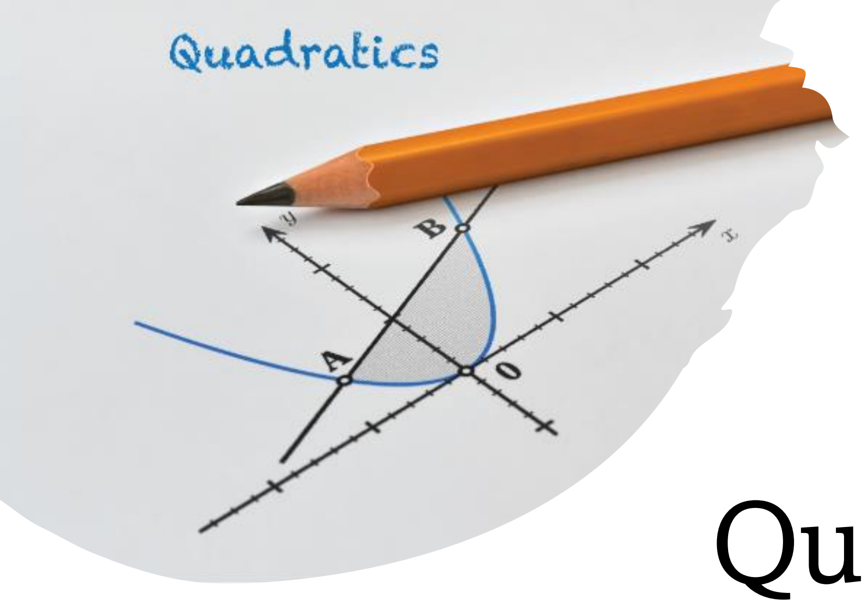


# Lecture 4:
# Quadratics and Circles

- Quadratics and Parabolas
- Methods of solving quadratic equations
  - Graphing
  - Completing the square
  - Quadratic Formula
- Circles

Ms. Togzhan Nurtayeva
Course Code: IT 161/A
Semester 1
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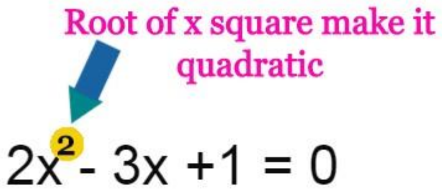
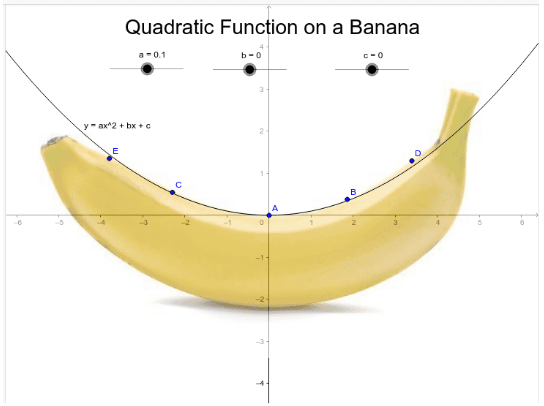
# Quadratics




- <u>Quadratic expressions</u>: A *quadratic expression* (*in* $x$) has general form:

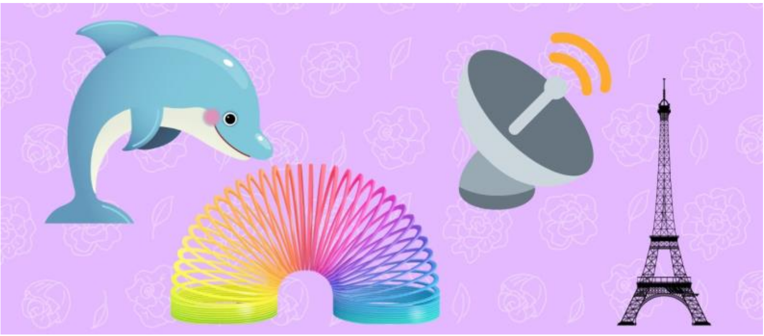


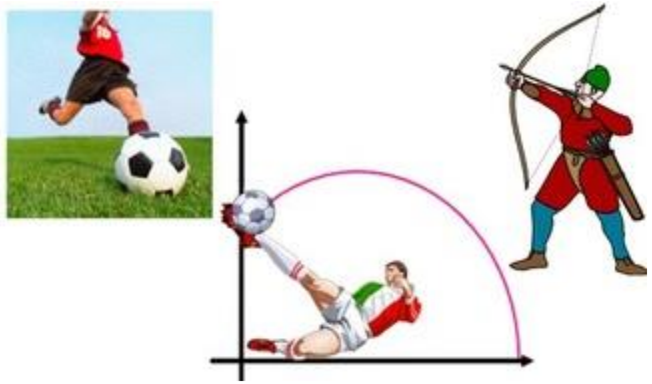

$$\boldsymbol{ax^2 + bx + c}$$

$$Ex. \quad 3x^2 - 16x + 36$$

$$Ex. \quad \frac{4}{7}x^2 - x + 5$$

... ... ... ... ... ...

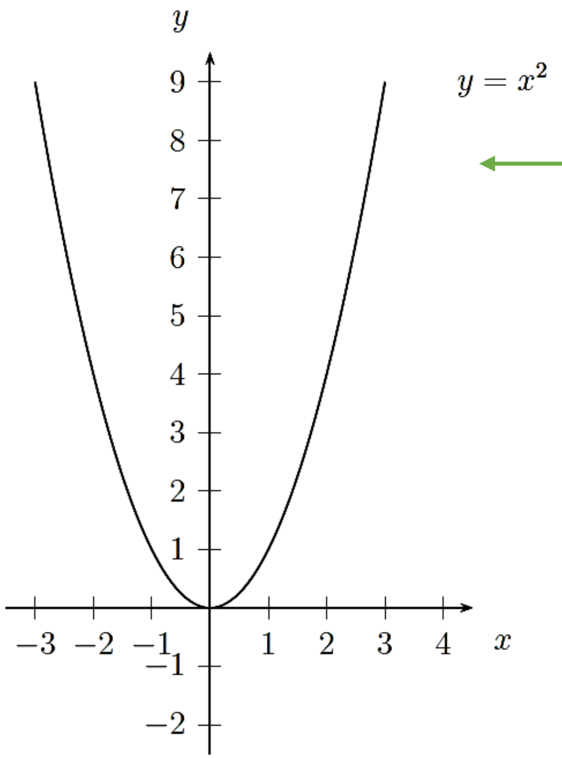




# Parabolas

- is a curve in the plane that is result of any combinations of (positive or negative) dilations, rotations or translations of the curve

$$\underline{y = x^2}$$




Parent function

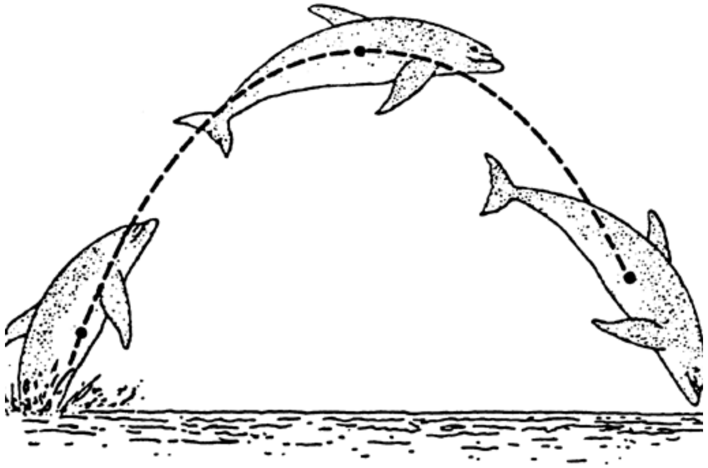
Facing upwards, a>0
Facing downwards, a<0

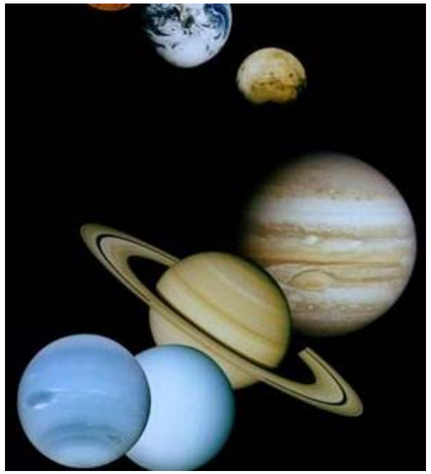
# Parabola

- Consider the satellite dish. These structures have a parabolic shape, allowing the reflection and focus of radio waves (parabolic shape helps receive and transmit signals).
- Parabolas in Spaceflight

# Quadratics and Parabolas in Real Life

- Calculating Room Areas
- Figuring a Profit (Business Profit)
- Quadratics in Athletics
- Finding a Speed
- River Cruise
- Resistors In Parallel (In Physics)
- For a parabolic mirror, a reflecting telescope or a satellite dish, the shape is defined by a quadratic equation.
- Quadratic equations are also needed when studying lenses and curved mirrors.
- And many questions involving time, distance and speed need quadratic equations.



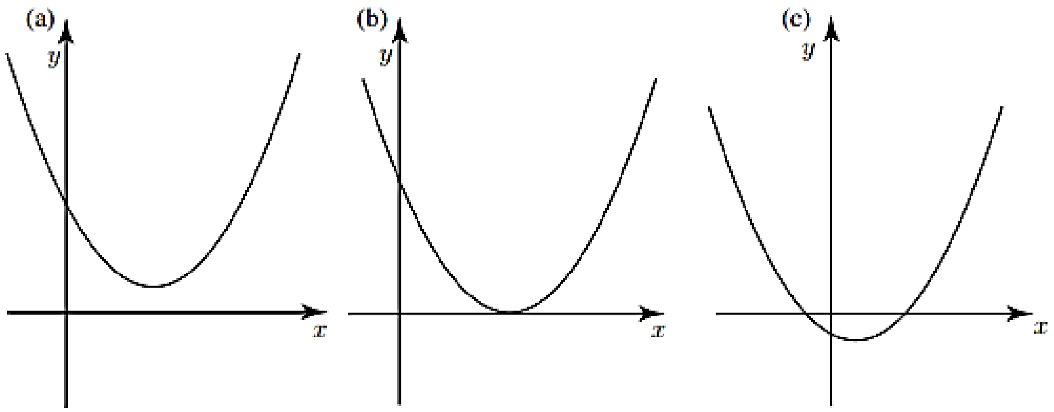
# Methods to solve quadratic equations

| Method | Can Be Used | Comments |
| --- | --- | --- |
| Graphing | Always | Not always exact; use only when an approximate solution is sufficient. |
| Factoring | Sometimes | Use if constant term is 0 or factors are easily determined. |
| Completing The Square | Always | Useful for equations of the form $x^2 + bx + c = 0$, where b is an even number. |
| Quadratic Formula | Always | Other methods may be easier to use in some cases but this method always gives accurate solutions. |

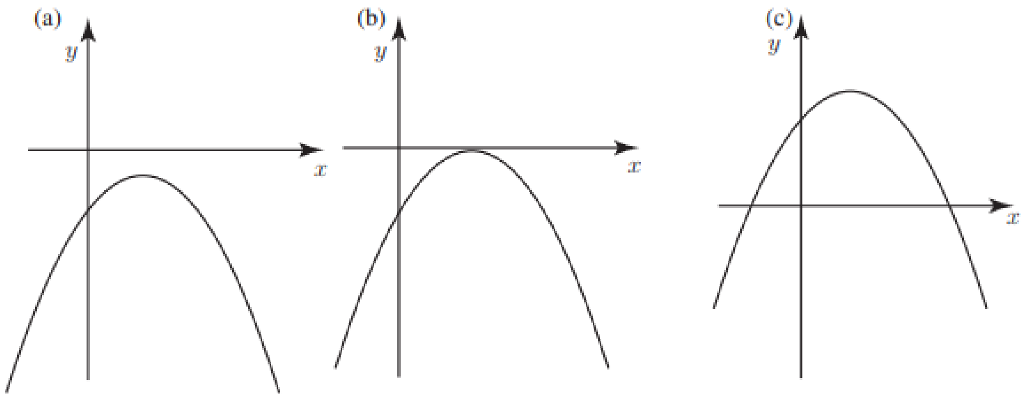
# Graphing

| Form | Vertex |
| --- | --- |
| $f(x) = a(x-h)^2 + k$ | $(h, k)$ |
| $f(x) = ax^2 + bx + c$ | $(-\frac{b}{2a}, f(-\frac{b}{2a}))$ |

## Graphs of $y = ax^2 + bx + c$ when $a$ is positive




## Graphs of $y = ax^2 + bx + c$ when $a$ is negative



# Vertex Form of Parabolas

$$y = a(x - h)^2 + k$$

**vertex:** $(h, k)$

$h$ = x-coordinate of vertex

$k$ = y-coordinate of vertex

$$f(x) = a(x-h)^2 + k$$

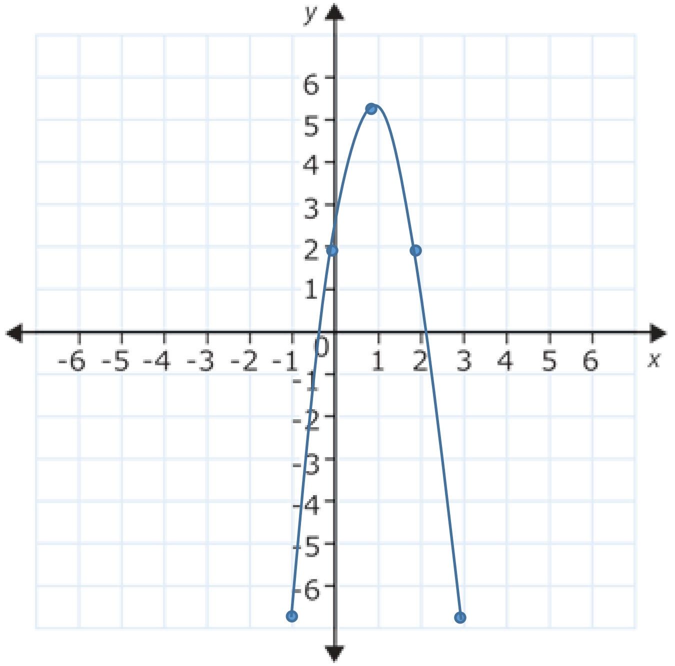
- $a \neq 0, if$ $a$ $is$ ...
  - $Positive: opens\ up$ ∪
  - $Negative: opens\ down$ ∩
- $Vertex$ $(h, k)$

$$y = -3(x-1)^2 + 5$$

- $a = -3 - opens\ down$
- $h = 1$ } $Vertex$
- $k = 5$
- So, $vertex\ is\ at\ point\ (1,5)$

$Vertex$

| $x$ | $y$ |
|---|---|
| -1 | -7 |
| 0 | 2 |
| 1 | 5 |
| 2 | 2 |
| 3 | -7 |



$$f(x) = ax^2 + bx + c$$

## Find the Vertex

$$f(x) = ax^2 + bx + c$$

$$x = \frac{-b}{2a}$$

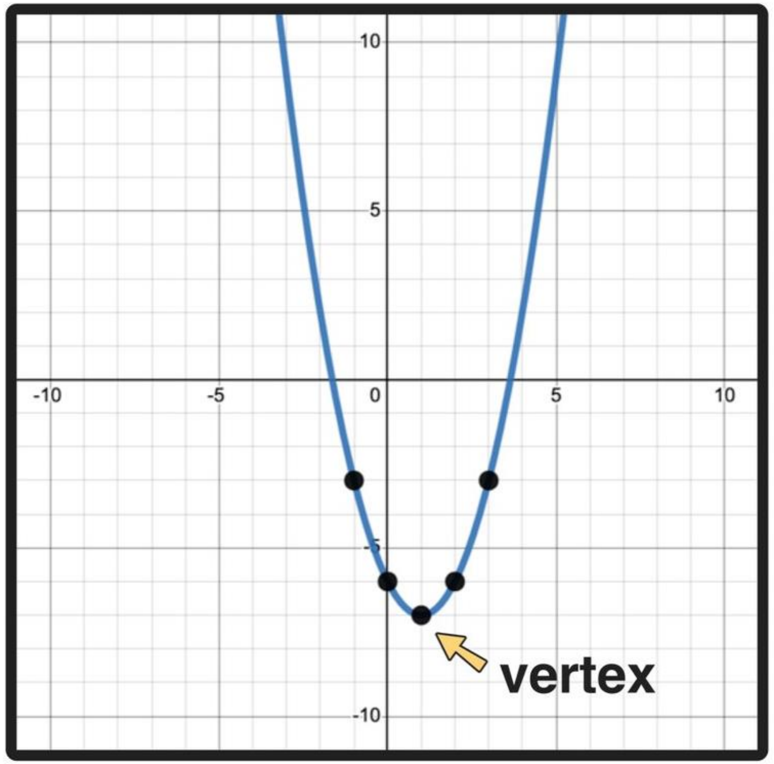
**(1,-7)**

## Build a Function Table

$$f(x) = x^2 - 2x - 6$$

| x | y |
|---|---|
| -1 | -3 |
| 0 | -6 |
| 1 | -7 |
| 2 | -6 |
| 3 | -3 |

## Plot Points and Graph



$$f(x) = ax^2 + bx + c$$

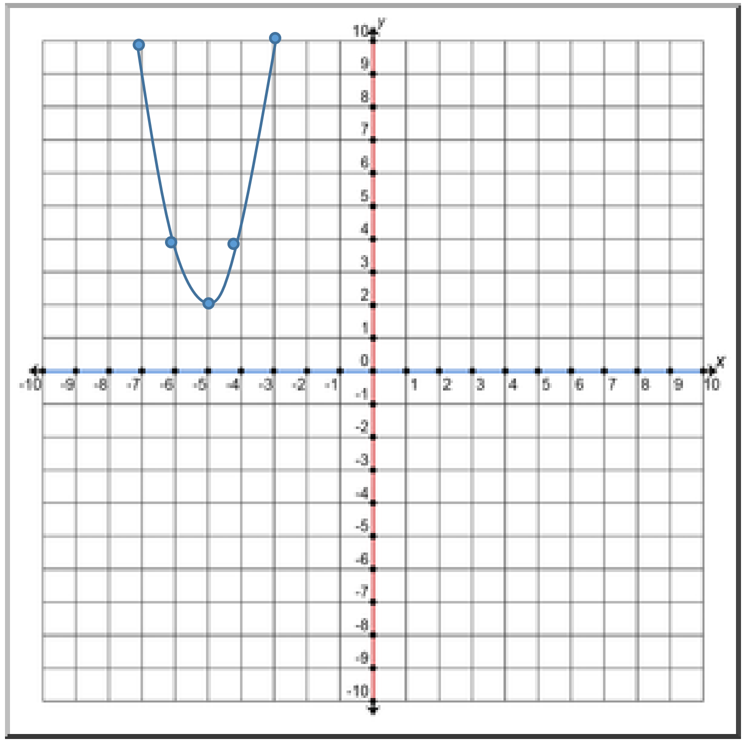
$Vertex: (-\frac{b}{2a}, f(-\frac{b}{2a}))$

$$y = 2x^2 + 20x + 52$$

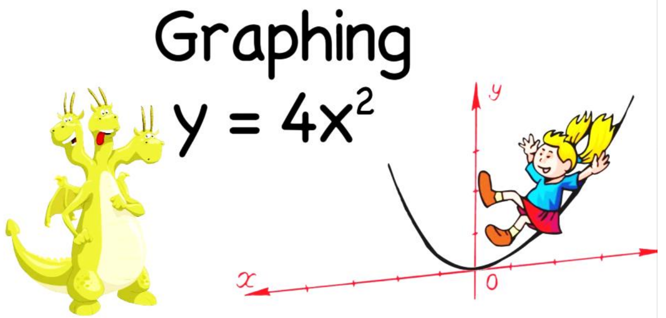
- $-\frac{b}{2a} = -\frac{20}{4} = -5$
- $f\left(-\frac{b}{2a}\right) = (2 \times (-5)^2) + 20 \times (-5) + 52 = 2$

*Vertex*

| $x$ | $y$ |
|---|---|
| -7 | 10 |
| -6 | 4 |
| -5 | 2 |
| -4 | 4 |
| -3 | 10 |



# Problems




(i) $y = \frac{1}{2}x^2$

(ii) $y = -3x^2$

(iii) $y = x^2 + 1$

(iv) $y = (x + 1)^2$

(v) $y = -(x - 2)^2$

(vi) $y = (x - 1)^2 - 2$

(i) $y = x^2 + x - 2$

(ii) $y = 2x^2 - 4x + 7$

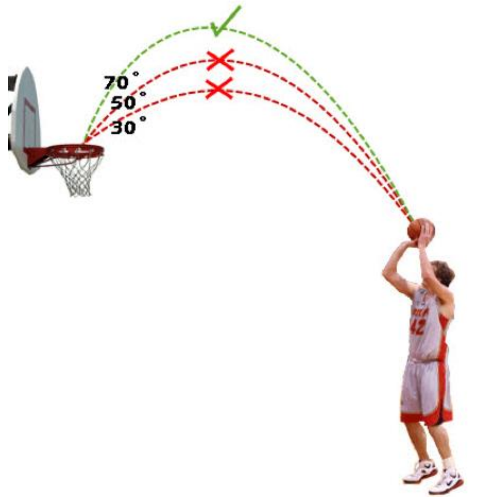
$y = x^2 + 12x + 38$

$y = 2x^2 - 16x + 30$

$y = 2x^2 + 6x$

$y = -x^2 - 10x - 26$

$y = -\frac{1}{2}x^2 + 4x - 6$



# Graphing Parabola Practice

5) $f(x) = -2x^2 + 4x$

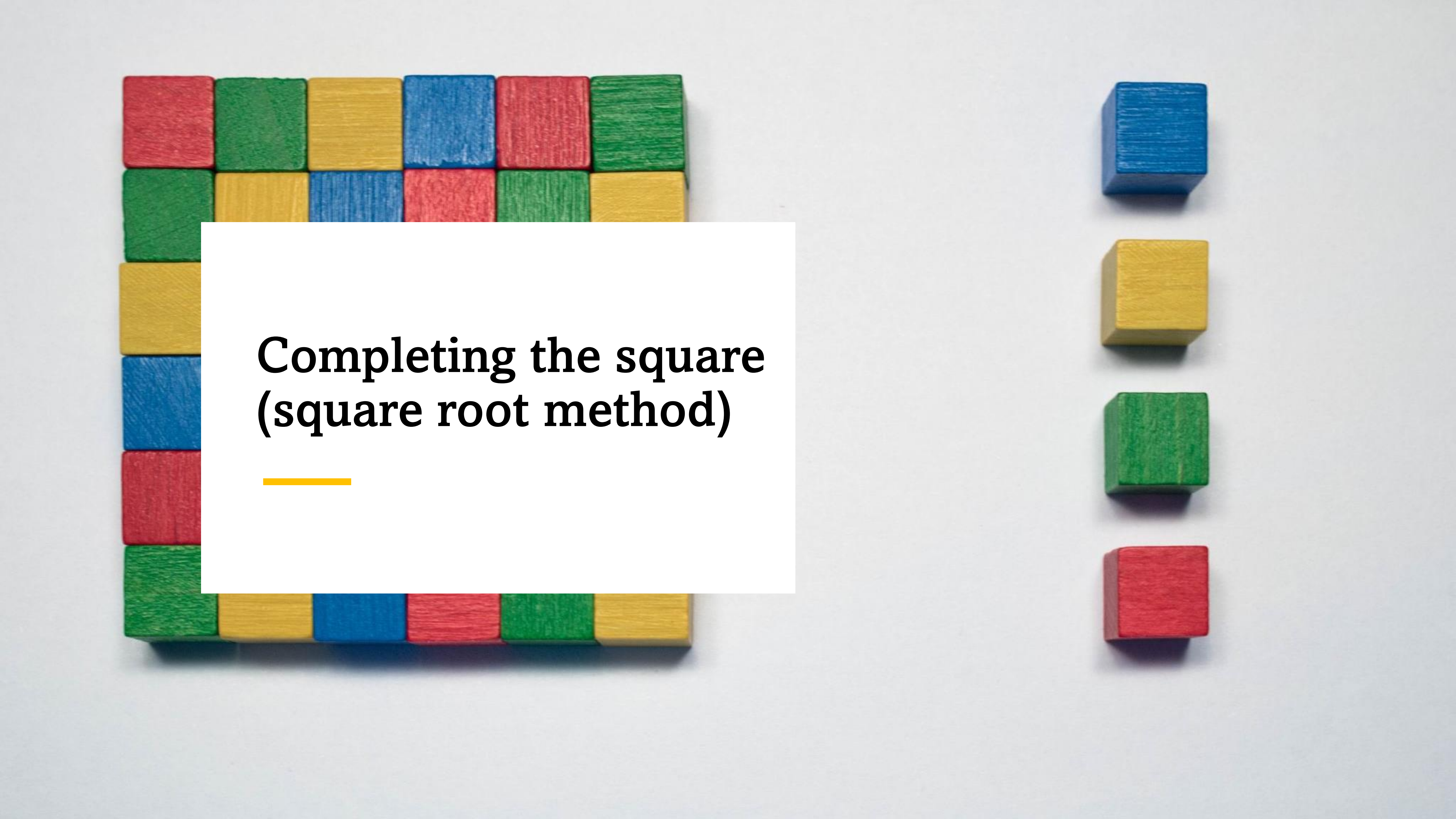
6) $f(x) = 2x^2 + 4x + 1$

7) $f(x) = -x^2 + 4x - 2$

8) $f(x) = 2x^2 - 4x - 2$

9) $f(x) = -2x^2 - 8x - 5$

10) $f(x) = x^2 - 2x + 5$



# Completing the square (square root method)

$x^2 + bx + c$

# Completing the Square

## Solve Quadratics

1. If a ≠ 1, divide the quadratic by a.

2. Write the quadratic in the form

$$x^2 + bx = c$$

3. Add $(b/2)^2$ to both sides of the equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

4. Factor the left side of the equation into a perfect square.

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

5. Square root both sides of the equation and solve for x.

$$x + \frac{b}{2} = \pm\sqrt{c + \left(\frac{b}{2}\right)^2}$$

$ax^2 + bx + c$

1. Factor $a$ (the coefficient of $x^2$) out of the first two terms only:

$$a\left(x^2 + \frac{b}{a}x\right) + c$$

2. In your head, divide the coefficient of $x$ by 2, square it, and whatever you get, add and subtract that inside the brackets (since you're adding AND then subtracting, nothing changes!)

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

3. Bring the new term you're subtracting outside of the brackets, but remember that it gets multiplied by $a$ when you do so.

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$

4. The terms inside the brackets are a perfect square!

$$a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2$$

## Step 1

### Rearrange if Necessary

*move the constants to one side

$$x^2 - 6x + 9 = 25$$
$$\quad\quad -9 \quad -9$$
$$x^2 - 6x = 16$$

## Step 2

### $+\left(b/2\right)^2$ to both sides

$$\left(\frac{-6}{2}\right)^2 \rightarrow (-3)^2 = 9$$

$$x^2 - 6x + 9 = 16 + 9$$
$$x^2 - 6x + 9 = 25$$

## Step 3

### Factor & Solve

$$x^2 - 6x + 9 = 25$$
$$(x-3)^2 = 25$$
$$\sqrt{(x-3)^2} = \sqrt{25}$$
$$x - 3 = \pm 5$$
$$x = 3 \pm 5$$

$x = 8$
$x = -2$

$$2x^2 + 7x + 6 = 0$$

$$2x^2 + 7x + 6 = 0 \quad | \div 2$$

$$x^2 + \frac{7}{2}x + 3 = 0$$

$$\left(x + \frac{7}{4}\right)^2 - \frac{1}{16} = 0$$

$$\left(x + \frac{7}{4}\right)^2 = \frac{1}{16}$$

$$x + \frac{7}{4} = \frac{1}{4} \quad \text{or} \quad x + \frac{7}{4} = -\frac{1}{4}$$

$$x = -\frac{3}{2} \quad \text{or} \quad x = -2.$$

# Example

- $3x^2 + x - 10 = 0$ – completing square $\quad a \neq 1,\quad \left(\frac{b}{2a}\right)^2 = \left(\frac{1}{2\times 3}\right)^2 = \left(\frac{1}{6}\right)^2 = \boxed{\frac{1}{36}}$

$\boldsymbol{Step\ 1.}\quad 3\left(x^2 + \frac{x}{3}\right) - 10$

$$3\left(x^2 + \frac{x}{3} + \frac{1}{36} - \frac{1}{36}\right) - 10$$

$$3(x + \frac{1}{6})^2 - 3 \times \frac{1}{36} - 10$$

$$3(x + \frac{1}{6})^2 - \frac{1}{12} - 10$$

$$3(x + \frac{1}{6})^2 - 10\frac{1}{12}$$

$\boldsymbol{Step\ 2.}$

$$3(x + \frac{1}{6})^2 - 10\frac{1}{12} = 0$$

$$3(x + \frac{1}{6})^2 = 10\frac{1}{12}$$

$$10\frac{1}{12} = \frac{121}{12}$$

$$3(x + \frac{1}{6})^2 = \frac{121}{12}$$

$$(x + \frac{1}{6})^2 = \frac{121}{36}$$

$$\sqrt{(x + \frac{1}{6})^2} = \sqrt{\frac{121}{36}}$$

$$x + \frac{1}{6} = \pm\frac{11}{6}$$

$$x_1 = -2$$

$$x_2 = \frac{5}{3}$$

# Example

- $y = x^2 + 4x + 4$ - *graph*

$Completing\ square\ method:$ $a = 1,$ $(\frac{b}{2})^2 = (\frac{4}{2})^2 = 2^2 = 4$

$perfect\ square$

$x^2 + 4x + 4 = x^2 + 4x + 4 + 4 - 4 = (x + 2)^2$

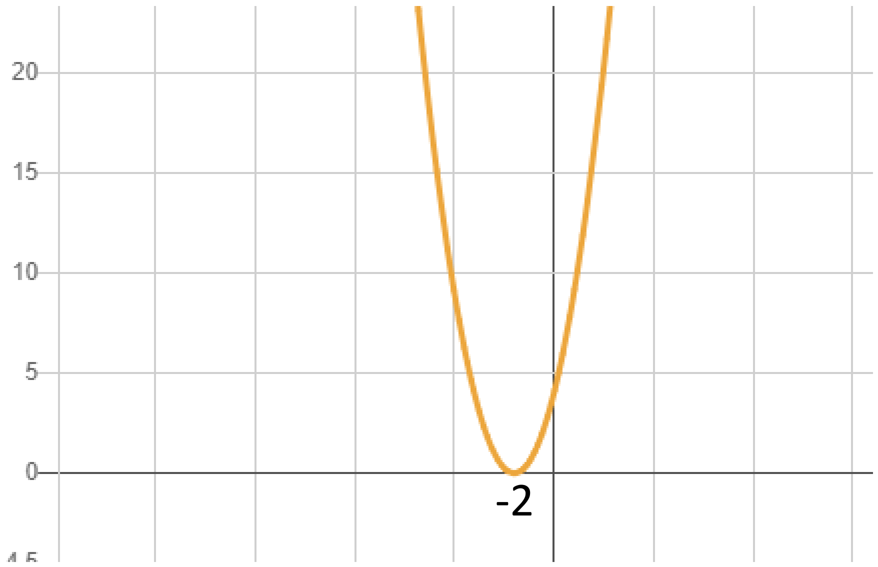
$y = (x + 2)^2$

↓

$-2\ units\ left\ by\ x - axis$

$(2\ values\ for\ line)\ but$
$5\ values\ for\ parabola:$

| $x$ | $y$ |
|---|---|
| 0 | 4 |
| -1 | 1 |
| -2 | 0 |
| -3 | 1 |
| -4 | 4 |



# Solving Quadratic Equations by Completing the Square

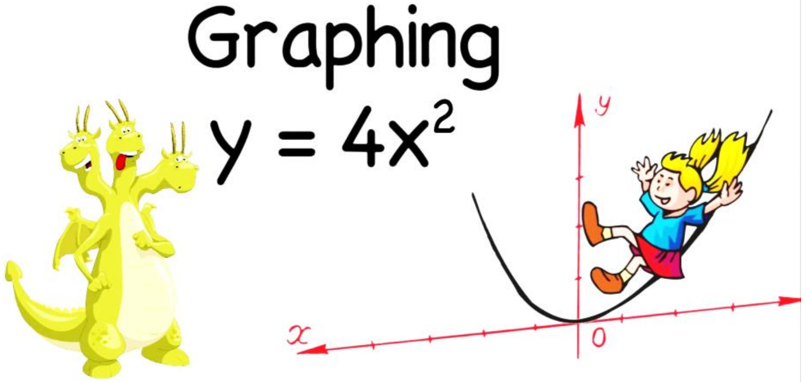
Try the following examples. Do your work on your paper and then check your answers.

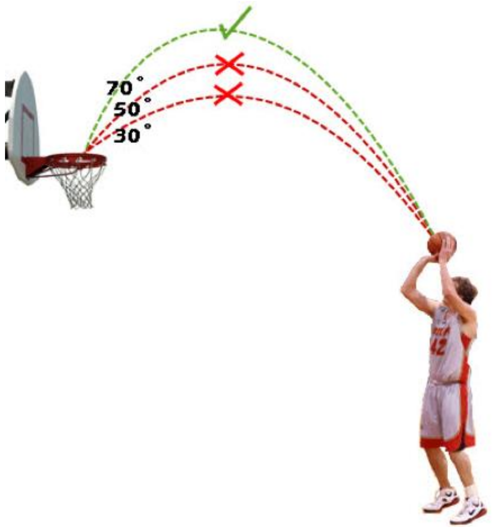
1. $x^2 + 2x - 63 = 0$
2. $x^2 + 8x - 84 = 0$
3. $x^2 - 5x - 24 = 0$
4. $x^2 + 7x + 13 = 0$
5. $3x^2 + 5x + 6 = 0$

1. $(-9, 7)$
2. $(6, -14)$
3. $(-3, 8)$
4. $\left(\frac{-7 \pm i\sqrt{3}}{2}\right)$
5. $\left(\frac{-5 \pm i\sqrt{47}}{6}\right)$

# Example

- $y = x^2 + 2$
- $y = x^2 + 4x + 4$
- $y = 2x - x^2$
- $2x^2 + 3x = 0$
- $x^2 + 2x + 8 = 0$
- $3x^2 + x - 10 = 0$



a) Show that $x^2 + 2x = (x+1)^2 - 1$.
Hence, use completing the square to solve $x^2 + 2x - 3 = 0$.

b) Show that $x^2 - 6x = (x-3)^2 - 9$.
Hence use completing the square to solve $x^2 - 6x = 5$.

c) Use completing the square to solve $x^2 - 5x + 1 = 0$.

d) Use completing the square to solve $x^2 + 8x + 4 = 0$.


Write in completed square form:

1. $x^2 + 6x$
2. $x^2 + 8x$
3. $x^2 - 8x$
4. $x^2 - 9x$
5. $x^2 - 3x$
6. $x^2 - 3x - 1$

9. $2x^2 + 6x + 2$
10. $2x^2 + 5x + 2$
11. $2x^2 + x + 2$
12. $3x^2 + x + 2$
13. $3x^2 + x - 10$
14. $4x^2 + x - 10$

# THE QUADRATIC FORMULA



If $ax^2 + bx + c = 0$ but $a \neq 0$

then $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

DISCRIMINANT

- $b^2 - 4ac > 0$ two real solutions
- $b^2 - 4ac = 0$ one real solutions
- $b^2 - 4ac < 0$ zero real solutions

# Quadratic Formula


$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $16x^2 - 56x + 49 = 0$

- $\frac{2}{x+1} - \frac{2}{x+2} = 7$

# Example

- $\frac{2}{x+1} - \frac{2}{x+2} = 7$ ⟹ $In\ order\ to\ solve\ this\ type\ of\ questions, we\ have\ to\ get\ the\ form\ of\ quadratic\ equation: ax^2 + bx + c$

$check\ for\ the\ denominator, should\ be\ the\ same$

$$\frac{2\times(x+2)}{(x+1)(x+2)} - \frac{2\times(x+1)}{(x+1)(x+2)} = 7$$

$$\frac{2x+4-2x-2}{(x+1)(x+2)} = 7$$

$Denominator\ cannot\ be\ zero, so\ \ x \neq -1, \qquad x \neq -2$

$$2 = 7(x+1)(x+2)$$

$$2 = 7(x^2 + 3x + 2)$$

$$7x^2 + 21x + 14 = 2$$

$$7x^2 + 21x + 12 = 0$$

$a = 7$ $\quad$ $\mathrm{b} = 21$ $\quad$ $\mathrm{c} = 12$

$Quadratic\ Formula:$

$$x = \frac{-21 \pm \sqrt{21^2 - 4\times 7\times 12}}{2\times 7}$$

$$x = \frac{-21 \pm \sqrt{105}}{14}$$

Write each quadratic equation in standard form then solve them.

1.) $3m^2 + 2m = 7$

2.) $3x^2 - 6 = -2x$

3.) $x^2 = -3x + 2$

4.) $(x + 2)(x + 9) = 0$

5.) $(x + 7)(x - 3) = 0$

6.) $(x - 8)(x - 1) = -7$

7.) $x^2 = 81$

8.) $4m^2 + 3m = 1$

9.) $4x^2 - 9 = -3x$

10.) $x^2 = -4x + 9$

11.) $2m^2 + 3m = 5$

12.) $7x^2 - 1 = -3x$

13.) $x^2 = -2x + 1$

14.) $(x + 1)(x + 3) = 0$

15.) $(x + 10)(x - 1) = 0$

16.) $(x - 11)(x - 2) = -4$

17.) $x^2 = 49$

18.) $3m^2 + 7m = 4$

19.) $4x^2 - 1 = -7x$

20.) $x^2 = -3x + 10$

Use the quadratic formula to solve the following quadratic equations.

a) $x^2 - 3x + 2 = 0$ b) $4x^2 - 11x + 6 = 0$ c) $x^2 - 5x - 2 = 0$ d) $3x^2 + 12x + 2 = 0$

e) $2x^2 = 3x + 1$ f) $x^2 + 3 = 2x$ g) $x^2 + 4x = 10$ h) $25x^2 = 40x - 16$

## *Try this!*

- ✓ A right triangle has one leg that is 3 m shorter than the other leg. The triangle has area of 54 $m^2$. Find the lengths of the legs.

- ✓ The hypotenuse of a right triangle is 3 in. longer than the longer leg. The shorter leg is 3 in. shorter than the longer leg. Find the lengths of the sides of the triangle.

- ✓ The hypotenuse of right-angled triangle is 2 more than twice of one of the other side while the third side is 13 more than half of the hypotenuse. Find the length of the median to the hypotenuse.

- ✓ The difference between the squares of the two numbers is 72. Eight times the numerically smaller number is 1 more than 5 times the other number. Find the numerically greater number.


a) $2x^2 - 11x + 14 = 0$

b) $x^2 + 2x - 63 = 0$

c) $x^2 - 16x + 48 = 0$

d) $x^2 + 7x - 1 = 0$

e) $2x^2 + 7x - 9 = 0$

f) $7 - 5x^2 + 2x = 0$

g) $x^2 + 3x + 7 = 0$

h) $4x^2 + 4x + 1 = 0$

i) $x^2 - 7x - 30 = 0$

j) $x^2 + 4x - 96 = 0$

k) $x^2 + 0.9x - 0.36 = 0$

l) $\frac{6}{x-1} + \frac{5}{x+1} = \frac{6}{x-2}$

m) $\frac{x-1}{x-2} + \frac{x-2}{x-1} = \frac{5}{2}$

n) $2x^2 + 9x = 0$

o) $3x^2 = 6x$

p) $12x = -8x^2$

q) $4x^2 - 64 = 0$

r) $16 - 7x^2 = 79$

s) $-3x^2 + 243 = 0$

t) $2(4x-1)(x+1) = (4x+1)(x-1) - 7$

u) $(6-x)(2x-5) + 30 = 0$

v) $(x+1)(x+2) = (2x-1)(2x-10)$

w) $(3x-2) - (x-3)^2 + 11 = 0$

x) $(x+7)(x-9) + (x-7)(x+9) + 76 = 0$

y) $\frac{x+3}{x-3} + \frac{x-6}{x+6} = \frac{11}{5}$

z) $\frac{25+x}{9+x} = \frac{13+x}{47-x}$

# Problems to practice (try to use all methods)

**Solve the following equations.**

16. $(x-4)^2-9=0$
17. $(x-10)^2-48=0$
18. $x^2+14x+45=0$
19. $x^2+6x-10=30$
20. $4x^2-100=0$
21. $6x^2-48x-54=0$
22. $4x^2+2x=12$
23. $9x^2+7x-4=0$
24. $3x^2+9x-6=0$

**Write Linear if the given equation is a Linear equation or Quadratic and solve them.**

1.) $2m + 9 = 15$

2.) $x^2 = 5x + 2$

3.) $(x + 3)\ (x + 2) = 0$

4.) $c = 2x + 1$

5.) $6 - 3x + 2x^2 = 0$

6.) $36 - r^2 = 4r$

7.) $3m + 4 = 5$

8.) $x^2 = -3x + 1$

9.) $(x + 2)(x + 1) = 0$

10.) $a = 5x + 3$

11.) $4x(x - 3) = 8$

12.) $5y + 2 = -3$

13.) $(x + 2)^2 = 1$

14.) $(w - 7)(w + 5) = 0$

15.) $2x + 3y = 4$

16.) $m^2 = 9$

17.) $5x(x - 6) = 1$

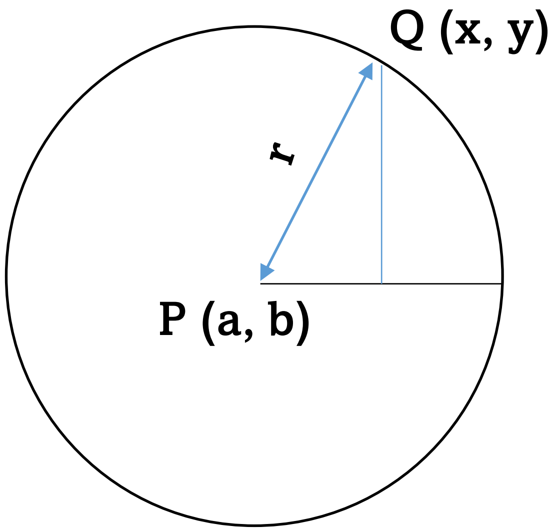
18.) $2y + 7 = -4$

19.) $(x + 5)^2 = 7$

20.) $(w - 1)(w + 2) = 0$

# Circle

- r - radius




By Pythagoras:

$$r^2 = (x-a)^2 + (y-b)^2$$

Equation of the circle

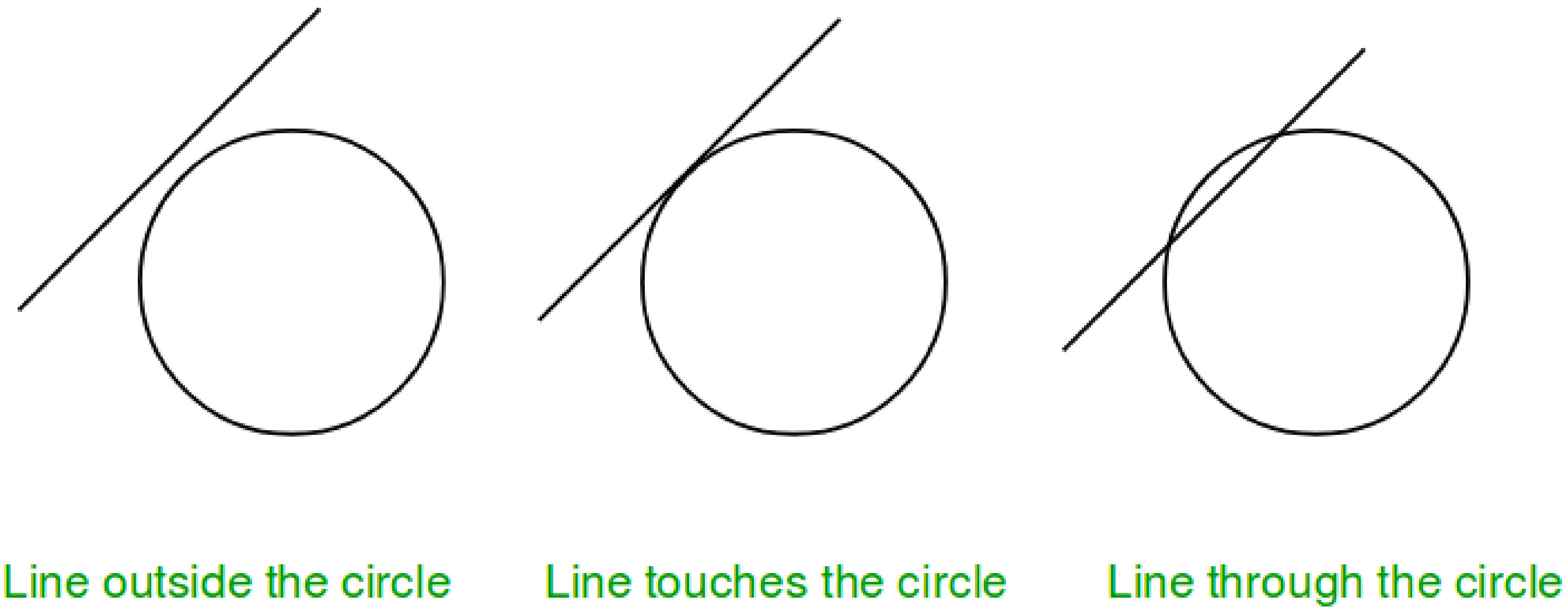
- Circle C centered at P (3, 4) with radius 5 units. Equation?

- Crossing $x$ and $y$ axes:

$x = 0$, touches y-axis

$y = 0$, touches x-axis

# Circle-Line Intersection



# Example $x, y = ?$

$$\begin{cases} (x-2)^2 + (y+3)^2 = 4 \\ 2x + 2y = -1 \end{cases}$$

***Step* 1.** $2x + 2y = -1$

$$2y = -2x - 1$$

$$y = -x - \frac{1}{2}$$

***Step* 2.** $(x-2)^2 + (y+3)^2 = 4$ $\quad$ $y \Leftrightarrow x$

$$(x-2)^2 + \left(-x - \frac{1}{2} + 3\right)^2 = 4$$

$$(x-2)^2 + \left(-x + \frac{5}{2}\right)^2 = 4$$

$$x^2 - 2 \times x \times 2 + 2^2 + x^2 - 2 \times x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 = 4$$

$$2x^2 - 9x + \frac{25}{4} = 0$$

***Step* 3.** *Any method of solving quadratic equations:*

$$x = \frac{9 \pm \sqrt{81 - 50}}{4} = \frac{9 \pm \sqrt{31}}{4} \quad (quadratic\ formula)$$

$$x_1 = \frac{9 + \sqrt{31}}{4}$$

$$x_2 = \frac{9 - \sqrt{31}}{4}$$

$$y_1 = -\frac{9 + \sqrt{31}}{4} - \frac{1}{2} = \frac{-11 - \sqrt{31}}{4}$$

$$y_2 = -\frac{9 - \sqrt{31}}{4} - \frac{1}{2} = \frac{-11 + \sqrt{31}}{4}$$


- **At what points given equations' graphs intersect?**

$$\begin{cases} x + y = 1 \\ (x-2)^2 + (y+1)^2 = 8 \end{cases}$$

$$\begin{cases} x^2 + y^2 = 9 \\ (x-1) + y = 2 \end{cases}$$

- **At what points does the line with equation $y = 2x + 5$ intersect a circle with radius 2 and center (0, 5)?**

# More problems

**Q1:** Consider the circle $(x-5)^2+(y+2)^2=25$. Is the line $y-3=0$ tangent to, intersecting, or disjoint from the circle?

- A disjoint from the circle
- B tangent to the circle
- C intersecting the circle

# Solve for 3 lines below:

$$\begin{cases} (x+9)^2 + (y+10)^2 = 100 \\ \bullet \ \_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$$

1 • $y = x + 1$

2 • $x = 1$

3 • $y = -x + 3$

- $-4y - 3x - 2 = 0$ **meets the circle** $x^2 + y^2 + 6x + 4y = 0$ **in points A and B. Find the coordinates of A and B.**

- $$\begin{cases} (x-2)^2 + (y+3)^2 = 4 \\ 2x + 2y = -1 \end{cases}$$


1.) **Find the coordinates of all points where the given parabola and line intersect each other.**

a) $\begin{cases} y = x^2 + 6x - 10 \\ y + 5 = 2x \end{cases}$ b) $\begin{cases} y = 2x^2 - 8x + 1 \\ y = -4x - 1 \end{cases}$ c) $\begin{cases} y = -x^2 + 3x - 5 \\ y = x + 1 \end{cases}$

2.) **Find the coordinates of all points where the given circle and line intersect each other.**

a) $\begin{cases} (x-4)^2 + (y+1)^2 = 20 \\ x + 3y = 11 \end{cases}$ c) $\begin{cases} (x-3)^2 + (y-1)^2 = 20 \\ x - y = -5 \end{cases}$

b) $\begin{cases} (x+5)^2 + (y-6)^2 = 10 \\ y = -\frac{1}{3}x + 1 \end{cases}$ d) $\begin{cases} (x+1)^2 + (y-2)^2 = 25 \\ 4y = 3x + 11 \end{cases}$