3.3.4 Derivation of the LHS of the equations of motion

The equations of motion relate the vehicle motion to the applied forces and moments:

<u>LHS</u>		<u>RHS</u>
Applied Forces and Moments	=	Observed Vehicle Motion

$$F_x = m(\dot{U} + QW - PV)$$
$$G_x = \dot{P}I_x + QR(I_z - I_y) - (\dot{R} + PQ)I_{yz}$$
etc.

The RHS of each of these six equations has been completely expanded in terms of easily measured quantities. The LHS must also be expanded in terms of convenient variables. In order to do this, we must be able to relate the orientation of the body axes (xyz) to the moving earth axes (XYZ). This is done through the use of Euler angles. The moving earth axis system is used because we will be concerned with the orientation of the aircraft with respect to the earth and not its position (location of the cg) with respect to the earth.

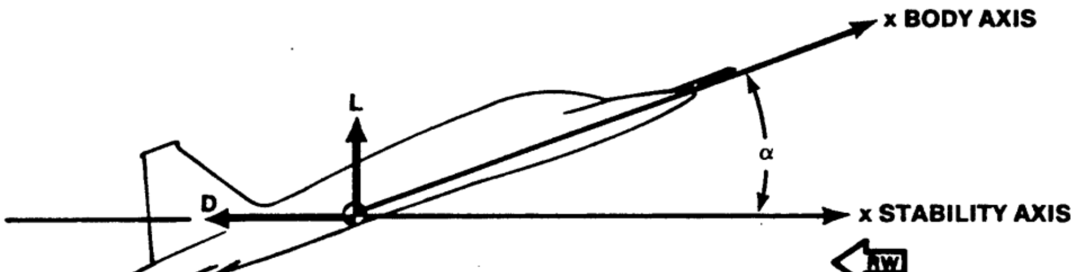
$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\Phi & C_\theta S_\Phi \\ 0 & -S_\Phi & C_\theta C_\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

With these equations it is now possible to transform the equations of motion written in body axis terms (U, V, W, P, Q, and R) in terms of the motion seen in the inertial (earth axis) system (u, V, W, ψ, θ, and ϕ).

In general, the applied forces and moments on the LHS can be broken up according to the sources shown below.

3.3.5 Aerodynamic Forces And Moments: They are most important forces and moments on the LHS of the equation are the aerodynamic terms. Unfortunately, they are also the most complex. As a result, certain simplifying assumptions are made, and several of the smaller terms are arbitrarily excluded to simplify the analysis.

$$\mathbf{Fx = L \sin \alpha - D \cos \alpha} \qquad \mathbf{(22)}$$



CHOICE OF AXIS SYSTEM

		SOURCE					
		Aero-dynamic	Direct Thrust	Gravity	Gyro-Scopic	Other	
LONGITUDINAL	F_x	X_A	X_T	X_g	0	X_{oth}	$= m\dot{U} + \cdots$ (19a)
	F_z	Z_A	Z_T	Z_g	0	Z_{oth}	$= m\dot{W} + \cdots$ (19b)
	G_y	M_A	M_T	0	M_{gyro}	M_{oth}	$= \dot{Q}I_y + \cdots$ (20a)
LATERAL-DIRECTIONAL	F_y	Y_A	Y_T	Y_g	0	Y_{oth}	$= m\dot{V} + \cdots$ (20b)
	G_x	L_A	L_T	0	L_{gyro}	L_{oth}	$= \dot{P}I_x + \cdots$ (21a)
	G_z	N_A	N_T	0	N_{gyro}	N_{oth}	$= \dot{R}I_z + \cdots$ (21b)

Notice that if the forces were summed along the x stability axis (Figure), it would be $\mathbf{Fx = - D}$ **(23)**

A small angle assumption enables us to do this: cos α = 1 sin α = 0

Using this assumption, equation **22** reduces to equation **23**

It should be noted that lift and drag are defined to be positive as illustrated. Thus these quantities have a negative sense with respect to the stability axis system. The aerodynamic terms will be developed using the stability axis system so that the equations assume the form,

$$\text{"DRAG"} \quad -D + X_T + X_g + X_{oth} = m\dot{U} + \text{-----} \quad (24)$$

$$\text{"LIFT"} \quad -L + Z_T + Z_g + Z_{oth} = m\dot{W} + \text{-----} \quad (25)$$

$$\text{"PITCH"} \quad M_A + M_T + M_{gyro} + M_{oth} = \dot{Q}I_y + \text{-----} \quad (26)$$

$$\text{"SIDE"} \quad Y_A + Y_T + Y_g + Y_{oth} = m\dot{V} + \text{-----} \quad (28)$$

$$\text{"ROLL"} \quad L_A + L_T + L_{gyro} + L_{oth} = \dot{P}I_x + \text{-----} \quad (29)$$

$$\text{"YAW"} \quad N_A + N_T + N_{gyro} + N_{oth} = \dot{R}I_z + \text{-----} \quad (30)$$

Expansion of Aerodynamic Terms. A stability and control analysis is concerned with how a vehicle responds to perturbation inputs. For instance, up elevator should cause the nose to come up; or for turbulence caused sideslip, the aircraft should realign itself with the relative wind. Intuitively, the aerodynamic terms have the most effect on the resulting motion of the aircraft.

The small perturbation theory is based on a simple technique used for linearizing a set of differential equations. In aircraft flight dynamics, the aerodynamic forces and moments are assumed to be functions of the instantaneous values of the perturbation velocities, control deflections, and of their derivatives.

They are obtained in the form of a Taylor series in these variables, and the expressions are linearized by excluding all higher-order terms. Analysis of certain unsteady motions may therefore require consideration of the time derivatives of the variables listed above. In other words:

		VARIABLE	FIRST DERIVATIVE	SECOND DERIVATIVE
D, L, M_A, Y_A, L_A, N_A	Are a Function of	$U \quad \alpha \quad \beta$	$\dot{U} \quad \dot{\alpha} \quad \dot{\beta}$	$\ddot{U} \quad \ddot{\alpha} \quad \ddot{\beta}$
		$P \quad Q \quad R$	$\dot{P} \quad \dot{Q} \quad \dot{R}$	$\ddot{P} \quad \ddot{Q} \quad \ddot{R}$
		$\delta_e \quad \delta_a \quad \delta_r$	$\dot{\delta}_e \quad \dot{\delta}_a \quad \dot{\delta}_r$	$\ddot{\delta}_e \quad \ddot{\delta}_a \quad \ddot{\delta}_r$
		$\rho \quad M \quad R_e \quad T$	assumed constant	------

The variables are considered to consist of some equilibrium value plus an incremental change, called the "perturbed value." The notation for these perturbed values is usually lower case. For example, $P = P_0 + p$, $U = U_0 + u$.

In summary, the small disturbance assumption is applied in three steps:

1. Assuming an initial (equilibrium) condition.
2. Assuming vehicle motion consists of small perturbations about this condition .
3. Using a first order Taylor series expansion

The vehicle motion can be thought of as two independent (decoupled) motions, each of which is a function only of the variables shown below.

1. <u>Longitudinal Motion</u> $(D, L, M_A) = f(U, \alpha, \dot{\alpha}, Q, \delta_e)$ (30B)

2. <u>Lateral-Directional Motion</u> $(Y_A, L_A, N_A) = f(\beta, \dot{\beta}, P, R, \delta_a, \delta_r)$ (30C)

Initial Conditions:

Steady Flight. Motion with zero rates of change of the linear and angular velocity components, i.e.,

$$\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = 0.$$

Straight Flight. Motion with zero angular velocity components, P, Q, and R = 0.

Symmetric Flight. Motion in which the vehicle plane of symmetry remains fixed in space throughout the maneuver. The unsymmetric variables **P, R, V, , and ß are all zero** in symmetric flight.

Some symmetric flight conditions are wings-level dives, climbs, and pull-ups with no sideslip.

Steady straight symmetric flight, the aircraft is assumed to be flying wings level with all components of velocity zero except Uo and Wo. Therefore, with reference to the body axis

$$V_T \simeq U_0 \simeq \text{constant}$$

$$W_0 \simeq \text{small constant} \therefore \alpha_0 \simeq \text{small constant}$$

$$V_0 \simeq 0 \therefore \beta_0 \simeq 0$$

$$P_0 \simeq Q_0 \simeq R_0 \simeq 0$$

We have already found that the equations of motion simplify considerably when the stability axis is used as the reference axis. This idea will again be employed and the final set of boundary conditions will result. This, therefore, is another

ASSUMPTION:

$$V_T \simeq U_0 \simeq \text{constant}$$

$$W_0 \simeq 0 \quad \therefore \quad \alpha_0 \simeq 0$$

$$V_0 \simeq 0 \quad \therefore \quad \beta_0 \simeq 0$$

$$P_0 \simeq Q_0 \simeq R_0 \simeq 0$$

$$(\rho,\ M,\ Re,\ \text{aircraft configuration}) = \text{constant}$$

Expansion By Taylor Series. An approximate solution is found by linearizing these equations using a Taylor Series expansion and neglecting higher ordered terms. The resulting Taylor Series expansion has the form

$$f(U)\Big|_{U_0 + \Delta U} = f(U_0) + \frac{\partial f(U_o)}{\partial U}\Delta U + \frac{1}{2!}\frac{\partial^2 f(U_o)}{\partial U^2}(\Delta U)^2 + \frac{1}{3!}\frac{\partial^3 f(U_o)}{\partial U^3}(\Delta U)^3$$

$$+ \ldots + \frac{1}{n!}\frac{\partial^n f(U_o)}{\partial U^n}(\Delta U)^n \qquad (31)$$

for small perturbed values of U, the function can be accurately approximated by

$$f(U)\Big|_{U_0 + \Delta U} = f(U_0) + \frac{\partial f(U_o)}{\partial U}\Delta U$$

In small perturbation theory, each of the variables is expressed as the sum of an initial value plus a small perturbed value. For example

$$U = U_0 + u, \text{ where } u = \Delta U = U - U_0$$

and

$$\frac{\partial u}{\partial U} = \frac{\partial(U - U_0)}{\partial U} = \frac{\partial U}{\partial U} - \frac{\partial U_0}{\partial U} = 1 \qquad \left(\frac{\partial U_0}{\partial U} \to 0\right)$$

Therefore

$$\frac{\partial L}{\partial U} = \frac{\partial L}{\partial u}\frac{\partial u}{\partial U} = \frac{\partial L}{\partial u}$$

and

$$\Delta U = u$$

$$\frac{\partial L}{\partial U}\,\Delta U \;=\; \frac{\partial L}{\partial u}\,u \qquad\qquad \frac{\partial L}{\partial Q}\,\Delta Q \;=\; \frac{\partial L}{\partial q}\;q$$

And all other terms follow. We also elect to let $\alpha = \Delta\alpha$, $\alpha'' = \Delta\alpha''$ and $\delta_e = \Delta\delta_e$. Dropping higher order terms involving u^2, q^2, etc.,

Lateral-directional motion: is a function of **ß, ß, P, R, δ_a , δ_r** and can be handled in a similar manner. For example, the aerodynamic terms for rolling moment become

$$L_A = L_{A_0} + \frac{\partial L_A}{\partial \beta}\beta + \frac{\partial L_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial L_A}{\partial p}p + \frac{\partial L_A}{\partial r}r + \frac{\partial L_A}{\partial \delta_a}\delta_a + \frac{\partial L_A}{\partial \delta_r}\delta_r \qquad \textbf{(32)}$$

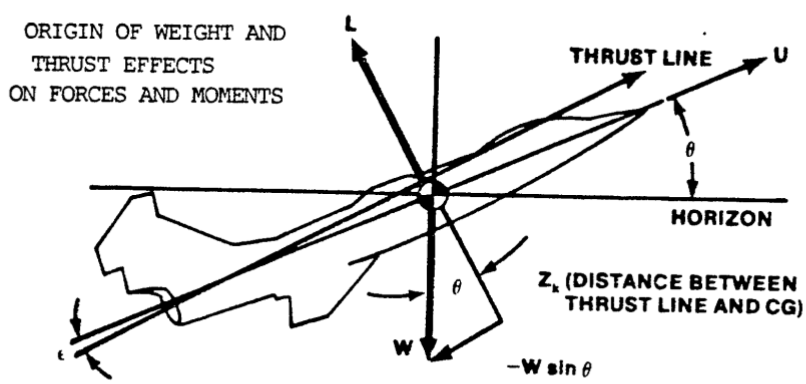
This development can be applied to all of the aerodynamic forces and moments. The equations resulting from this development can now be substituted into the LHS of the equations of motion.

3.3.6 Direct Thrust Forces and Moments:

Since thrust does not always pass through the cg, its effects on both the force and moment equations must be considered (Figure). The component of the thrust vector along the x-axis is

$$X_T = T\cos\varepsilon$$

The component of the thrust vector along the z-axis is $\mathbf{Z_T = -T\sin\mathcal{E}}$



The pitching moment component is, $\mathbf{M_T = T(Z_k) = T\,Z_k}$,

where Z_k is the perpendicular distance from the thrust line to the cg and $\mathcal{E}$ is the thrust angle.

For small disturbances, $T = T\,(U, \delta_{RPM})$

$$T = T_0 + \frac{\partial T}{\partial u}\,u + \frac{\partial T}{\partial \delta_{RPM}}\,\delta_{RPM}$$

Thrust effects will be considered in the longitudinal equations only since the thrust vector is normally in the vertical plane of symmetry and does not affect the lateral-directional motion.

X_T and Z_T will be referred to as "drag due to thrust" and "lift due to thrust. ($\alpha = 0$ assumption in order to use the stability axes). They are components of thrust in the drag (x) and lift (z) directions. Thus:

$$X_T = \left(T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}\right)(\cos \varepsilon)$$

$$Z_T = \left(T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}\right)(\sin \varepsilon)$$

$$M_T = \left(T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}\right)(Z_k) \qquad (33)$$

3.3.7 **Gravity Forces:** Gravity acts through the **cg** of an aircraft and, as a result, has no effect on the aircraft moments.

Effect of weight on the x-axis. **$X_g = -mg \sin \theta$**

Effect of weight on they-axis $X_y = mg \cos\theta \sin\phi$

Effect of weight on the z-axis $X_z = mg \cos\theta \cos\phi$

Since **m** and **g** are considered constant, **θ** is the only pertinent variable. using the small perturbation assumption as:

$$X_g = X_{g_0} + \frac{\partial X_g}{\partial \theta}\theta \qquad (X_{g_0} = \text{equilibrium condition of } X_g)$$

Xg will be referred to as drag due to weight, **(D_{wt}).**

$$D_{wt} = D_{0_{wt}} + \frac{\partial D_{wt}}{\partial \theta}\theta$$

Likewise the z-force can be expressed as negative lift due to weight (L), and the expanded term becomes

$$L_{wt} = L_{0_{wt}} + \frac{\partial L_{wt}}{\partial \theta}\theta$$

The effect of gravity on side force depends solely on bank angle (ϕ),assuming small ϕ. Therefore,

$$Y_{wt} = Y_{0_{wt}} + \frac{\partial Y_{wt}}{\partial \phi} \phi$$

Expanded LHS Equations:

"DRAG" $$-[D_0 + \frac{\partial D}{\partial u}u + \frac{\partial D}{\partial \alpha}\alpha + \frac{\partial D}{\partial \dot{\alpha}}\dot{\alpha} + \frac{\partial D}{\partial q}q + \frac{\partial D}{\partial \delta_e}\delta_e] + [T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}]\ (\cos\ \varepsilon) - [D_{0_{wt}} + \frac{\partial D}{\partial \theta}\theta]$$

"LIFT" $$-[L_0 + \frac{\partial L}{\partial u}u + \frac{\partial L}{\partial \alpha}\alpha + \frac{\partial L}{\partial \dot{\alpha}}\dot{\alpha} + \frac{\partial L}{\partial q}q + \frac{\partial L}{\partial \delta_e}\delta_e] + [-T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}]\ (\sin\ \varepsilon) + [L_{0_{wt}} + \frac{\partial L}{\partial \theta}\theta]$$

"PITCH" $$M_{A_0} + \frac{\partial M_A}{\partial u}u + \frac{\partial M_A}{\partial \alpha}\alpha + \frac{\partial M_A}{\partial \dot{\alpha}}\dot{\alpha} + \frac{\partial M_A}{\partial q}q + \frac{\partial M_A}{\partial \delta_e}\delta_e] + [T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}](Z_k)$$

"SIDE" $$Y_{A_0} + \frac{\partial Y_A}{\partial \beta}\beta + \frac{\partial Y_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial Y_A}{\partial p}p + \frac{\partial Y_A}{\partial r}r + \frac{\partial Y_A}{\partial \delta_a}\delta_a + \frac{\partial Y_A}{\partial \delta_r}\delta_r + Y_{0_{wt}} + \frac{\partial Y}{\partial \phi}\phi$$

"ROLL" $$L_{A_0} + \frac{\partial L_A}{\partial \beta}\beta + \frac{\partial L_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial L_A}{\partial p}p + \frac{\partial L_A}{\partial r}r + \frac{\partial L_A}{\partial \delta_a}\delta_a + \frac{\partial L_A}{\partial \delta_r}\delta_r$$

"YAW" $$N_{A_0} + \frac{\partial N_A}{\partial \beta}\beta + \frac{\partial N_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial N_A}{\partial p}p + \frac{\partial N_A}{\partial r}r + \frac{\partial N_A}{\partial \delta_a}\delta_a + \frac{\partial N_A}{\partial \delta_r}\delta_r$$

RHS in terms of small perturbations:

Start with the RHS of Equation.

$$F_z = m\ (\dot{W} + PV - QU)$$

Substitute the initial plus perturbed values for each variable.

$$F_z = m\ [\dot{W}_0 + \dot{w} + (P_0 + p)\ (V_0 + v) - (Q_0 + q)\ (U_0 + u)]$$

Applying the boundary conditions, (assumptions from subsection 4.6.4.3.3), simplifies the equation to

$$F_z = m\,[\dot{w} + pv - qU]$$

Using this same technique, the set of RHS equations become:

Longitudinal

$$\begin{aligned} &\text{"DRAG":} && m\,(\dot{u} + qw - rv) \\ &\text{"LIFT":} && m\,(\dot{w} + pv - qU) \\ &\text{"PITCH":} && \dot{q}\,I_y - pr\,(I_z - I_x) + (p^2 - r^2)\,I_{xz} \end{aligned} \qquad \textbf{(34)}$$

Lateral-Directional

$$\begin{aligned} &\text{"SIDE":} && m\,(\dot{v} + rU - pw) \\ &\text{"ROLL":} && \dot{p}\,I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\,I_{xz} \\ &\text{"YAW":} && \dot{r}\,I_z + pq\,(I_y - I_x) + (qr - \dot{p})\,I_{xz} \end{aligned} \qquad \textbf{(35)}$$

REDUCTION OF EQUATIONS TO A USABLE FORM

Normalization Of Equations

To put the linearized expressions into a more usable form, each equation is multiplied by a "normalization factor." This factor is different for each equation and is picked to simplify the first term on the RHS of the equation. It is desirable to have the first term of the RHS be either a pure acceleration (u, p, q, or r), or angular rate (a, or ß) and these terms were previously identified in equations (30A) and (30B) as the longitudinal or lateral-directional variables.

$$\alpha \simeq \frac{W}{V_T} \qquad \text{and} \qquad \beta \simeq \frac{V}{V_T}$$

Since we have assumed that $V_T \simeq U_0$ and $V_0 \simeq W_0 \simeq 0$, we have

$$\dot{\alpha} \simeq \frac{\dot{w}}{U_0} \qquad \text{and} \qquad \dot{\beta} \simeq \frac{\dot{v}}{U_0}$$

Equation	Normalizing Factor	First Term is Now Pure Accel/Ang Rate	Units
"DRAG"	$\frac{1}{m}$	$-\frac{D}{m} + \frac{X_T}{m} + - - - - = \dot{u}$ —	$[\frac{ft}{sec^2}]$
"LIFT"	$\frac{1}{mU_0}$	$-\frac{L}{mU_0} + \frac{Z_T}{mU_0} + - - - - = \dot{\alpha} +$ —	$[\frac{rad}{sec}]$
"PITCH"	$\frac{1}{I_y}$	$\frac{M_A}{I_y} + \frac{M_T}{I_y} + - - - - = \dot{q}$ —	$[\frac{rad}{sec^2}]$
"SIDE"	$\frac{1}{mU_0}$	$\frac{Y_A}{mU_0} + \frac{Y_T}{mU_0} + - - - - = \dot{\beta} +$ —	$[\frac{rad}{sec}]$
"ROLL"	$\frac{1}{I_x}$	$\frac{L_A}{I_x} + \frac{L_T}{I_x} + - - - - = \dot{p} +$ —	$[\frac{rad}{sec^2}]$
"YAW"	$\frac{1}{I_z}$	$\frac{N_A}{I_z} + \frac{N_T}{I_z} + - - - - = \dot{r} +$ —	$[\frac{rad}{sec^2}]$

Now, if we introduce the small-disturbance notation into the equations of motion, we can simplify these equations. As an example, consider the X force equation:

$$X - mg \sin \theta = m(\dot{u} + qw - rv)$$

Substituting the small-disturbance variables into this equation yields

$$\begin{aligned} &X_0 + \Delta X - mg \sin(\theta_0 + \Delta\theta) \\ &\quad = m\left[\frac{d}{dt}(u_0 + \Delta u) + (q_0 + \Delta q)(w_0 + \Delta w) - (r_0 + \Delta r)(v_0 + \Delta v)\right] \end{aligned}$$

If we neglect products of the disturbance and assume that

$$w_0 = v_0 = p_0 = q_0 = r_0 = \Phi_0 = \psi_0 = 0$$

then the X equation becomes

$$X_0 + \Delta X - mg \sin(\theta_0 + \Delta\theta) = m\, \Delta\dot{u}$$

$$X_0 + \Delta X - mg(\sin \theta_0 + \Delta\theta \cos \theta_0) = m\, \Delta\dot{u}$$

If all the disturbance quantities are set equal to 0 in these equation, we have the reference flight condition

$$X_0 - mg \sin \theta_0 = 0$$

This reduces the X-force equation to

$$\Delta X - mg \, \Delta\theta \cos \theta_0 = m \, \Delta\dot{u}$$

The force AX is the change in aerodynamic and propulsive force in the x direction and can be expressed by means of a Taylor series in terms of the perturbation variables. If we assume that ΔX is a function only of u, w, **δ_e**, and **δ_T**, then ΔX can be expressed as

$$\Delta X = \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_e} \Delta\delta_e + \frac{\partial X}{\partial \delta_T} \Delta\delta_T$$

where $\partial X/\partial u$, $\partial X/\partial w$, $\partial X/\partial \delta_e$, and $\partial X/\partial \delta_T$, called stability derivatives,

The variables **δ_e**, and **δ_T**, are the change in elevator angle and throttle setting, respectively.

Substituting the expression for AX into the force equation yields:

$$\frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_e} \Delta\delta_e + \frac{\partial X}{\partial \delta_T} \Delta\delta_T - mg \, \Delta\theta \cos \theta_0 = m \, \Delta\dot{u}$$

or on rearranging

$$\left(m \frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial X}{\partial u}\right) \Delta u - \left(\frac{\partial X}{\partial w}\right) \Delta w + (mg \cos \theta_0) \, \Delta\theta = \frac{\partial X}{\partial \delta_e} \Delta\delta_e + \frac{\partial X}{\partial \delta_T} \Delta\delta_T$$

The equation can be rewritten in a more convenient form by dividing through by the mass m:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - X_u\right) \Delta u - X_w \, \Delta w + (g \cos \theta_0) \, \Delta\theta = X_{\delta_e} \Delta\delta_e + X_{\delta_T} \Delta\delta_T$$

where Xu = dX/du/m, X, = dX/dw/m, and so on are aerodynamic derivatives divided by the airplane's mass.

The complete set of linearized equations of motion is presented in Table 3.2. below

Longitudinal equations

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\,\Delta\delta_e + X_{\delta_T}\,\Delta\delta_T$$

$$-Z_u\,\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\,\Delta\delta_e + Z_{\delta_T}\,\Delta\delta_T$$

$$-M_u\,\Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta_e}\,\Delta\delta_e + M_{\delta_T}\,\Delta\delta_T$$

Lateral equations

$$\left(\frac{d}{dt} - Y_v\right)\Delta v - Y_p\,\Delta p + (u_0 - Y_r)\,\Delta r - (g\cos\theta_0)\,\Delta\phi = Y_{\delta_r}\,\Delta\delta_r$$

$$-L_v\,\Delta v + \left(\frac{d}{dt} - L_p\right)\Delta p - \left(\frac{I_{yz}}{I_x}\frac{d}{dt} + L_r\right)\Delta r = L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

$$-N_v\,\Delta v - \left(\frac{I_{xz}}{I_z}\frac{d}{dt} + N_p\right)\Delta p + \left(\frac{d}{dt} - N_r\right)\Delta r = N_{\delta_a}\,\Delta\delta_a + N_{\delta_r}\,\Delta\delta_r$$

Summery

Therefore the resultant components of total force acting on the rigid body are given by

$$m(\dot{U} - rV + qW) = X$$

$$m(\dot{V} - pW + rU) = Y$$

$$m(\dot{W} - qU + pV) = Z$$

where m is the total mass of the body.

Thus the moment equations simplify to the following:

$$I_x\dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) = L$$

$$I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) = M$$

$$I_z\dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) = N$$

Bringing together the total force and moment equations they may be written to include these contributions as follow

$$m(\dot{U} - rV + qW) = X_a + X_g + X_c + X_p + X_d$$
$$m(\dot{V} - pW + rU) = Y_a + Y_g + Y_c + Y_p + Y_d$$
$$m(\dot{W} - qU + pV) = Z_a + Z_g + Z_c + Z_p + Z_d$$

$$I_x\dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) = L_a + L_g + L_c + L_p + L_d$$
$$I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) = M_a + M_g + M_c + M_p + M_d$$
$$I_z\dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) = N_a + N_g + N_c + N_p + N_d$$

Stability derivatives used in equations of equations

$$X_u = \frac{\mathring{X}_u}{\frac{1}{2}\rho V_0 S} = -2C_D - V_0\frac{\partial C_D}{\partial V} + \frac{1}{\frac{1}{2}\rho V_0 S}\frac{\partial \tau}{\partial V} \qquad X_w = \frac{\mathring{X}_w}{\frac{1}{2}\rho V_0 S} = \left(C_L - \frac{\partial C_D}{\partial \alpha}\right)$$

$$Z_w = \frac{\mathring{Z}_w}{\frac{1}{2}\rho V_0 S} = -\left(\frac{\partial C_L}{\partial \alpha} + C_D\right) \qquad Z_u = \frac{\mathring{Z}_u}{\frac{1}{2}\rho V_0 S} = -2C_L - V_0\frac{\partial C_L}{\partial V}$$

$$M_u = \frac{\mathring{M}_u}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = V_0\frac{\partial C_m}{\partial V} \qquad M_w = \frac{\mathring{M}_w}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = \frac{\partial C_m}{\partial \alpha}$$

$$X_q = \frac{\mathring{X}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = -\overline{V}_T\frac{\partial C_{D_T}}{\partial \alpha_T}$$

$$\overline{V}_T = \frac{S_T l_T}{S\overline{\overline{c}}} \qquad Z_q = \frac{\mathring{Z}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = -\overline{V}_T a_1$$

$$M_q = \frac{\mathring{M}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}^2} = -\overline{V}_T\frac{l_T}{\overline{\overline{c}}}a_1 \equiv \frac{l_T}{\overline{\overline{c}}}Z_q \qquad X_{\dot{w}} = \frac{\mathring{X}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}} = -\overline{V}_T\frac{\partial C_{D_T}}{\partial \alpha_T}\frac{d\varepsilon}{d\alpha} \equiv X_q\frac{d\varepsilon}{d\alpha}$$

$$Z_{\dot{w}} = \frac{\mathring{Z}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}} = -\overline{V}_T a_1\frac{d\varepsilon}{d\alpha} \equiv Z_q\frac{d\varepsilon}{d\alpha} \qquad M_{\dot{w}} = \frac{\mathring{M}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}^2} = -\overline{V}_T\frac{l_T}{\overline{\overline{c}}}a_1\frac{d\varepsilon}{d\alpha} \equiv M_q\frac{d\varepsilon}{d\alpha}$$

$$X_\eta = \frac{\mathring{X}_\eta}{\frac{1}{2}\rho V_0^2 S} = -2\frac{S_T}{S}k_T C_{L_T} a_2 \qquad Z_\eta = \frac{\mathring{Z}_\eta}{\frac{1}{2}\rho V_0^2 S} = -\frac{S_T}{S}a_2$$

Problems

1. Given (F = d mV_T/dt) , F is a force vector, m is a constant mass, and V_T is the velocity vector of the mass center. Find Fx , Fy , and Fz (if V_T = Ui + Vj + Wk and w = Pi + Qj + Rk) with respect to the fixed earth axis system.
2. Given H = Jv pA (r X V) dv where pAdV is the mass of a particle, with r as its radius vector from the eg, and V as its velocity, with respect to the eg. Find Hx with respect to the fixed earth axis system.
3. Define: L , M , N, P, Q, R
4. Define ψ, ϕ,θ,What are they used for? in what sequence must they be used? Explain the difference between ψ and ß.
5. What are the expressions for P, Q, R, in terms of Euler angles?
6. D, L, M = f(, , , ,)

Y, L, N = f(, , , , ,)