  1  0  0

bus = [0 1 1 0 ; 1 0 0 1 ; 1 0 0 0 ; 0 1 1 0]

bus

>4×4 Array{Int64,2}:
 0  1  1  0
 1  0  0  1
 1  0  0  0
 0  1  1  0

together = train * bus

together

>4×4 Array{Int64,2}:
 2  0  0  1
 1  2  2  0
 1  1  1  1
 1  0  0  1
```

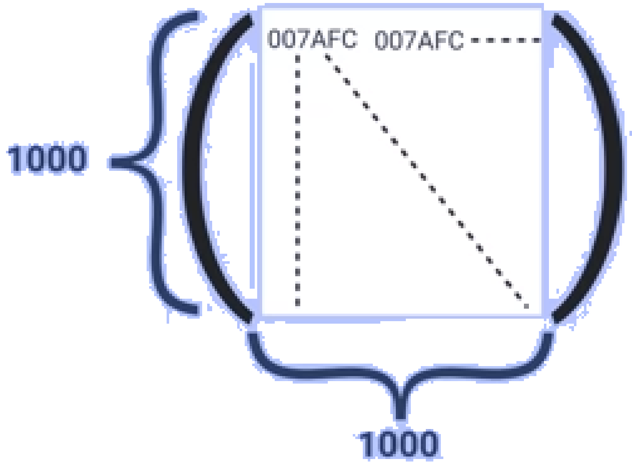
Another example of matrices for calculating scores of millions of students at once.



Matrix management structures in organizations





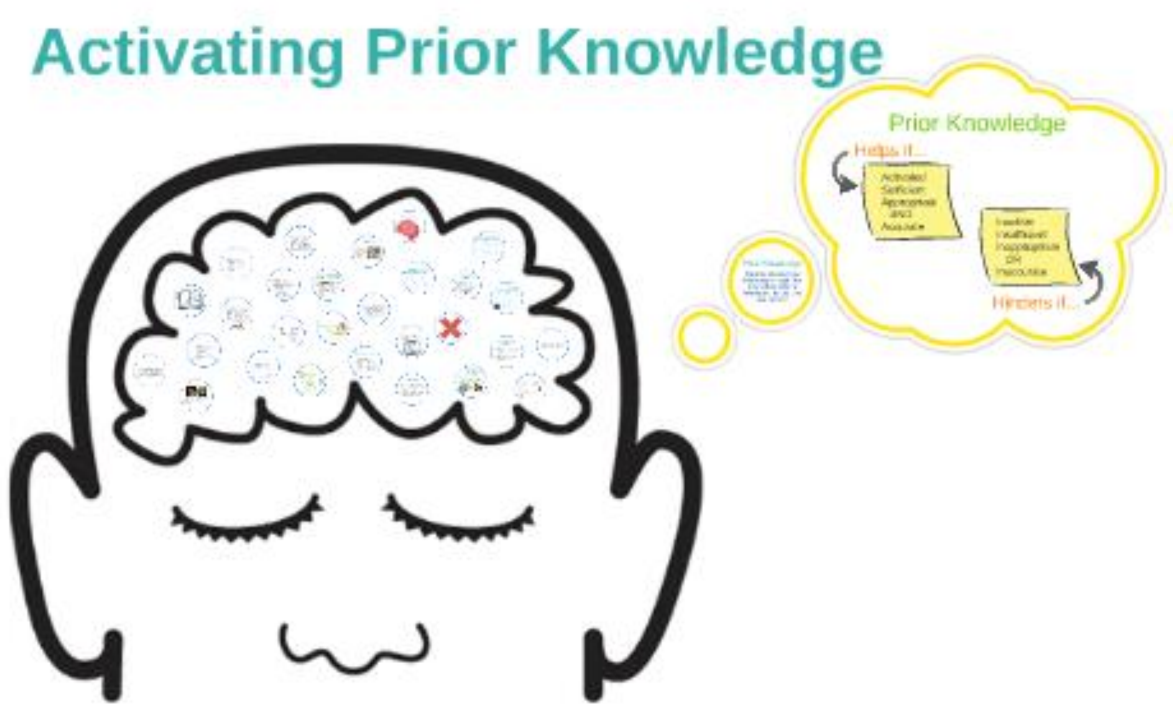


1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	0	0	1	1	1	1
1	1	1	0	0	0	1	1	1	1
1	1	1	1	0	0	1	1	1	1
1	1	1	1	0	0	1	1	1	1
1	1	1	1	0	0	1	1	1	1
1	1	1	0	0	0	0	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

Add and subtract matrices

$$\begin{bmatrix}8 & 5\\ 2 & \end{bmatrix} + \begin{bmatrix}9 & 5\\ & \end{bmatrix} = \begin{bmatrix}17 & 10\\ & \end{bmatrix}$$

$$\begin{bmatrix}8 & \\ 2 & \end{bmatrix}$$



...r multiplication

$$\ldots\Big] = \begin{bmatrix}2\cdot 10 & 2\cdot 6\\ 2\cdot 4 & 2\cdot 3\end{bmatrix}$$

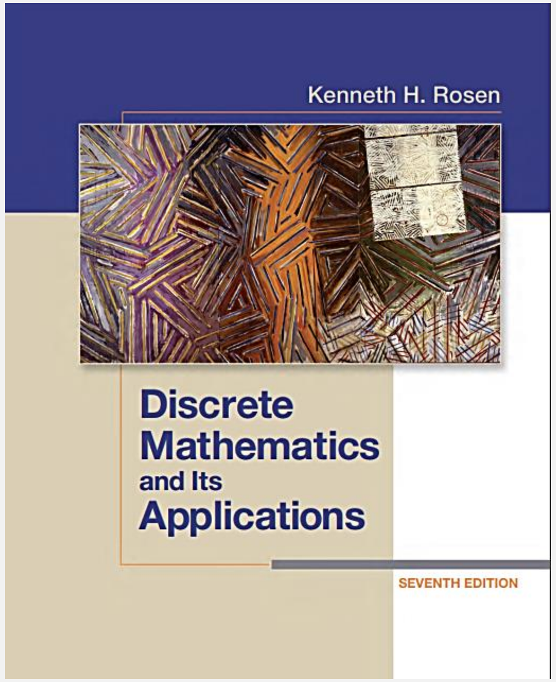
Matrix ...

$$\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix} \times \begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix} = \begin{bmatrix} & \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2\end{bmatrix}$$

Reference

Discrete Mathematics and Its Applications, 7th edition by Kenneth H. Rosen

Chapter 9: Relations, pages 573-597



- What is the binary relation type of a set: {(1,1), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2)}?

- Let R1 be a relation from A = {1, 3, 5, 7} to B = {2, 4, 6, 8} and R2 be another relation from B to C = {1, 2, 3, 4} as defined below:
  a) An element x in A is related to an element y in B (under R1) if x + y is divisible by 3.
  b) An element x in B is related to an element y in C (under R2) if x + y is even but not divisible by 3.

List ordered pairs of R1 and R2, find their types of relations.

- Let R_1, R_2 are relation defined on Z such that $aR_1b \Leftrightarrow (a-b)$ is divisible by 3 and $aR_2b \Leftrightarrow (a-b)$ is divisible by 4. Then what are: $(R_1 \cup R_2)$, $(R_1 \cap R_2)$ and $(R_2 - R_1)$?

- What is the Cartesian product of A = {1, 2} and B = {a, b}?

- What is the Cardinality of the set {0, 1, 2}?

- The Cartesian Product B x A is equal to the Cartesian product A x B.
  a) True
  b) False

- Which of the following two sets are equal?
  a) A = {1, 2} and B = {1}
  b) A = {1, 2} and B = {1, 2, 3}
  c) A = {1, 2, 3} and B = {2, 1, 3}
  d) A = {1, 2, 4} and B = {1, 2, 3}

- The members of the set S = {x | x is the square of an integer and x < 100} is ______________
  a) {0, 2, 4, 5, 9, 58, 49, 56, 99, 12}
  b) {0, 1, 4, 9, 16, 25, 36, 49, 64, 81}
  c) {1, 4, 9, 16, 25, 36, 64, 81, 85, 99}
  d) {0, 1, 4, 9, 16, 25, 36, 49, 64, 121}

$N = \{0, 1, 2, 3, \dots\}$ and $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$

- Let R be the relation on $N \times Z \times N \times Z$ consisting of 4-tuples (a, b, c, d) such that $(a + b \neq c + d) \wedge (a + b + c + d = 0)$. Give examples for this relation and find its degree.

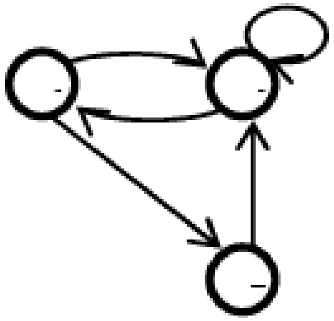
The relation has degree 4

$(5\ -11, 3, 3) \in R$

$(0, -1, 1, 0) \in R$

$(6, 6, 3, 9) \notin R$

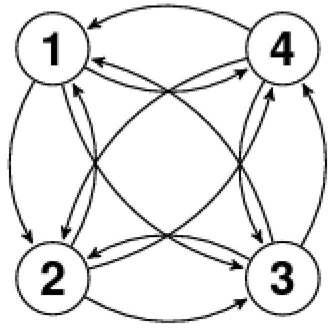
- Let A={1,4,5} and R be given by the digraph shown below. Find the relation determined by the following digraph:

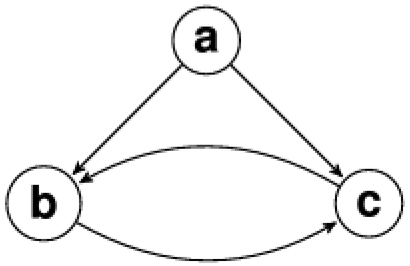


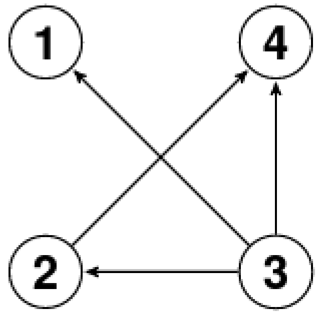
- $$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

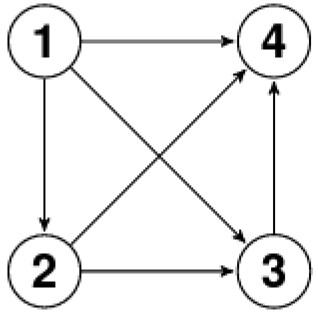
- Let A={1,2,3,4} and R={(1,1), (1,2),(2,1),(2,2),(2,3), (2,4),(3,4),(4,1)} is a relation on A. Draw the digraph of R.

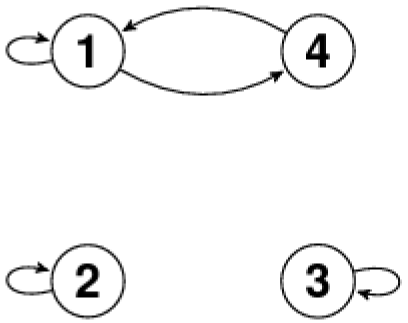
Determine types of all these relations.

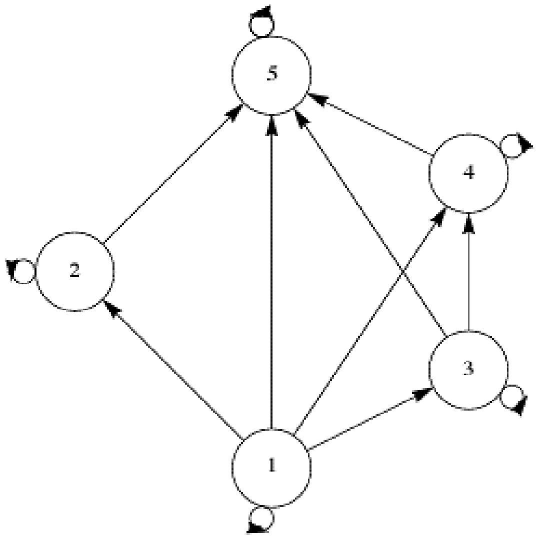
- 

- 

- 

- 

- 

- 

Determine their types.

- State whether each set of ordered pairs represents a function.

1) {(10, 9), (–2, –16), (–6, 7), (5, 8), (8, –16), (–11, 9)}

2) {(–7, 4), (–8, 3), (–7, 7), (–20, 8), (5, 9), (3, 1), (2, 6)}

3) {(–13, 4), (7, –15), (–13, 9), (6, –12), (–18, 0)}

4) {(15, –3), (–6, 9), (–3, 0), (–1, 16)}

5) {(–4, 3), (5, –9), (11, 4), (9, 6), (5, –3), (8, –9), (1, 4)}

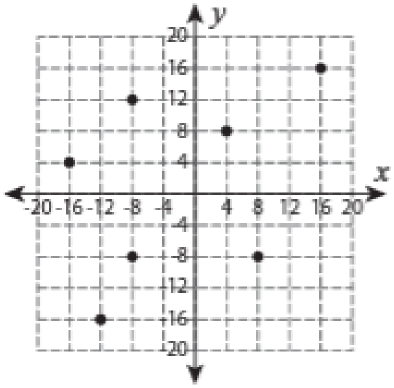
6) {(12, –18), (15, 1), (12, 5), (0, 9), (–5, –17)}

7) {(6, 0), (–12, –16), (–6, 10), (20, –7)}

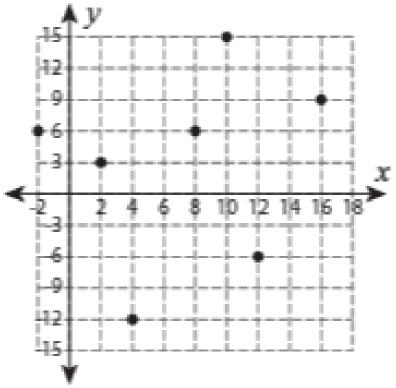
8) {(–2, –4), (–8, 3), (–7, –4), (–2, –8), (11, 8), (9, –4)}

- State whether each set of ordered pairs on the graph represents a function.

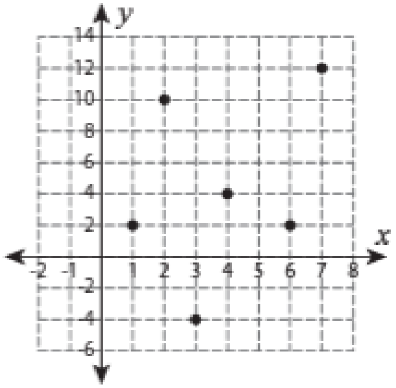
1)



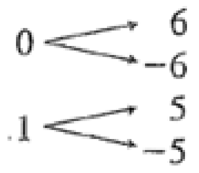
2)



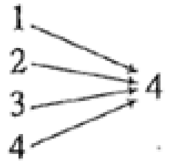
3)



4. Input Output



5. Input Output



6. Input Output