Tishk International University
Architectural Engineering Department
First Grade
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Calculus

Lecture -5-
Trigonometric Functions

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Contents

1. What's a trigonometric function?
2. Systems of measuring angles
3. Trigonometric functions in right angles
4. Trigonometric functions unit circle approach

1. What's a trigonometric function?

2. Systems of measuring angles

1. What's A Trigonometric Function?

- The word 'trigonometry' is derived from the Greek words' 'trigon' and 'metron' which means measuring the sides of a triangle.
- Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums.
- In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions.
- Usually, we follow two types of conversions for measuring angles, i.e., (1) Sexagesimal system (2) Circular system. (3) Centesimal system
- In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

2. Systems of measuring angles
(i) Sexagesimal system

- The Sexagesimal system, also known as the English system, is the most preferred angle measurement system. Degrees, Minutes and Seconds are the units of measure in the sexagesimal system.
- In this system, the right angle is split into 90 equally divided parts. Each part is called a degree. (1°). Furthermore, one degree is split into 60 equally divided parts. Each part is known as the sexagesimal minute (1'). Every single minute is split into 60 equally divided parts, known as the sexagesimal second (1").
- This system is prevalent and widely used in practical applications of Trigonometry.

In short,

- 1 right angle = 90 degrees (or 90°)
- 1 degree (or 1°) = 60 minutes (or $60'$)
- 1 minute (or $1'$) = 60 seconds (or $60''$)

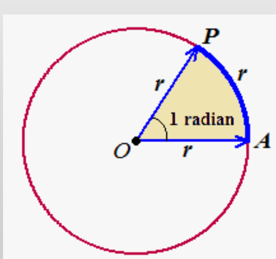
A complete rotation describes 360°, which forms a full circle.

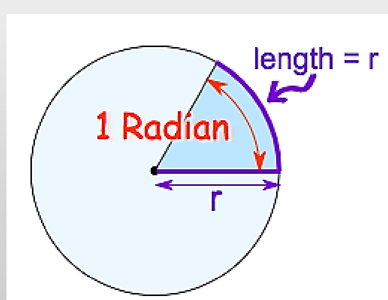
Example

Let's convert 40° 4 minutes 2 seconds :
40° will remain the same.
1° = 60 minutes
Therefore, 4 minutes = $\frac{4}{60} = 0.067$ minutes
2 seconds = $\frac{2}{60} = 0.033$ seconds
40° 4 minutes 2 seconds = 40 degree + 0.067 minutes + 0.033 seconds = 40.1

(ii) Circular system

- In this System, an angle is measured in radians. In higher mathematics angles are usually measured in a circular system. In this system a radian is considered as the unit for the measurement of angles.
- A radian is an angle subtended at the center of a circle by an arc whose length is equal to the radius.
- In any circle, the angle subtended at its centre by an arc of the circle whose length is equal to the radius of the circle is called a radian.





A/ Conversion – Sexagesimal System to Circular System

In sexagesimal system, 1 right angle = 90° and in circular system, 1 right angle = $\frac{\pi}{2}$ radian

$=> 90^\circ = \frac{\pi}{2} radian => 1^\circ = \frac{\pi}{180} radian$

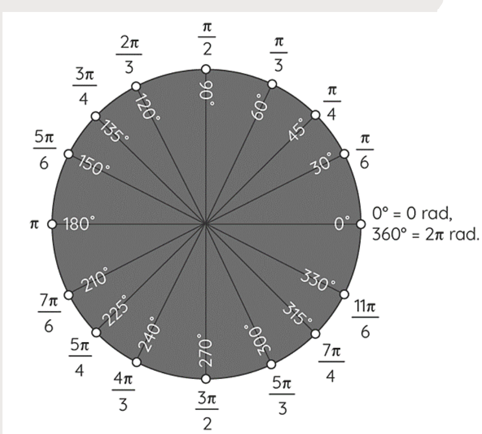
For example $30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6} radian$

B/ Conversion – Circular System to Sexagesimal System

In circular system, 1 right angle = $\frac{\pi}{2}$ and in sexagesimal system, 1 right angle = 90°

For example, $\frac{\pi}{4} radian = \left(\frac{\pi}{4}\right) \times \left(\frac{180}{\pi}\right) = 45^\circ$

- If an angle is given without mentioning units, it is assumed to be in radians. The relation between degree measures and circular (radian) measures of some standard angles are given below:



Degree (°)	Radian (rad)
0°	0
30°	π/6
45°	π/4
60°	π/3
90°	π/2
120°	(2π)/3
135°	(3π)/4
150°	(5π)/6
180°	π
270°	(3π)/2
360°	2π

Example-1-

Convert the angles from radians to degrees.

a. $\frac{3\pi}{2}$ b. $\frac{7\pi}{6}$

c. $\frac{\pi}{2}$ d. $\frac{11\pi}{6}$

Solution:

a. $\frac{3\pi}{2} \cdot \frac{180}{\pi} = \frac{540}{2} = 270°$

b. $\frac{7\pi}{6} \cdot \frac{180}{\pi} = \frac{1260}{6} = 210°$

c. $\frac{\pi}{2} \cdot \frac{180}{\pi} = \frac{180}{2} = 90°$

d. $\frac{11\pi}{6} \cdot \frac{180}{\pi} = \frac{1980}{6} = 330°$

Example-2-

Convert the angles from degrees to radians.

a. 120° b. 210°

c. 150° d. 315°

Solution:

a. $120° \cdot \frac{\pi}{180} = \frac{120\pi}{180} = \frac{2\pi}{3}$

b. $210° \cdot \frac{\pi}{180} = \frac{210\pi}{180} = \frac{7\pi}{6}$

c. $150° \cdot \frac{\pi}{180} = \frac{150\pi}{180} = \frac{5\pi}{6}$

d. $315° \cdot \frac{\pi}{180} = \frac{315\pi}{180} = \frac{7\pi}{4}$

Example-3-

Convert 18°30’42’’ into the degree.

Solution:

$18°30'42'' = 18° + (30/60)° + (42/(60 \times 60))°$

$= 18° + 0.5° + 0.01166°$

$= 18.51167°$

Example-4-

Convert the degree measure to radians, or the radian measure to degrees.

1. $135°$
2. $\frac{5\pi}{4}$
3. $-50°$

3. Trigonometric functions in right angles

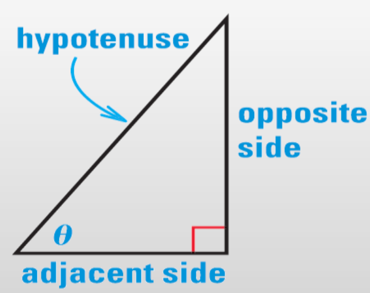
4. Trigonometric functions unit circle approach

3. Trigonometric functions in right angles

If you have a right triangle, there are six ratios of sides that are always constant.

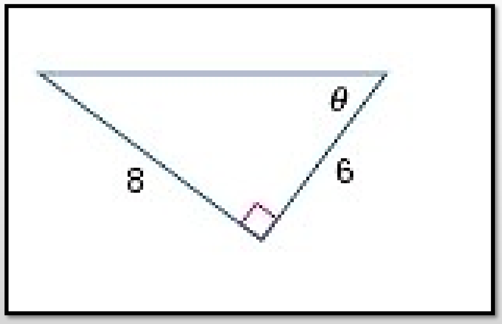
- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

- $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$
- $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
- $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$



Example-5-

Find the values of the six trigonometric functions for angle θ.



15

Solution :

Opposite side = 8
Adjacent Side = 6
Let x be the hypotenuse.
By the Pythagorean theorem,

$$x = \sqrt{8^2 + 6^2}$$
$$= 10$$

Therefore, hypotenuse = 10.

The trigonometric ratios are:

$$\sin\theta = \frac{\text{Opposite Side}}{\text{Hypotenuse}}$$
$$\cos\theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse}}$$
$$\tan\theta = \frac{\text{Opposite Side}}{\text{Adjacent Side}}$$
$$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite Side}}$$
$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent Side}}$$
$$\cot\theta = \frac{\text{Adjacent Side}}{\text{Opposite Side}}$$

Substitute:

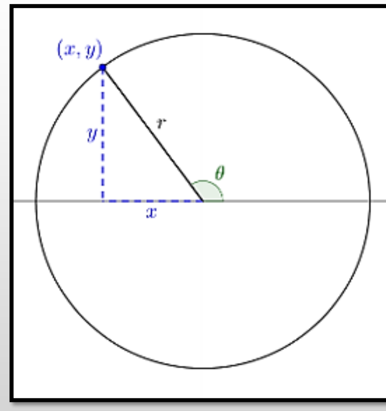
$$\sin\theta = \frac{8}{10} \text{ or } \frac{4}{5}$$
$$\cos\theta = \frac{6}{10} \text{ or } \frac{3}{5}$$
$$\tan\theta = \frac{8}{6} \text{ or } \frac{4}{3}$$
$$\csc\theta = \frac{10}{8} \text{ or } \frac{5}{4}$$
$$\sec\theta = \frac{10}{6} \text{ or } \frac{5}{3}$$
$$\cot\theta = \frac{6}{8} \text{ or } \frac{3}{4}$$

16

4. Trigonometric functions unit circle approach

- A **unit circle** has a center at (0,0) and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle 1.
- Comparing the unit circle formulas and the right triangle formulas develops the formulas for any angle. For example, consider sin θ.

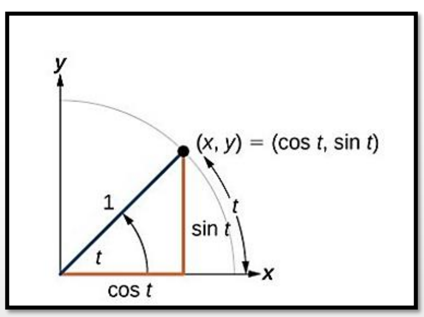
$\sin\theta = y$	Unit Circle
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$	Right Triangle
$\sin\theta = \frac{y}{r}$	Apply the right triangle formula for the acute angle by the origin.



- Notice the last equation matches the unit circle formula with $r = 1$. All the unit circle formulas can be similarly modified.

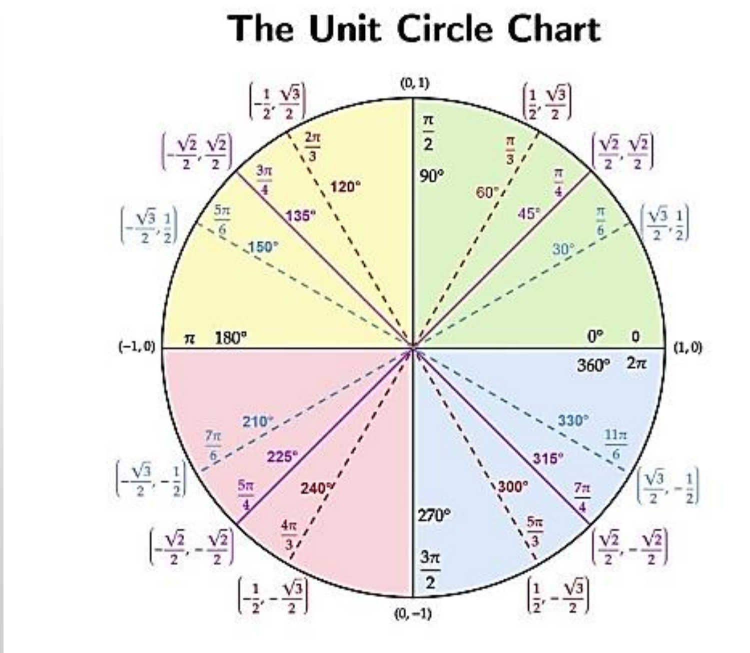
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

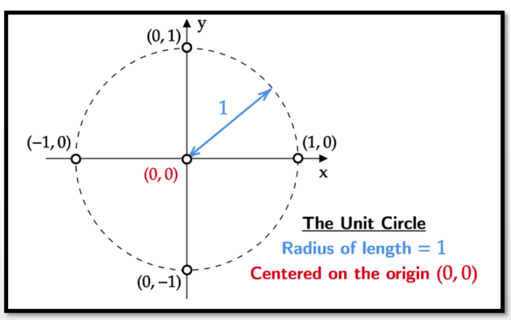
- where θ is an angle in standard position with point (x, y) on the terminal side and $r = \sqrt{x^2 + y^2}$



- The **sine function** of an angle *t* equals the *y*-value of the endpoint on the unit circle of an arc of length *t*.
- The **cosine function** of an angle *t* equals the *x*-value of the endpoint on the unit circle of an arc of length *t*.

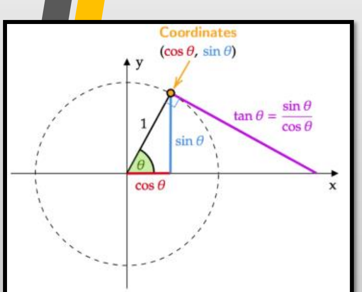
$$\cos t = x$$
$$\sin t = y$$

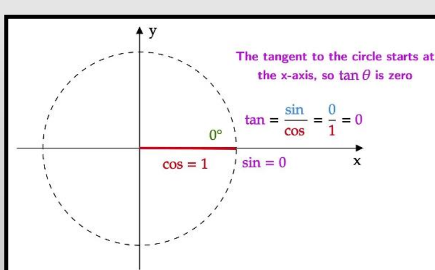


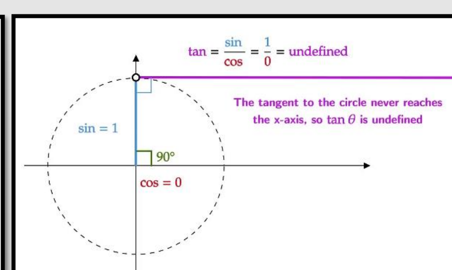


What are the sin, cos, and tan of an angle of 0°?

And what about 90°?







Special Angles: 30°, 45°, and 60°

The angles **30°, 45°, and 60°** have special properties for sin, cos and tan.

- Memorizing sounds like a pain, but don't worry, there are some tricks to help. Let's start with the values for sin.

	0°	30°	45°	60°	90°
sin goes: 0, 1, 2, 3, 4	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

- These are the only values you need to memorize. Can you see why?

	0°	30°	45°	60°	90°
cos goes 4, 3, 2, 1, 0	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

- The final trick is this: $\tan = \frac{\sin}{\cos}$

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$$

$$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Always rationalize the denominator so you don't lose marks.

$$\tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = 1$$

$$\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$$

$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} = \text{undefined}$$

- *Tan* also leads to a nice pattern, although it doesn't include 0° and 90° like *sin* and *cos* do.

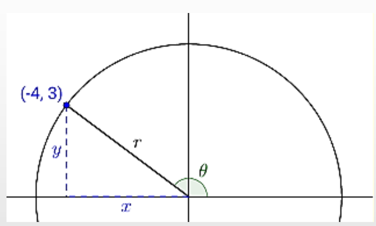
	30°	45°	60°
$\tan = \left(\frac{\sin}{\cos}\right)$	$\frac{(\sqrt{3})^1}{3}$	$\frac{(\sqrt{3})^2}{3}$	$\frac{(\sqrt{3})^3}{3}$

- Put these all together and you get the table of special trigonometric values, or the unit circle table:

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan or $\left(\frac{\sin}{\cos}\right)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

Example-6-

Let (−4, 3) be a point on the terminal side of angle θ. Evaluate the six trigonometric functions of θ.



Solution
Find *r*.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-4)^2 + 3^2}$$
$$r = 5$$

$$\sin\theta = \frac{y}{r} = \frac{3}{5} \qquad \csc\theta = \frac{r}{y} = \frac{5}{3}$$
$$\cos\theta = \frac{x}{r} = -\frac{4}{5} \qquad \sec\theta = \frac{r}{x} = -\frac{5}{4}$$
$$\tan\theta = \frac{y}{x} = -\frac{3}{4} \qquad \cot\theta = \frac{y}{r} = -\frac{4}{3}$$

Example-7-

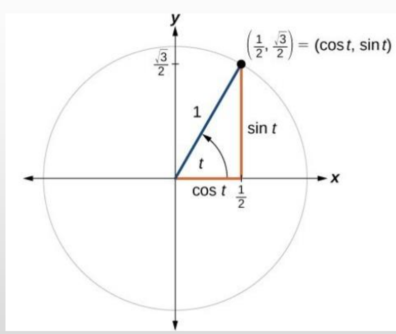
If (4, −8) is a point on the terminal side of angle α in standard position, evaluate the six trigonometric functions of α.

Answers

$\sin\alpha = -\frac{2\sqrt{5}}{5}$	$\csc\alpha = -\frac{\sqrt{5}}{2}$
$\cos\alpha = \frac{\sqrt{5}}{5}$	$\sec\alpha = \sqrt{5}$
$\tan\alpha = -2$	$\cot\alpha = -\frac{1}{2}$

Example-8-

Point P is a point on the unit circle corresponding to an angle of t, as shown in figure. Find cos(t) and sin(t).



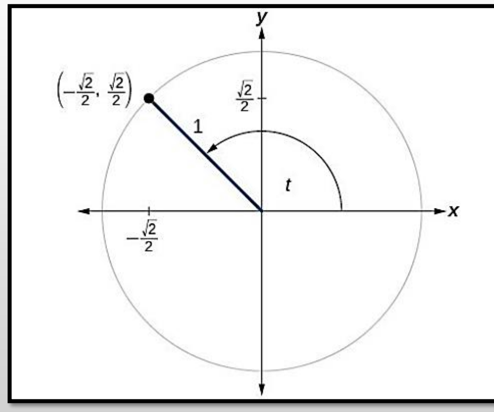
We know that *cos(t)* is the *x*-coordinate of the corresponding point on the unit circle and *sin(t)* is the *y*-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$

$$y = \sin t = \frac{\sqrt{3}}{2}$$

Example-9-

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure. Find cos$t$ and sin$t$.

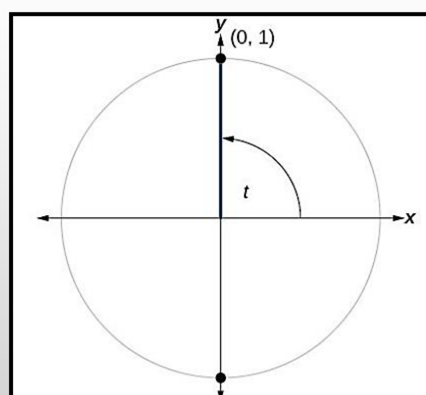


Solution:

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

Example-10-

Find cos(90∘) and sin(90∘).



Solution:

Moving 90∘ counterclockwise around the unit circle from the positive *x*-axis brings us to the top of the circle, where the (x,y) coordinates are (0, 1), as shown in Figure. Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^{\circ}) = 0$$
$$y = \sin t = \sin(90^{\circ}) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Example-11-

Find cosine and sine of the angle π.

Solution:

cos(π)=−1, sin(π)=0

Example-12-

For the following exercises, find the exact value of each trigonometric function.

1. $\sin\frac{\pi}{2}$, $\tan\frac{\pi}{6}$
2. $\sin\frac{\pi}{3}$, $\sec\frac{\pi}{6}$
3. $\cos\frac{\pi}{2}$, $\csc\frac{\pi}{6}$
4. $\cos\frac{\pi}{3}$, $\csc\frac{\pi}{6}$
5. $\sin\frac{\pi}{4}$, $\sec\frac{\pi}{6}$
6. $\cos\frac{\pi}{4}$, $\cot\frac{\pi}{6}$
7. $\sin\frac{\pi}{6}$, $\tan\frac{\pi}{4}$
8. $\sin\pi$, $\sec\frac{\pi}{4}$
9. $\sin\frac{3\pi}{2}$, $\csc\frac{\pi}{4}$
10. $\cos\pi$, $\tan\pi$
11. $\cos\frac{\pi}{6}$, $\sec\frac{\pi}{3}$
12. cos0, tan0

Example-13-

For the following exercises, use the given point on the unit circle to find the value of all six trigonometric functions of t.

1. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
2. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
3. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
4. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
5. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
6. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
7. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
8. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
9. $(1, 0)$
10. $(-1, 0)$

Example-14-

Evaluate the six trigonometric functions for the given angles.

a. $\theta = \pi$
b. $\theta = \frac{\pi}{4}$
c. $\theta = \frac{4\pi}{3}$
d. $\theta = \frac{11\pi}{6}$

Solution

a.Use the angle on the unit circle to find the corresponding *x* and *y*-coordinates. For π, *x* = −1 and *y* = 0.

$$\sin \pi = y = 0$$
$$\cos \pi = x = -1$$
$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

$$\csc \pi = \frac{1}{y} = \frac{1}{0} = \text{undefined}$$
$$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$
$$\cot \pi = \frac{x}{y} = \frac{-1}{0} = \text{undefined}$$

b. Use the angle on the unit circle to find the corresponding *x* and *y*-coordinates.
For $\frac{\pi}{4}$, $x=\frac{\sqrt{2}}{2}$ and $y=\frac{\sqrt{2}}{2}$

$$\sin\frac{\pi}{4}=y=\frac{\sqrt{2}}{2}$$
$$\cos\frac{\pi}{4}=x=\frac{\sqrt{2}}{2}$$
$$\tan\frac{\pi}{4}=\frac{y}{x}=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=1$$

$$\csc\frac{\pi}{4}=\frac{1}{y}=\frac{1}{\frac{\sqrt{2}}{2}}=\sqrt{2}$$
$$\sec\frac{\pi}{4}=\frac{1}{x}=\frac{1}{\frac{\sqrt{2}}{2}}=\sqrt{2}$$
$$\cot\frac{\pi}{4}=\frac{x}{y}=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=1$$

c. Use the angle on the unit circle to find the corresponding *x* and *y*-coordinates.
For $\frac{4\pi}{3}$, $x=-\frac{1}{2}$ and $y=-\frac{\sqrt{3}}{2}$

$$\sin\frac{4\pi}{3}=y=-\frac{\sqrt{3}}{2}$$
$$\cos\frac{4\pi}{3}=x=-\frac{1}{2}$$
$$\tan\frac{4\pi}{3}=\frac{y}{x}=\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}=\sqrt{3}$$

$$\csc\frac{4\pi}{3}=\frac{1}{y}=\frac{1}{-\frac{\sqrt{3}}{2}}=-\frac{2\sqrt{3}}{3}$$
$$\sec\frac{4\pi}{3}=\frac{1}{x}=\frac{1}{-\frac{1}{2}}=-2$$
$$\cot\frac{4\pi}{3}=\frac{x}{y}=\frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{3}$$

d. Use the angle on the unit circle to find the corresponding *x* and *y*-coordinates.
For $\frac{11\pi}{6}$, $x=\frac{\sqrt{3}}{2}$ and $y=-\frac{1}{2}$

$$\sin\frac{11\pi}{6}=y=-\frac{1}{2}$$
$$\cos\frac{11\pi}{6}=x=\frac{\sqrt{3}}{2}$$
$$\tan\frac{11\pi}{6}=\frac{y}{x}=\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=-\frac{\sqrt{3}}{3}$$

$$\csc\frac{11\pi}{6}=\frac{1}{y}=\frac{1}{-\frac{1}{2}}=-2$$
$$\sec\frac{11\pi}{6}=\frac{1}{x}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2\sqrt{3}}{3}$$
$$\cot\frac{11\pi}{6}=\frac{x}{y}=\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}=-\sqrt{3}$$

References

- Thomas-Calculus-14th -Edition
- Internet sources