Tishk International University
Architecture Department
First Grade
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Calculus

Lecture -2-
Linear Functions

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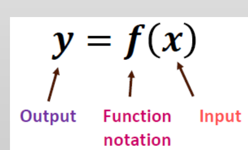
1. Functions
2. Domain and Ranges

1. Domain and Ranges of Functions

- A ***function*** is a relationship between two variables such that each value of the first variable is paired with exactly one value of the second variable.
- The ***domain*** is the set of permitted *x* values.
- The ***range*** is the set of found values of *y*.
- Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description



1. Domain and Ranges of Functions

- The temperature at which water boils depends on the elevation above sea level.
- The area of a circle depends on the radius of the circle.
- The distance an object travels depends on the elapsed time.
- In each case, the value of one variable quantity, say y, depends on the value of another variable quantity, which we often call x.
- We say that "y is a function of x" and write this symbolically as

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"}).$$

- The symbol f represents the function, the letter x is the independent variable representing the input value to f, and y is the dependent variable or output value of f at x.

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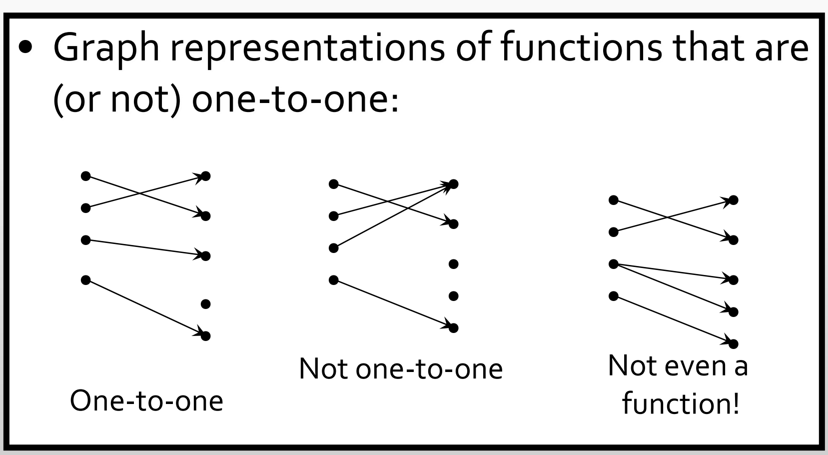
1. Domain and Ranges of Functions

- Graph representations of functions that are (or not) one-to-one:



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Example -1-

- Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

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Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root: $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input that is assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives nonnegative real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

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Example -2-

Functional Notation: Find the following

$f(x)=3x^2-x+2$

$f(-3)$

$$3(-3)^2-(-3)+2$$
$$27+3+2$$
$$30+2$$
$$32$$

$f(x)=x^2-x+2$

$f(m+3)$

$$(m+3)^2-(m+3)+2$$
$$(m+3)(m+3)-m-3+2$$
$$m^2+3m+3m+9-m-3+2$$
$$m^2+5m+8$$

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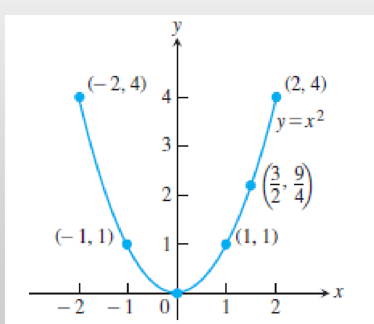
Example -3-

- Graph the function y = x^2 over the interval [-2, 2].

Solution:

Make a table of xy-pairs that satisfy the equation y = x^2. Plot the points (x, y) whose coordinates appear in the table, and draw a smooth curve (labeled with its equation) through the plotted points.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



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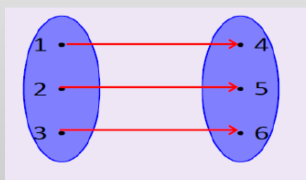
Example -4-

- Determine which of the following is a function?

x	$f(x)$
1	4
2	5
3	6

Solution:

Each input has only one outputs, so it's a function.



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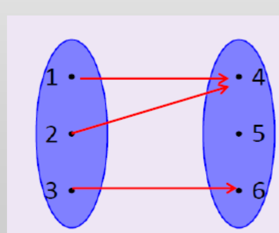
Example -5-

- Determine which of the following is a function?

x	$f(x)$
1	4
2	4
3	6

Solution:

Each input has only one outputs, even two of them has the same output, but its still a function.



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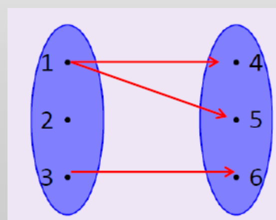
Example -6-

- Determine which of the following is a function?

x	$f(x)$
1	4
1	5
3	6

Solution:

One input has two outputs, so its not a function .



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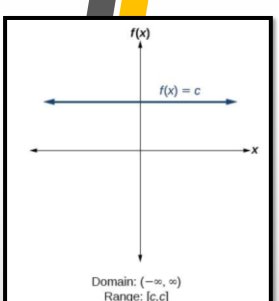
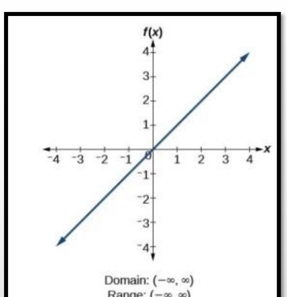
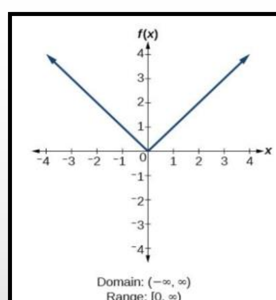
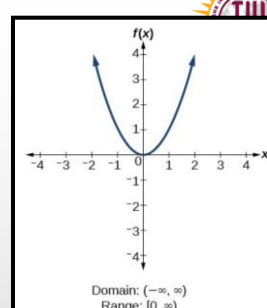
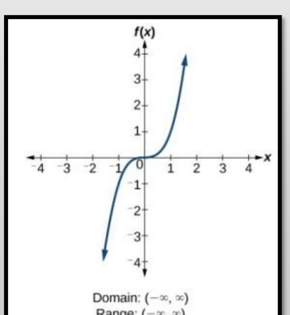
Example -7-











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3. Graphs of The Functions

3. Graphs of The Functions

- The graph of a function *f* is the set of all points in the plane of the form *(x, f(x))*. We could also define the graph of *f* to be the graph of the equation *y = f(x)*. So, the graph of a function is a special case of the graph of an equation.
- There are lots of ways to visualize or picture a function in your head. You can think of it as a machine accepting inputs and shooting out outputs, or whatever way you come up with.
- However, by far the most important way to visualize a function is through its *graph*. By looking at a graph in the xy-plane we can usually find the domain and range of the graph, discover asymptotes, and know whether or not the graph is actually a function.

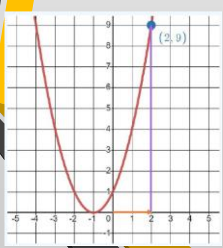
Example -8-

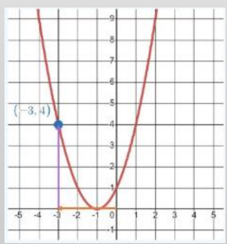
Use the graph below to determine the following values for $f(x) = (x+1)^2$

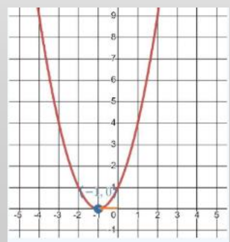
a. $f(2)$
b. $f(-3)$
c. $f(-1)$

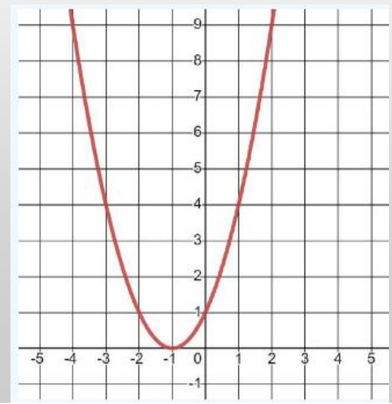
Solution:

After determining these values, compare your answers to what you would get by simply plugging the given values into the function.









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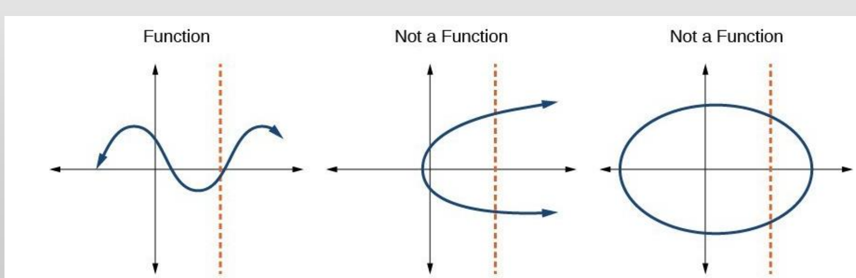
A. The Vertical Line Test

- The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value.
- A curve in the xy-plane is a function if and only if no vertical line intersects the curve more than once.
- Remember that a function can only take on one output for each input. We cannot plug in a value and get out two values. The Vertical Line Test will show this.

For instance,



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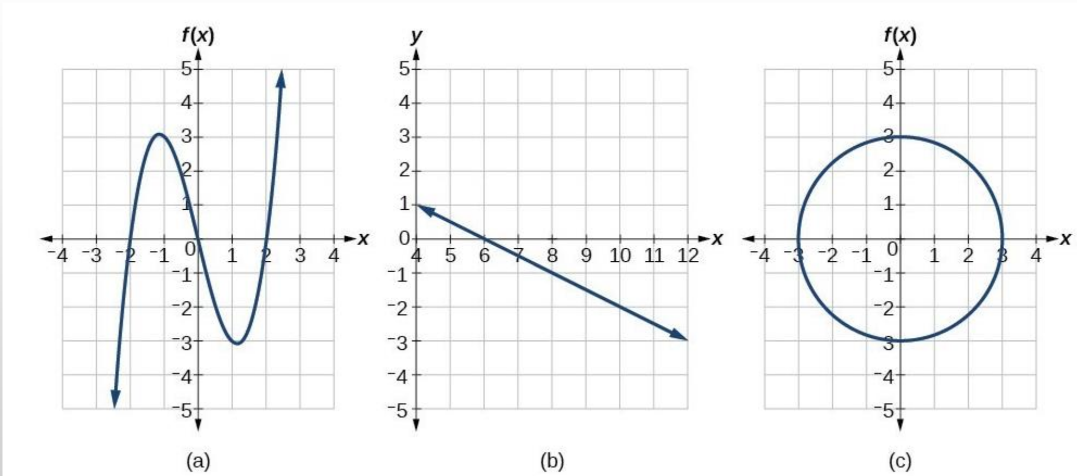
What do you think?



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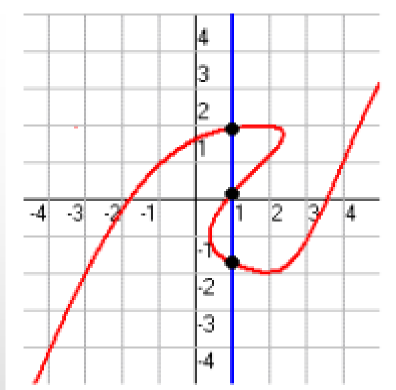

- This red graph is NOT a function as it fails the Vertical Line Test in blue. We can draw a vertical line and it hits more than one point on our function.
- For this function in red when we plug in x = 1 we get 3 values out. And we know that a function can only have one output for each input. Thus, it isn't a function.

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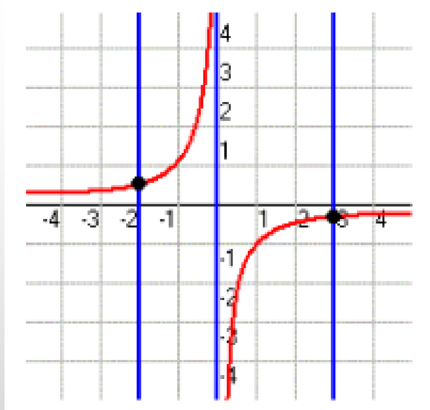
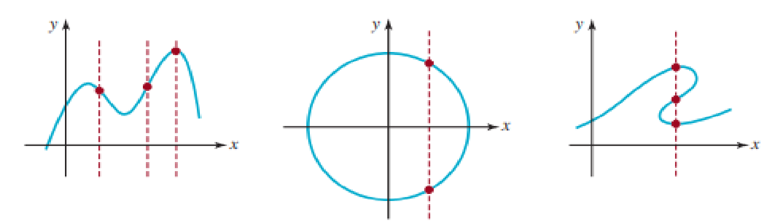
However, this graph,



is a function as we can draw any vertical line (blue) and it does not intersect the function (red) at more than one point.



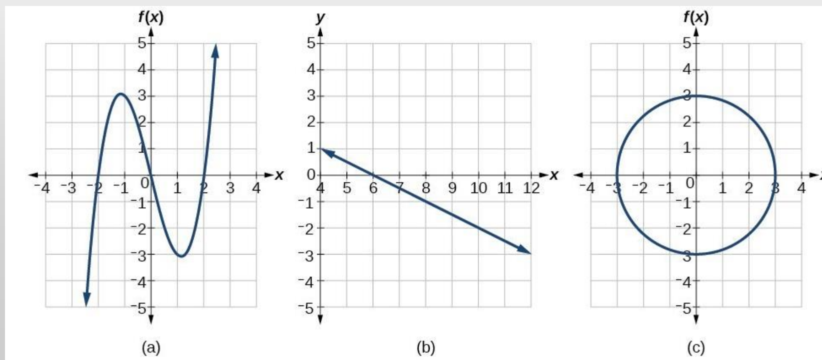
(a) Function (b) Not a function (c) Not a function

- *Vertical Line Test* From the definition of a function we know that for each x in the domain of f there corresponds only one value $f(x)$ in the range. This means a vertical line that intersects the graph of a function $y=f(x)$ (this is equivalent to choosing an x) can do so in at most one point. Conversely, if every vertical line that intersects a graph of an equation does so in at most one point, then the graph is the graph of a function. The last statement is called the vertical line test for a function.
- On the other hand, if some vertical line intersects a graph of an equation more than once, then the graph is not that of a function.

B. The Horizontal Line Test

- Once we have determined that a graph defines a function, an easy way to determine if it is a **one-to-one function** is to use the horizontal line test.
- Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a **one-to-one function**.

Are either of the functions one-to-one?



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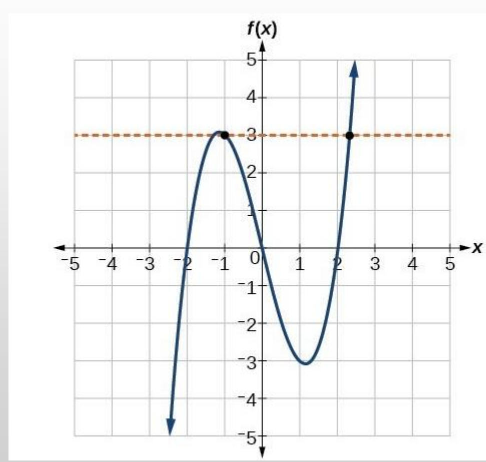
Are either of the functions one-to-one?

The function in (a) is not one-to-one. The horizontal line intersects the graph of the function at two points



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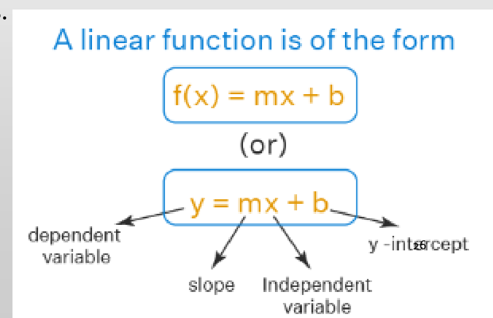
4. Graphing Linear Functions

4. Graphing Linear Functions

- A linear function is a function that represents a straight line on the coordinate plane. For example, *y* = 3x - 2 represents a straight line on a coordinate plane and hence it represents a linear function.
- Since *y* can be replaced with *f(x)*, this function can be written as *f(x)* = 3x - 2.
- Linear function only have at most a degree of 1. This means it has at most one *x* raised to a power of 1. They follow the form: *f(x) = mx + b.*
- There are two basic methods of graphing linear functions.

a. *The first* is by plotting points and then drawing a line through the points.

a. *The second* is by using the y -intercept and slope.



a. Graphing a Function by Plotting Points

- To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid.
- In general, we must evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x)=2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point (1,2). Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point (2,4).
- Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

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a. Graphing a Function by Plotting Points

- Given a linear function, graph by plotting points, the followings are steps:

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

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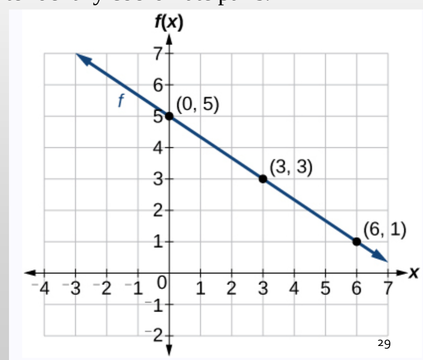
Example -9-

Graph $f(x)=-\frac{2}{3}x+5$ by plotting points.

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value and use the output value to identify coordinate pairs.

$$x=0 \qquad f(0)=-\frac{2}{3}(0)+5=5\rightarrow(0,5)$$
$$x=3 \qquad f(3)=-\frac{2}{3}(3)+5=3\rightarrow(3,3)$$
$$x=6 \qquad f(6)=-\frac{2}{3}(6)+5=1\rightarrow(6,1)$$

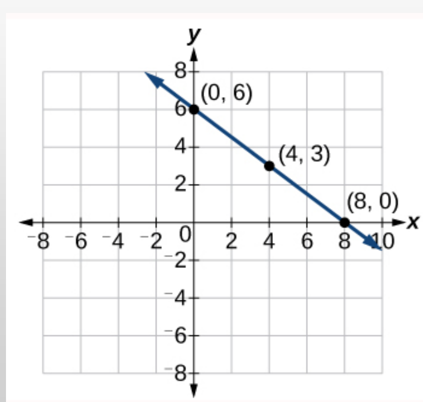


The graph of the linear function $f(x)=-\frac{2}{3}x+5$.

Example -10-

Graph $f(x)=-\frac{3}{4}x+6$ by plotting points.

Answer;



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b. Graphing a Function Using y- Intercept and Slope

- Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its y-intercept. To find the y-intercept, we can set x=0 in the equation. The other characteristic of the linear function is its slope m, which is the constant rate of change, which can be calculated using any two points on the graph. Let's consider the following function. $f(x) = \frac{1}{2}x + 1$
- The slope is 1/2. Because the slope is positive, we know the graph will slant upward from left to right.
- The y-intercept is the point on the graph when x=0. The graph crosses the y-axis at (0,1). Now we know the slope and the y-intercept.
- We can begin graphing by plotting the point (0,1) We know that the slope is rise over run, m=rise/run.
- From our example, we have m=1/2, which means that the rise is 1 and the run is 2. So, starting from our y-intercept (0,1), we can rise 1 and then run 2, or run 2 and then rise 1.
- We repeat until we have a few points, and then we draw a line through the points

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b. Graphing a Function Using y- Intercept and Slope

Steps:

In the equation $f(x) = mx + b$

- b is the y-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the y-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

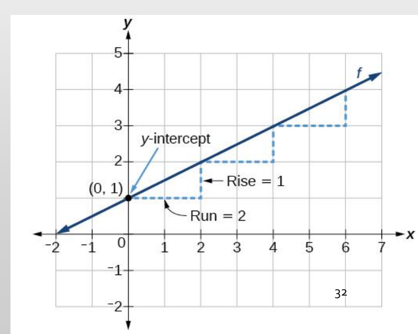
1.Evaluate the function at an input value of zero to find the y-intercept.

2.Identify the slope as the rate of change of the input value.

3.Plot the point represented by the y-intercept.

4.Use rise/run to determine at least two more points on the line.

5.Sketch the line that passes through the points.



Graph of $f(x)$ showing slope and y-intercept

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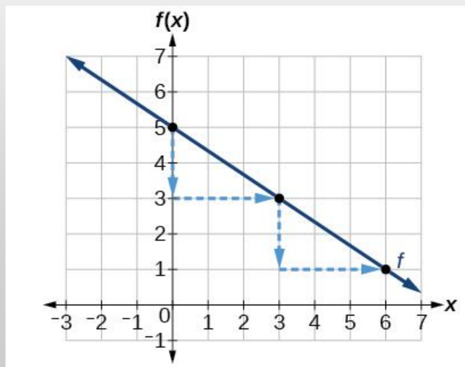
Example -11-

Graph $f(x) = -\frac{2}{3}x + 5$ using the y-intercept and slope.

Evaluate the function at $x = 0$ to find the y-intercept. The output value when $x = 0$ is 5, so the graph will cross the y-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$, or $\frac{-2}{3}$. This tells us that for each vertical decrease in the "rise" of –2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the y-intercept on the graph in Figure 3.4.2.4. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.



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Example -12-

Find the slope of the line passing through (-3,-5) and (2,1).

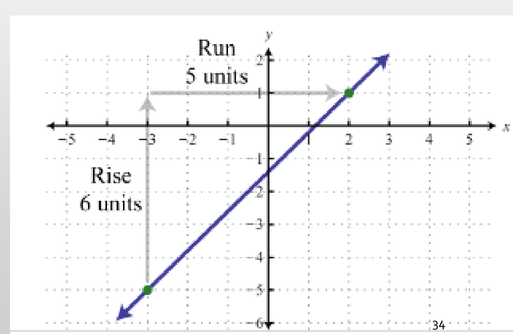
Solution:

Given (−3, −5) and (2, 1), calculate the difference of the *y*-values divided by the difference of the *x*-values. Take care to be consistent when subtracting the coordinates:

(x_1, y_1)	(x_2, y_2)
$(-3, -5)$	$(2, 1)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-5)}{2-(-3)} = \frac{1+5}{2+3} = \frac{6}{5}$$

Answer: $m = \frac{6}{5}$



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Example -13-

Find the *y*-value for which the slope of the line passing through (6,−3) and (−9,y) is -2/3.

Solution: Substitute the given information into the slope formula.

After substituting in the given information, the only variable left is *y*. Solve.

Slope	(x_1, y_1)	(x_2, y_2)
$m=-\frac{2}{3}$	$(6, -3)$	$(-9, y)$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$-\frac{2}{3}=\frac{y-(-3)}{-9-6}$$

$$-\frac{2}{3}=\frac{y+3}{-15}$$

$$-15\left(-\frac{2}{3}\right)=-15\left(-\frac{y+3}{15}\right)$$

$$10=y+3$$

$$7=y$$

Answer: $y=7$

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5. *Writing the Equation for a Function from the Graph of a Line*

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5. Writing the Equation for a Function from the Graph of a Line

- Recall that in Linear Functions, we wrote the equation for a linear function from a graph.
- Now we can extend what we know about graphing linear functions to analyze graphs a little more closely.
- Begin by looking at the graph, we can see right away that the graph crosses the y-axis at the point (0,4) so this is the y-intercept.

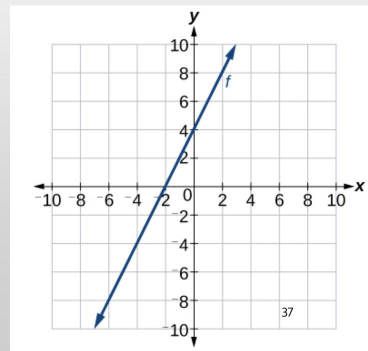
Example -14-

- Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point (−2,0). To get from this point to the y-intercept, we must move up 4 units (rise) and to the right 2 units (run). So, the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$$

- Substituting the slope and y-intercept into the slope-intercept form of a line gives

$$y = 2x + 4$$

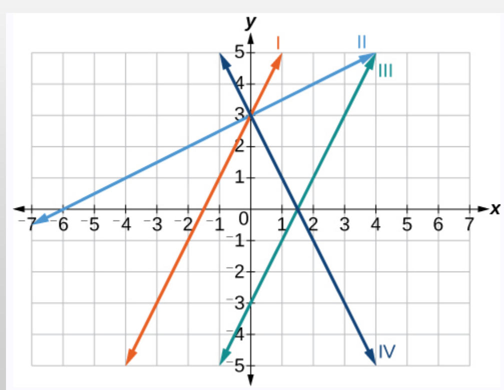


Graph of a linear function

Example -15-

- Match each equation of the linear functions with one of the lines.

a. $f(x) = 2x + 3$
b. $g(x) = 2x - 3$
c. $h(x) = -2x + 3$
d. $j(x) = \frac{1}{2}x + 3$



Graph of four linear functions

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- Solution

Analyze the information for each function.

1.This function has a slope of 2 and a y-intercept of 3. It must pass through the point (0,3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function *g* has the same slope, but a different y-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through (0,3) so *f* must be represented by line I.

2.This function also has a slope of 2, but a y-intercept of −3. It must pass through the point (0,−3) and slant upward from left to right. It must be represented by line III.

3.This function has a slope of –2 and a y-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.

4.This function has a slope of 1/2 and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through (0,3) but the slope of *j* is less than the slope of *f* so the line for *j* must be flatter. This function is represented by Line II.

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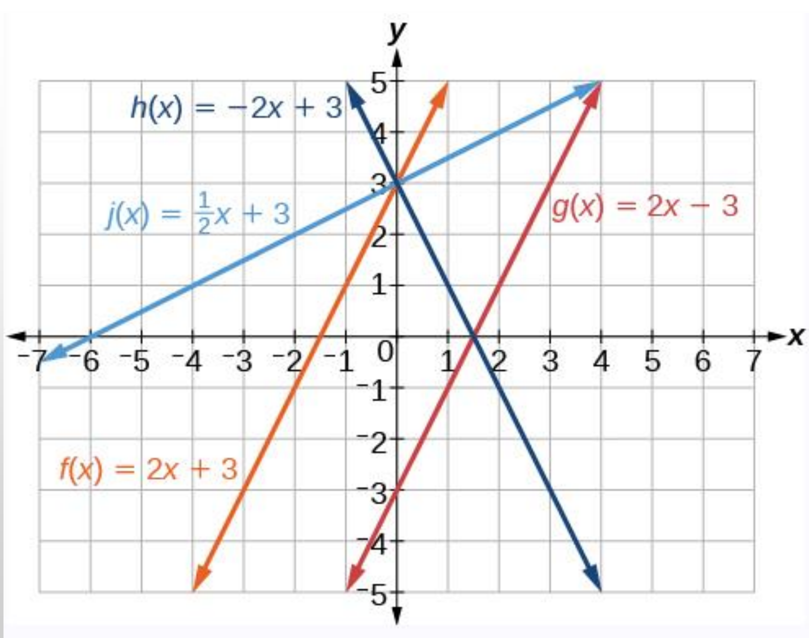

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Example -16-

Find the linear function that has two points (-1, 15) and (2, 27) on it.

Solution:

The given points are $(x_1, y_1) = (-1, 15)$ and $(x_2, y_2) = (2, 27)$.

Step 1: Find the slope of the function using the slope formula:

$m = (y_2 - y_1) / (x_2 - x_1) = (27 - 15) / (2 - (-1)) = 12/3 = 4.$

Step 2: Find the equation of linear function using the point slope form.

$y - y_1 = m (x - x_1)$

$y - 15 = 4 (x - (-1))$

$y - 15 = 4 (x + 1)$

$y - 15 = 4x + 4$

$y = 4x + 19$

Therefore, the equation of the linear function is, $f(x) = 4x + 19$.

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6. Identifying a Linear Function

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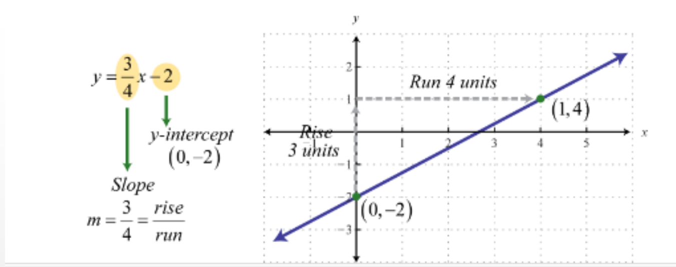
6. Identifying a Linear Function

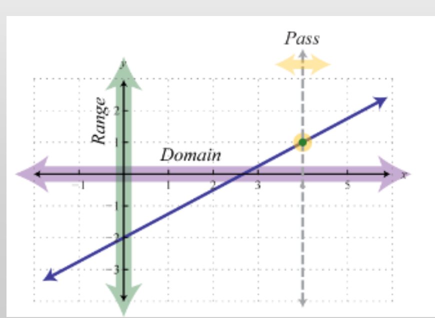
Given any linear equation in **standard form**, $ax + by = c$, we can solve for y to obtain **slope-intercept form**, $y = mx + b$. For example,

$$\begin{aligned} 3x - 4y &= 8 && \leftarrow \textit{Standard Form} \\ -4y &= -3x + 8 \\ y &= \frac{-3x+8}{-4} \\ y &= \frac{-3x}{-4} + \frac{8}{-4} \\ y &= \frac{3}{4}x - 2 && \leftarrow \textit{Slope-Intercept Form} \end{aligned}$$

Where $x = 0$, we can see that $y = -2$ and thus $(0, -2)$ is an ordered pair solution. This is the point where the graph intersects the y-axis and is called the **y-intercept**. We can use this point and the slope as a means to quickly graph a line. For example, to graph $y = \frac{3}{4}x - 2$, start at the y-intercept $(0, -2)$ and mark off the slope to find a second point. Then use these points to graph the line as follows:



The vertical line test indicates that this graph represents a function. Furthermore, the domain and range consists of all real numbers.



In general, a **linear function** is a function that can be written in the form

$$f(x) = mx + b \; \textit{Linear Function}$$

where the slope m and b represent any real numbers. Because $y = f(x)$, we can use y and $f(x)$ interchangeably, and ordered pair solutions on the graph (x, y) can be written in the form $(x, f(x))$.

$$(x, y) \quad \Leftrightarrow \quad (x, f(x))$$

We know that any y-intercept will have an x-value equal to zero. Therefore, the y-intercept can be expressed as the ordered pair $(0, f(0))$. For linear functions,

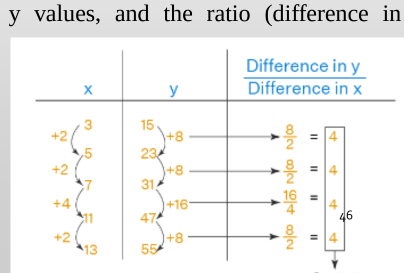
$$\begin{aligned} f(0) &= m(0) + b \\ &= b \end{aligned}$$

Example -17-

Determine whether the following data from the following table represents a linear function.

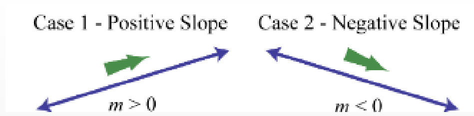
x	y
3	15
5	23
7	31
11	47
13	55

Solution: We will compute the differences in x-values, differences in y values, and the ratio (difference in y)/(difference in x) every time and see whether this ratio is a constant.



Since all numbers in the last column are equal to a constant, the data in the given table represents a linear function.

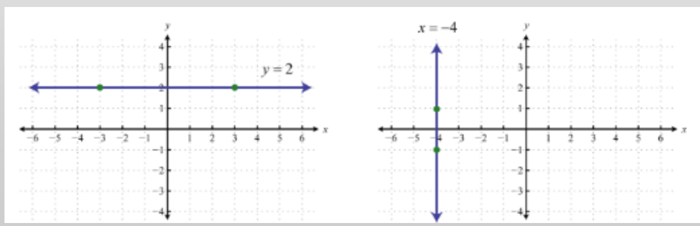
- There are four geometric cases for the value of the slope.



- Reading the graph from left to right, lines with an upward incline have positive slopes and lines with a downward incline have negative slopes. The other two cases involve horizontal and vertical lines. Recall that if k is a real number we have

$y = k$ *Horizontal Line*
$x = k$ *Vertical Line*

- For example, if we graph $y=2$ we obtain a horizontal line, and if we graph $x=-4$ we obtain a vertical line.



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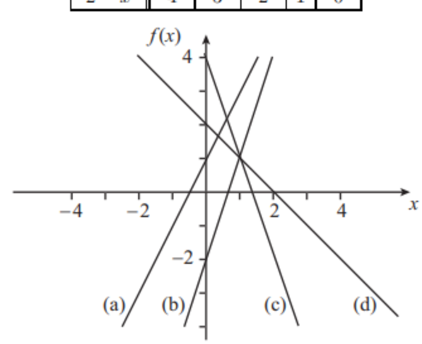
Example -18-

By drawing up a table of values, plot the following linear functions on the same axes:

(a) $f(x) = 2x + 1$ (b) $f(x) = 3x - 2$ (c) $f(x) = 4 - 3x$ (d) $f(x) = 2 - x$

Solution

x	-2	-1	0	1	2
$2x+1$	-3	-1	1	3	5
$3x-2$	-8	-5	-2	1	4
$4-3x$	10	7	4	1	-2
$2-x$	4	3	2	1	0



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Example -19-

Find the gradient and the vertical intercept for each of the following linear functions by rearranging them into the form $f(x) = ax + b$ (note: $y = f(x)$)

(a) $2y + 4x = 12$ (b) $5x - y = 9$ (c) $-3x = 1 - 4y$ (d) $2 - y/3 = x$
(e) $3 = 3y/4 - 2x/3$ (f) $12x - 4 = y/3 + 3$

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Solution

(a)

$$\begin{aligned} 2y + 4x &= 12 \\ 2y &= 12 - 4x \\ y &= 6 - 2x \\ y &= -2x + 6 \\ f(x) &= -2x + 6 \end{aligned}$$

Gradient $= -2$, vertical intercept $= 6$.

(b)

$$\begin{aligned} 5x - y &= 9 \\ 5x &= 9 + y \\ 5x - 9 &= y \\ y &= 5x - 9 \\ f(x) &= 5x - 9 \end{aligned}$$

Gradient $= 5$, vertical intercept $= -9$.

(c)

$$\begin{aligned} -3x &= 1 - 4y \\ 4y - 3x &= 1 \\ 4y &= 3x + 1 \\ y &= \tfrac{3}{4}x + \tfrac{1}{4} \\ f(x) &= \tfrac{3}{4}x + \tfrac{1}{4} \end{aligned}$$

(d)

$$\begin{aligned} 2 - \frac{y}{3} &= x \\ 2 &= x + \frac{y}{3} \\ 2 - x &= \frac{y}{3} \\ \frac{y}{3} &= 2 - x \\ y &= 6 - 3x \\ f(x) &= -3x + 6 \end{aligned}$$

Gradient $= -3$, vertical intercept $= 6$.

(e)

$$\begin{aligned} 3 &= \frac{3y}{4} - \frac{2x}{3} \\ 3 + \frac{2x}{3} &= \frac{3y}{4} \\ 12 + \frac{8x}{3} &= 3y \\ 4 + \frac{8x}{9} &= y \\ y &= \frac{8x}{9} + 4 \\ f(x) &= \frac{8x}{9} + 4 \end{aligned}$$

Gradient $= \frac{8}{9}$, vertical intercept $= 4$.

(f)

$$\begin{aligned} 12x - 4 &= \frac{y}{3} + 3 \\ 12x - 7 &= \frac{y}{3} \\ 36x - 21 &= y \\ f(x) &= 36x - 21 \end{aligned}$$

Gradient $= 36$, vertical intercept $= -21$.

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Example -20-

Write down three different functions in which all the graphs are represented by parallel lines.

Solution

$f(x) = 2x - 1, \quad f(x) = 2x + 5, \quad f(x) = 2x - \frac{1}{2}.$

Write down three different functions in which all the graphs have the same vertical intercept.

Solution

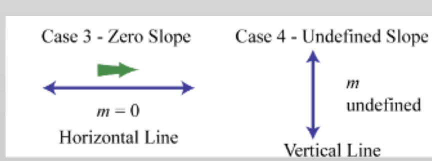
$f(x) = 4x - 5, \quad f(x) = 2x - 5, \quad f(x) = 25x - 5.$

Example -21-

From the graphs we can determine two points and calculate the slope using the slope formula.

Horizontal Line	*Vertical Line*
(x_1, y_1) (x_2, y_2) $(-3, 2)$ $(3, 2)$ $m = \frac{y_2-y_1}{x_2-x_1} = \frac{2-(2)}{3-(-3)} = \frac{2-2}{3+3} = \frac{0}{6} = 0$	(x_1, y_1) (x_2, y_2) $(-4, -1)$ $(-4, 1)$ $m = \frac{y_2-y_1}{x_2-x_1} = \frac{1-(-1)}{-4-(-4)} = \frac{1+1}{-4+4} = \frac{2}{0}$ *Undefined*

Notice that the points on the horizontal line share the same *y*-values. Therefore, the rise is zero and hence the slope is zero. The points on the vertical line share the same *x*-values. Consequently, the run is zero, leading to an undefined slope. In general,



References

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- Internet sources