EOM

3.1 A) Linear Transformation Relationship

Roll Angle ϕ : $\vec{v}' = R_1(\phi)\vec{v}$

Pitch Angle θ : $\vec{v}' = R_2(\theta)\vec{v}$

Yaw Angle ψ : $\vec{v}' = R_3(\psi)\vec{v}$

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Roll-Pitch-Yaw: $\vec{v}_{RPY} = R_1(\phi)R_2(\theta)R_3(\psi)\vec{v}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{V_B = D\ V_E}$$

$$\mathbf{D} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

Matrix D transforms from V_E to V_B (from Earth to Body system

By inverting the direction cosine matrix D the transformation from (V_B) to (V_E) is obtained as given by:

$$\mathbf{V_E = \ = D^{-1}\,V_B = D^T V_B} = \boldsymbol{R_3^{-1}(\psi)\ R_2^{-1}(\theta)\ R_1^{-1}(\phi)}\ \mathbf{V_B}$$

$$\mathbf{D}^{-1} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

B) Rotational Transformation Matrix

The relationship between the angular velocities in the body frame (p, q, and r) and the Euler rates (**ψ' , θ', and ϕ'**) also can be determined from

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\Phi & C_\theta S_\Phi \\ 0 & -S_\Phi & C_\theta C_\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Above Equation can be solved for the Euler rates in terms of the body angular velocities:

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S_\Phi \tan\theta & C_\Phi \tan\theta \\ 0 & C_\Phi & -S_\Phi \\ 0 & S_\Phi \sec\theta & C_\Phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

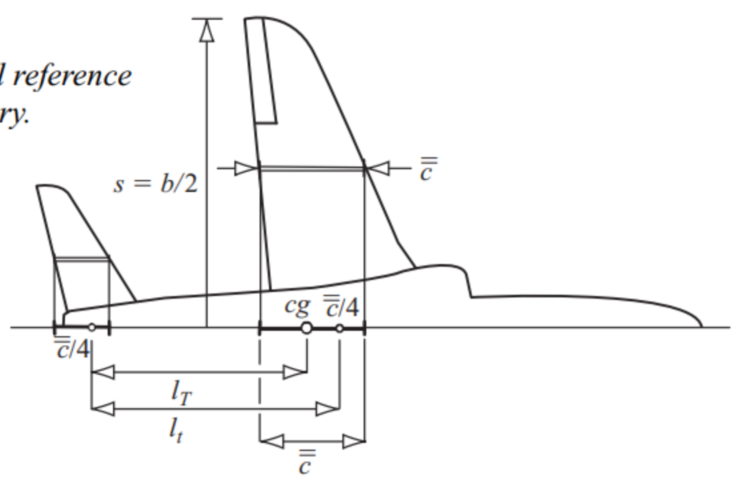
By integrating these equations, one can determine the Euler angles (**ψ' , θ', and ϕ'**)

3.1.1 Wing area: The reference area is usually the gross plan area of the wing, including that part within the fuselage, and is denoted S:

$$S = b\bar{c}$$

where b is the wing span and $\bar{c}$ is the standard mean chord of the wing.

Longitudinal reference geometry.



3.1.2 Standard mean chord (smc)

For a straight tapered wing equation simplifies to $\bar{c} = \frac{S}{b}$

3.1.3 Aspect ratio: The aspect ratio of the aircraft wing is a measure of its spanwise slenderness and is denoted A and is defined as follows:

$$A = \frac{b^2}{S} = \frac{b}{\bar{c}}$$

3.1.4 Centre of gravity location: The centre of gravity, cg, of an aircraft is usually located on the reference chord as indicated in Fig. above. Its position is quoted as a fraction of (c), denoted h, and is measured from the leading edge of the reference chord as shown. The cg position varies as a function of aircraft loading, the typical variation being in the range 10–40% of c. Or, equivalently, $0.1 \leq h \leq 0.4$.

EXAMPLE 1: Useful application of the direction cosine matrix is to calculate height perturbations in terms of aircraft motion. Above Equation (D^{-1}) may be used to relate the velocity components in aircraft axes to the corresponding components in earth axes as follows:

$$\begin{bmatrix} U_E \\ V_E \\ W_E \end{bmatrix} = \mathbf{D}^{-1} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$\begin{bmatrix} U_E \\ V_E \\ W_E \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & \begin{array}{c}\cos\psi\sin\theta\sin\phi \\ -\sin\psi\cos\phi\end{array} & \begin{array}{c}\cos\psi\sin\theta\cos\phi \\ +\sin\psi\sin\phi\end{array} \\ \sin\psi\cos\theta & \begin{array}{c}\sin\psi\sin\theta\sin\phi \\ +\cos\psi\cos\phi\end{array} & \begin{array}{c}\sin\psi\sin\theta\cos\phi \\ -\cos\psi\sin\phi\end{array} \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

<u>Problems</u>

1. With the aid of a diagram showing a generalized set of aircraft body axes, define the parameter notation used in the mathematical modeling of aircraft motion1.
2. In the context of aircraft motion, what are the Euler angles? If the standard right handed aircraft axis set is rotated through pitch θ and yaw ψ angles only, show that the initial vector quantity (x0, y0, z0) is related to the transformed vector quantity (x, y, z) as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

3. Define the span, gross area, aspect ratio and mean aerodynamic chord of an aircraft wing.

4. Distinguish between the center of pressure and the aerodynamic centre of an aerofoil. Explain why the pitching moment about the quarter chord point of an aerofoil is nominally constant in subsonic flight.

3.2 Degree of stability

The condition for stable trim at incidence **αe** may be expressed:

$$Cm = 0$$

and $$dCm/ d\alpha < 0$$

This is usually an acceptable approximation for subsonic aircraft.
However, this argument becomes increasingly inappropriate with increasing Mach number.
The rather more complex analysis by Gates and Lyon (1944) takes speed effects into account and defines **a general requirement for longitudinal static stability** as:

$dC_m/ dC_L < 0$ this because α and CL are linearly related by the lift curve slope

$$\mathbf{a = dC_L/d\alpha.}$$

In a similar way the conditions for lateral–directional static stability may be deduced as

$$dCl/ d\varphi < 0 \quad \text{and} \quad dCn /d\beta > 0$$

where Cl and Cn are rolling moment and yawing moment coefficients respectively and φ and β are roll angle and sideslip angle respectively

3.3 Equations of Motions

An aircraft has six degrees of freedom (if it is assumed to be rigid), which means is has six paths it is free to follow: it can move forward, sideways, and down; and it can rotate about its axes with yaw, pitch, and roll. In order to describe the state of a system that has six degrees of freedom, values for six variables (unknowns) are necessary. To solve for these six unknowns, six simultaneous equations are necessary. For an aircraft, these are known as the aircraft equations of motion.

The derivation of the equations, however, follows a very simple pattern starting from Newton's second law for translational and rotational motions. Newton's second law for translational motions is

$$\bar{F} = \frac{d}{dt} (m \bar{V})$$

where F is the sum of the externally applied forces and mV is linear momentum. Newton's second law for rotational motions is

$$\bar{G} = \frac{d}{dt}(\bar{H})$$

where G is the sum of the externally applied moments and H is angular momentum.

In order to derive the equations of motion, each side of Newton's equations are expanded to yield the following six nonlinear differential equations:

$$\text{Longitudinal} \begin{cases} F_x = m(\dot{U} + QW - RV) \\ F_z = m(\dot{W} + PV - QU) \\ G_y = \dot{Q} I_y - PR(I_z - I_x) + (P^2 - R^2) I_{xz} \end{cases}$$

$$\text{Lateral-Directional} \begin{cases} F_y = m(\dot{V} + RU - PW) \\ G_x = \dot{P} I_x + QR(I_z - I_y) - (\dot{R} + PQ) I_{xz} \\ G_z = \dot{R} I_z + PQ(I_y - I_x) + (QR - \dot{P}) I_{xz} \end{cases}$$

The Left-Hand Side (LHS) of these equations represent the applied forces and moments on the aircraft while the Right-Hand Side (RHS) stands for the aircraft's response to these forces and moments.

Small perturbation theory will be used to linearize these equations so they can be solved.

The equations will also be used to derive aircraft transfer functions which will be a fundamental part of the mathematical modeling of the aircraft and its control system in later Lectures.

3.3.1 Derivation of the right hand side (rhs) of the equations of motion

ASSUMPTION;

1. The aircraft is a rigid body.
2. The earth and atmosphere are fixed in inertial space.
3. ASSUMPTION; Mass (m) is constant (dm/dt =0).
4. In addition, most motion of interest in stability and control takes place in a relatively short time.

3.3.2 Translational Force Relations: The vector equation for the aircraft translation from Newton's second law

$$\bar{F} = \left.\frac{d(m\bar{V}_T)}{dt}\right|_{XYZ} \qquad (1)$$

where V_T is the true velocity of the aircraft. Figure below shows how this vector changes in both magnitude and direction with respect to the xyz (body) and XYZ (fixed earth) axes.

From vector analysis, the derivative of the velocity VT in the inertial (fixed earth) coordinate system is related to the derivative of V in the body axis system through the relationship

$$\left.\frac{d\bar{V}_T}{dt}\right|_{XYZ} = \left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T \qquad (2)$$

$$\bar{F} = m\left[\left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T \right] \qquad (3)$$

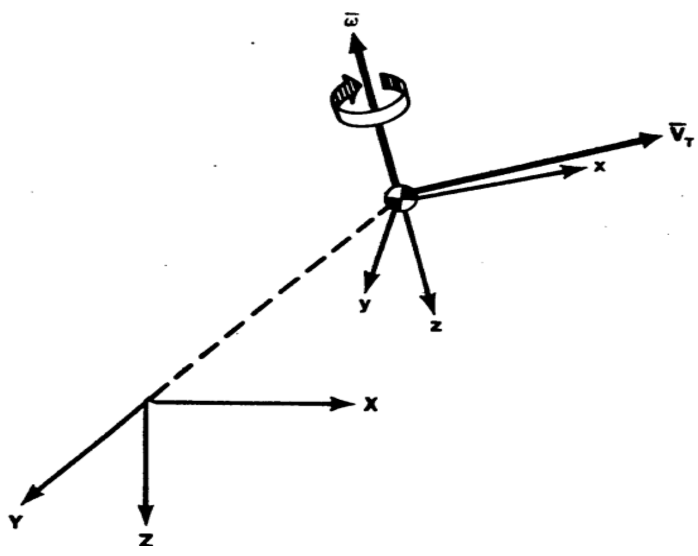
$\mathbf{V_T}$ and $\boldsymbol{\omega}$ are two of the four vectors used in the equations of motion to describe the vehicle motion (F and G are the other two).

They are defined as follows: $\mathbf{V_T = Ui + Vj + Wk}$ **(4)**

where **U** = forward velocity, **V** = **side velocity, W** = vertical velocity

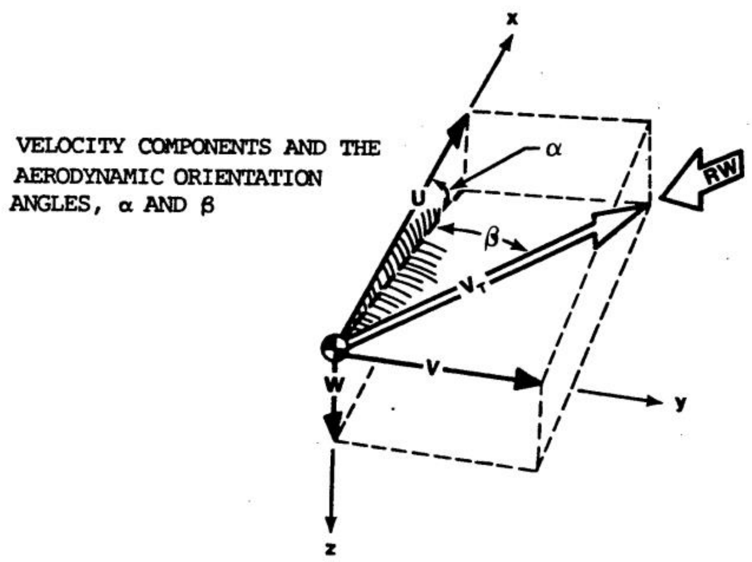
and $\boldsymbol{\omega} = Pi + Qj + Rk$ (5)

where P - roll rate , Q pitch rate, R - yaw rate



TRUE VELOCITY IN
BODY AND FIXED EARTH AXES

The relationship of the true velocity and its components to α and β and the body axis coordinate system is shown in Figure below:



$$\sin\alpha = \frac{W}{V_T\cos\beta}$$

If ß is small (assumption), then **cos ß = 1**,

$$\sin\alpha \simeq \frac{W}{V_T}$$

If α is also small (assumption), then

$$\alpha \simeq \frac{W}{V_T}$$

For angle of sideslip $\sin\beta = \frac{V}{VT} \Rightarrow \beta = \frac{V}{VT}$ **, for β is small.**

Using equations (3) and (4) the translational equation (3) can now be written in component form as

$$\bar{F} = m\left[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ U & V & W \end{vmatrix}\right] \tag{6}$$

Expanding

$$\bar{F} = m\,[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + (QW - RV)\bar{i} - (PW - RU)\bar{j} + (PV - QU)\bar{k}]$$

Rearranging

$$\bar{F} = m\,[(\dot{U} + QW - RV)\bar{i} + (\dot{V} + RU - PW)\bar{j} + (\dot{W} + PV - QU)\bar{k}]$$

In component form, the sum of forces in the body axis system is

$$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k} \tag{7}$$

this results in three component translational equations:

$$F_x = m(\dot{U} + QW - RV)$$

$$F_y = m(\dot{V} + RU - PW)$$

$$F_z = m(\dot{W} + PV - QU) \qquad (8)$$

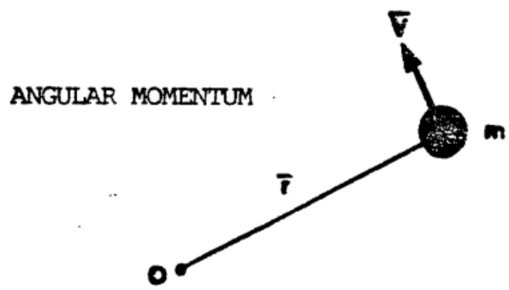
3.3.3 Rotational Equations

Once again from Newton's second law

$$\bar{G} = \left.\frac{d(\bar{H})}{dt}\right|_{XYZ} \qquad (9)$$

This equation states the change in angular momentum, H, is equal to the applied moments, G.

Angular moment; can be thought of as linear momentum with a moment arm included. Consider a ball swinging on the end of a string, at any instant of time, as shown in Figure.



The linear momentum of this system would be:

Linear Momentum = mV

Angular momentum is defined as H, where H = r X Linear Momentum and,

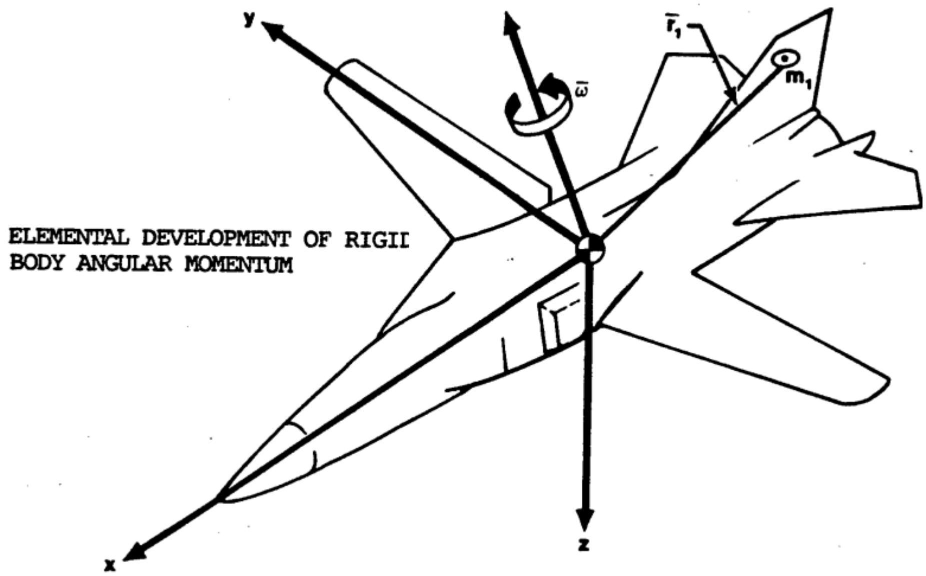
since in the example of Figure above, the angle between r and V is 90 degrees, the magnitude of the angular momentum is **mrV**.

Just as a force F changes linear momentum, (F = d(mV)/dt), a moment G will change angular momentum (G = dH/dt). A moment is related to a force in the same manner that angular momentum is related to linear momentum:

$$\overline{\text{Moment}} = \bar{r} \times \overline{\text{Force}}$$

$$\overline{\text{Angular Momentum}} = \bar{r} \times \overline{\text{Linear Momentum}} \qquad (10)$$

In order for us to determine the angular momentum of the aircraft, consider a small element of mass **m1**, somewhere in the aircraft, a distance **r1** from the cg.



The angular momentum of **m1** is

$$\mathbf{H_{m1} = r1 \; X \; mV1 = m1 \; (r1 \; XV1)}$$

and $$\bar{V}_1 = \left.\frac{d\bar{r}_1}{dt}\right|_{XYZ} \quad \text{(i.e., in the inertial coordinate system)}$$

Again from vector analysis, the rate of change of the radius vector r can be related to the body axis system (xyz) by

$$\bar{V}_1 = \left.\frac{d\bar{r}_1}{dt}\right|_{XYZ} = \left.\frac{d\bar{r}_1}{dt}\right|_{xyz} + \bar{\omega} \; X \; \bar{r}_1$$

since the aircraft is a rigid body **r** does not change with time (assuming no aeroelastic effects). Therefore, the first term can be excluded, and the inertial velocity of the element m, is

$$\mathbf{V1 = \omega \; x \; r1}$$

Hence $$\mathbf{H_{m1} = = m1 \; [r1 \; X \; (\omega \; X \; r1)]} \qquad (11)$$

This is the angular momentum of the elemental mass m1 . In order to find the angular momentum of the whole aircraft, we integrate over the aircraft volume (V). $\boldsymbol{\rho_A}$ is the mass density of the aircraft.

$$\bar{H} = \int_V \rho_A [\bar{r} \times (\bar{\omega} \times \bar{r})]\, dV$$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then
$$\bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ x & y & z \end{vmatrix}$$

The determinant can be expanded to give

$$\bar{\omega} \times \bar{r} = (Qz - Ry)\bar{i} + (Rx - Pz)\bar{j} + (Py - Qx)\bar{k}$$

So the components of **H** are

$$H_x = \int_V \rho_A y(Py - Qx)dV - \int_V \rho_A z(Rx - Pz)dV$$
$$H_y = \int_V \rho_A z(Qz - Ry)dV - \int_V \rho_A x(Py - Qx)dV$$
$$H_z = \int_V \rho_A x(Rx - Pz)dV - \int_V \rho_A y(Qz - Ry)dV$$

Rearranging the equations

$$H_x = P \int_V \rho_A (y^2 + z^2)dV - Q \int_V \rho_A xydV - R \int_V \rho_A xzdV$$

$$H_y = Q \int_V \rho_A (x^2 + z^2)dV - R \int_V \rho_A yzdV - P \int_V \rho_A xydV$$

$$H_z = R \int_V \rho_A (x^2 + y^2)dV - P \int_V \rho_A xzdV - Q \int_V \rho_A yzdV \qquad \textbf{(12)}$$

The integrals are now recognizable as moments and products of inertia. The moments of inertia are defined as

$$I_x = \int_V \rho_A (y^2 + z^2)dV$$
$$I_y = \int_V \rho_A (x^2 + z^2)dV$$
$$I_z = \int_V \rho_A (x^2 + y^2)dV \qquad (13)$$

The products of inertia are defined as

$$I_{xy} = I_{yx} = \int_V \rho_A xy\, dV$$

$$I_{yz} = I_{zy} = \int_V \rho_A yz\, dV$$

$$I_{xz} = I_{zx} = \int_V \rho_A xz\, dV \qquad (14)$$

Products of inertia are measures of symmetry. They are zero for views having a plane of symmetry. Substituting into equations (12), we find the angular momentum of a rigid body is

$$\bar{H} = H_x \bar{i} + H_y \bar{j} + H_z \bar{k}$$

So that

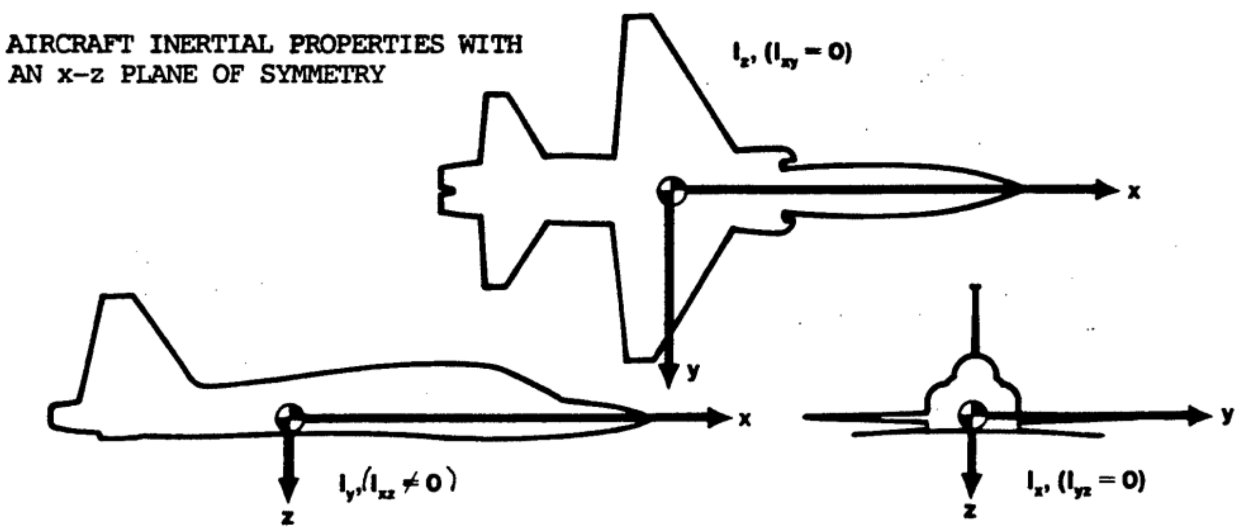
$$H_x = PI_x - QI_{xy} - RI_{xz}$$

$$H_y = QI_y - RI_{yz} - PI_{xy}$$

$$H_z = RI_z - PI_{xz} - QI_{yz} \qquad (15)$$

ASSUMPTION; The xz-plane is a plane of symmetry. This causes two products of inertia, $\mathbf{I_{xy}}$ and $\mathbf{I_{yz}}$ to be zero.

$$\mathbf{H = (PI_x - RI_{xz})\, i + QI_y\, j + (RI_z - PI_{xz})\, k} \qquad \mathbf{(16)}$$



The equation for angular momentum can now be substituted into the moment equation. Remember

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{XYZ}$$

applies only with respect to inertial space. Expressed in the fixed body axis system, the equation becomes:

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{xyz} + \bar{\omega} \times \bar{H}$$

$$\text{which is} \qquad \bar{G} = \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ H_x & H_y & H_z \end{vmatrix}$$

Remember, for a symmetric aircraft,

$$\mathbf{H = (PIxx - RIxz)\, i + QI_y\, j + (RI_z - PI_{xz})\, k}$$

Since the body axis system is used, the moments of inertia and the products of inertia are constant. Therefore, by differentiating and substituting, the moment equation becomes

$$\bar{G} = (\dot{P}I_x - \dot{R}I_{xz})\,\bar{i} + \dot{Q}I_y\,\bar{j} + (\dot{R}I_z - \dot{P}I_{xz})\,\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ (PI_x - RI_{xz}) & QI_y & (RI_z - PI_{xz}) \end{vmatrix} \quad \mathbf{(17)}$$

Therefore, the rotational component equations are

$$\begin{aligned} G_x &= \dot{P}I_x + QR\,(I_z - I_y) - (\dot{R} + PQ)\, I_{xz} \\ G_y &= \dot{Q}I_y - PR\,(I_z - I_x) + (P^2 - R^2)\, I_{xz} \\ G_z &= \dot{R}I_z + PQ\,(I_y - I_x) + (QR - \dot{P})\, I_{xz} \end{aligned} \quad \mathbf{(18)}$$

This completes the development of the RHS of the six equations (equations 8, and 18)

3.3.4 Derivation of the LHS of the equations of motion

The equations of motion relate the vehicle motion to the applied forces and moments:

<u>LHS</u> <u>RHS</u>

Applied Forces and Moments = Observed Vehicle Motion

$$\begin{aligned} F_x &= m(\dot{U} + QW - PV) \\ G_x &= \dot{P}I_x + QR(I_z - I_y) - (\dot{R} + PQ)I_{yz} \end{aligned}$$

etc.

The RHS of each of these six equations has been completely expanded in terms of easily measured quantities. The LHS must also be expanded in terms of convenient variables. In order to do this, we must be able to relate the orientation of the body axes (xyz) to the moving earth axes (XYZ). This is done through the use of Euler angles. The moving earth axis system is used because we will be concerned with the orientation of the aircraft with respect to the earth and not its position (location of the cg) with respect to the earth.

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\Phi & C_\theta S_\Phi \\ 0 & -S_\Phi & C_\theta C_\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

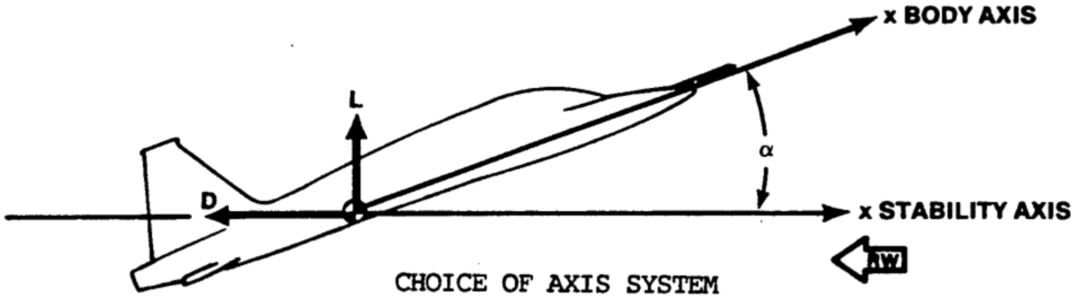
With these equations it is now possible to transform the equations of motion written in body axis terms (U, V, W, P, Q, and R) in terms of the motion seen in the inertial (earth axis) system (u, V, W, ψ, θ, and ϕ).

In general, the applied forces and moments on the LHS can be broken up according to the sources shown below.

3.3.5 Aerodynamic Forces And Moments: They are most important forces and moments on the LHS of the equation are the aerodynamic terms. Unfortunately, they are also the most complex. As a result, certain simplifying assumptions are made, and several of the smaller terms are arbitrarily excluded to simplify the analysis.

$$\mathbf{Fx = L \sin \alpha - D \cos \alpha} \qquad \mathbf{(22)}$$

		SOURCE					
		Aero-dynamic	Direct Thrust	Gravity	Gyro-Scopic	Other	
LONGITUDINAL	F_x	X_A	X_T	X_g	0	X_{oth}	$= m\dot{U} + \cdots$ (19a)
	F_z	Z_A	Z_T	Z_g	0	Z_{oth}	$= m\dot{W} + \cdots$ (19b)
	G_y	M_A	M_T	0	M_{gyro}	M_{oth}	$= \dot{Q}I_y + \cdots$ (20a)
LATERAL-DIRECTIONAL	F_y	Y_A	Y_T	Y_g	0	Y_{oth}	$= m\dot{V} + \cdots$ (20b)
	G_x	L_A	L_T	0	L_{gyro}	L_{oth}	$= \dot{P}I_x + \cdots$ (21a)
	G_z	N_A	N_T	0	N_{gyro}	N_{oth}	$= \dot{R}I_z + \cdots$ (21b)



CHOICE OF AXIS SYSTEM

Notice that if the forces were summed along the x stability axis (Figure), it would be **Fx = - D** **(23)**

A small angle assumption enables us to do this: cos a = 1 sin a = 0

Using this assumption, equation **22** reduces to equation **23**

It should be noted that lift and drag are defined to be positive as illustrated. Thus these quantities have a negative sense with respect to the stability axis system. The aerodynamic terms will be developed using the stability axis system so that the equations assume the form,

$$\text{"DRAG"} \quad -D + X_T + X_g + X_{oth} = m\dot{U} + - - - - - \qquad (24)$$

$$\text{"LIFT"} \quad -L + Z_T + Z_g + Z_{oth} = m\dot{W} + - - - - - \qquad (25)$$

$$\text{"PITCH"} \quad M_A + M_T + M_{gyro} + M_{oth} = \dot{Q}I_y + - - - - - \qquad (26)$$

$$\text{"SIDE"} \quad Y_A + Y_T + Y_g + Y_{oth} = m\dot{V} + - - - - - \qquad (28)$$

$$\text{"ROLL"} \quad L_A + L_T + L_{gyro} + L_{oth} = \dot{P}I_x + - - - - - \qquad (29)$$

$$\text{"YAW"} \quad N_A + N_T + N_{gyro} + N_{oth} = \dot{R}I_z + - - - - - \qquad (30)$$

Expansion of Aerodynamic Terms.

A stability and control analysis is concerned with how a vehicle responds to perturbation inputs. For instance, up elevator should cause the nose to come up; or for turbulence caused sideslip, the aircraft should realign itself with the relative wind. Intuitively, the aerodynamic terms have the most effect on the resulting motion of the aircraft.

The small perturbation theory is based on a simple technique used for linearizing a set of differential equations. In aircraft flight dynamics, the aerodynamic forces and moments are assumed to be functions of the instantaneous values of the perturbation velocities, control deflections, and of their derivatives.

They are obtained in the form of a Taylor series in these variables, and the expressions are linearized by excluding all higher-order terms. Analysis of certain unsteady motions may therefore require consideration of the time derivatives of the variables listed above. In other words:

		VARIABLE	FIRST DERIVATIVE	SECOND DERIVATIVE
D, L, M_A, Y_A, L_A, N_A	Are a Function of	$U \quad \alpha \quad \beta$	$\dot{U} \quad \dot{\alpha} \quad \dot{\beta}$	$\ddot{U} \quad \ddot{\alpha} \quad \ddot{\beta}$
		$P \quad Q \quad R$	$\dot{P} \quad \dot{Q} \quad \dot{R}$	$\ddot{P} \quad \ddot{Q} \quad \ddot{R}$
		$\delta_e \quad \delta_a \quad \delta_r$	$\dot{\delta}_e \quad \dot{\delta}_a \quad \dot{\delta}_r$	$\ddot{\delta}_e \quad \ddot{\delta}_a \quad \ddot{\delta}_r$
		$\rho \quad M \quad R_e \quad T$	assumed constant	------

The variables are considered to consist of some equilibrium value plus an incremental change, called the "perturbed value." The notation for these perturbed values is usually lower case. For example, $P = P_0 + p$, $U = U_0 + u$.

In summary, the small disturbance assumption is applied in three steps:

1. Assuming an initial (equilibrium) condition.
2. Assuming vehicle motion consists of small perturbations about this condition .
3. Using a first order Taylor series expansion

The vehicle motion can be thought of as two independent (decoupled) motions, each of which is a function only of the variables shown below.

1. Longitudinal Motion $(D, L, M_A) = f(U, \alpha, \dot{\alpha}, Q, \delta_e)$

2. Lateral-Directional Motion $(Y_A, L_A, N_A) = f(\beta, \dot{\beta}, P, R, \delta_a, \delta_r)$

Initial Conditions:

Steady Flight. Motion with zero rates of change of the linear and angular velocity components, i.e.,

$$\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = 0.$$

Straight Flight. Motion with zero angular velocity components, P, Q, and R = 0.

Symmetric Flight. Motion in which the vehicle plane of symmetry remains fixed in space throughout the maneuver. The unsymmetric variables **P, R, V, , and ß are all zero** in symmetric flight.

Some symmetric flight conditions are wings-level dives, climbs, and pull-ups with no sideslip.

S**teady straight symmetric flight**, the aircraft is assumed to be flying wings level with all components of velocity zero except Uo and Wo. Therefore, with reference to the body axis

$$V_T \simeq U_0 \simeq \text{constant}$$
$$W_0 \simeq \text{small constant} \therefore \alpha_0 \simeq \text{small constant}$$
$$V_0 \simeq 0 \therefore \beta_0 \simeq 0$$
$$P_0 \simeq Q_0 \simeq R_0 \simeq 0$$

We have already found that the equations of motion simplify considerably when the stability axis is used as the reference axis. This idea will again be employed and the final set of boundary conditions will result. This, therefore, is another

$$\underline{\text{ASSUMPTION:}} \qquad V_T \simeq U_0 \simeq \text{constant}$$

$$W_0 \simeq 0 \quad \therefore \quad \alpha_0 \simeq 0$$

$$V_0 \simeq 0 \quad \therefore \quad \beta_0 \simeq 0$$

$$P_0 \simeq Q_0 \simeq R_0 \simeq 0$$

$$(\rho,\ M,\ Re,\ \text{aircraft configuration}) = \text{constant}$$

Expansion By Taylor Series. An approximate solution is found by linearizing these equations using a Taylor Series expansion and neglecting higher ordered terms. The resulting Taylor Series expansion has the form

$$f(U)\Big|_{U_0} + \Delta U = f(U_0) + \frac{\partial f\ (U_o)}{\partial U}\Delta U + \frac{1}{2!}\frac{\partial^2 f\ (U_o)}{\partial U^2}(\Delta U)^2 + \frac{1}{3!}\frac{\partial^3 f\ (U_o)}{\partial U^3}(\Delta U)^3$$

$$+ \ldots + \frac{1}{n!}\frac{\partial^n f\ (U_o)}{\partial U^n}(\Delta U)^n \tag{31}$$

for small perturbed values of U, the function can be accurately approximated by

$$f(U)\Big|_{U_0} + \Delta U = f(U_0) + \frac{\partial f(U_o)}{\partial U}\Delta U$$

In small perturbation theory, each of the variables is expressed as the sum of an initial value plus a small perturbed value. For example

$$U = U_0 + u,\ \text{where } u = \Delta U = U - U_0$$

and

$$\frac{\partial u}{\partial U} = \frac{\partial (U - U_0)}{\partial U} = \frac{\partial U}{\partial U} - \cancelto{0}{\frac{\partial U_0}{\partial U}} = 1$$

Therefore

$$\frac{\partial L}{\partial U} = \frac{\partial L}{\partial u}\frac{\partial u}{\partial U} = \frac{\partial L}{\partial u}$$

and

$$\Delta U = u$$

$$\frac{\partial L}{\partial U}\Delta U = \frac{\partial L}{\partial u}u \qquad \frac{\partial L}{\partial Q}\Delta Q = \frac{\partial L}{\partial q}q$$

And all other terms follow. We also elect to let $\alpha = \Delta\alpha$, $\alpha'' = \Delta\alpha''$ and $\delta_e = \Delta\delta_e$. Dropping higher order terms involving u^2, q^2, etc.,

Lateral-directional motion: is a function of **ß, ß, P, R, δ_a , δ_r** and can be handled in a similar manner. For example, the aerodynamic terms for rolling moment become

$$L_A = L_{A_0} + \frac{\partial L_A}{\partial \beta}\beta + \frac{\partial L_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial L_A}{\partial p}p + \frac{\partial L_A}{\partial r}r + \frac{\partial L_A}{\partial \delta_a}\delta_a + \frac{\partial L_A}{\partial \delta_r}\delta_r \quad \textbf{(32)}$$

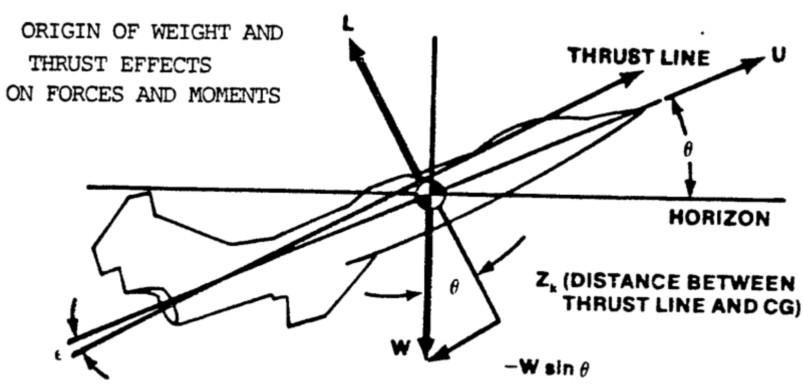
This development can be applied to all of the aerodynamic forces and moments. The equations resulting from this development can now be substituted into the LHS of the equations of motion.

3.3.6 Direct Thrust Forces and Moments:

Since thrust does not always pass through the cg, its effects on both the force and moment equations must be considered (Figure). The component of the thrust vector along the x-axis is

$$X_T = T \cos \varepsilon$$

The component of the thrust vector along the z-axis is **$Z_T = -T \sin \varepsilon$**



The pitching moment component is, **$M_T = T(Z_k) = T Z_k$,**

where Z_k is the perpendicular distance from the thrust line to the cg and ε is the thrust angle.

For small disturbances, $T = T(U, \delta_{RPM})$

$$T = T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}$$

Thrust effects will be considered in the longitudinal equations only since the thrust vector is normally in the vertical plane of symmetry and does not affect the lateral-directional motion.

X_T and Z_T will be referred to as "drag due to thrust" and "lift due to thrust. ($\alpha = 0$ assumption in order to use the stability axes). They are components of thrust in the drag (x) and lift (z) directions. Thus:

$$X_T = (T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}) (\cos \varepsilon)$$

$$Z_T = (T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}) (\sin \varepsilon)$$

$$M_T = (T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}) (Z_k)$$

3.3.7 **Gravity Forces:** Gravity acts through the eg of an aircraft and, as a result, has no effect on the aircraft moments.

Effect of weight on the x-axis. **Xg = -mg sin θ**

Since m and g are considered constant, **θ** is the only pertinent variable. using the small perturbation assumption as:

$$X_g = X_{g_0} + \frac{\partial X_g}{\partial \theta} \theta \qquad (X_{g_0} = \text{equilibrium condition of } X_g)$$

Xg will be referred to as drag due to weight, **(D_{wt}).**

$$D_{wt} = D_{0_{wt}} + \frac{\partial D_{wt}}{\partial \theta} \theta$$

Likewise the z-force can be expressed as negative lift due to weight (L), and the expanded term becomes

$$L_{wt} = L_{0_{wt}} + \frac{\partial L_{wt}}{\partial \theta} \theta$$

The effect of gravity on side force depends solely on bank angle (ϕ),assuming small **θ**. Therefore,

$$Y_{wt} = Y_{0_{wt}} + \frac{\partial Y_{wt}}{\partial \phi} \phi$$