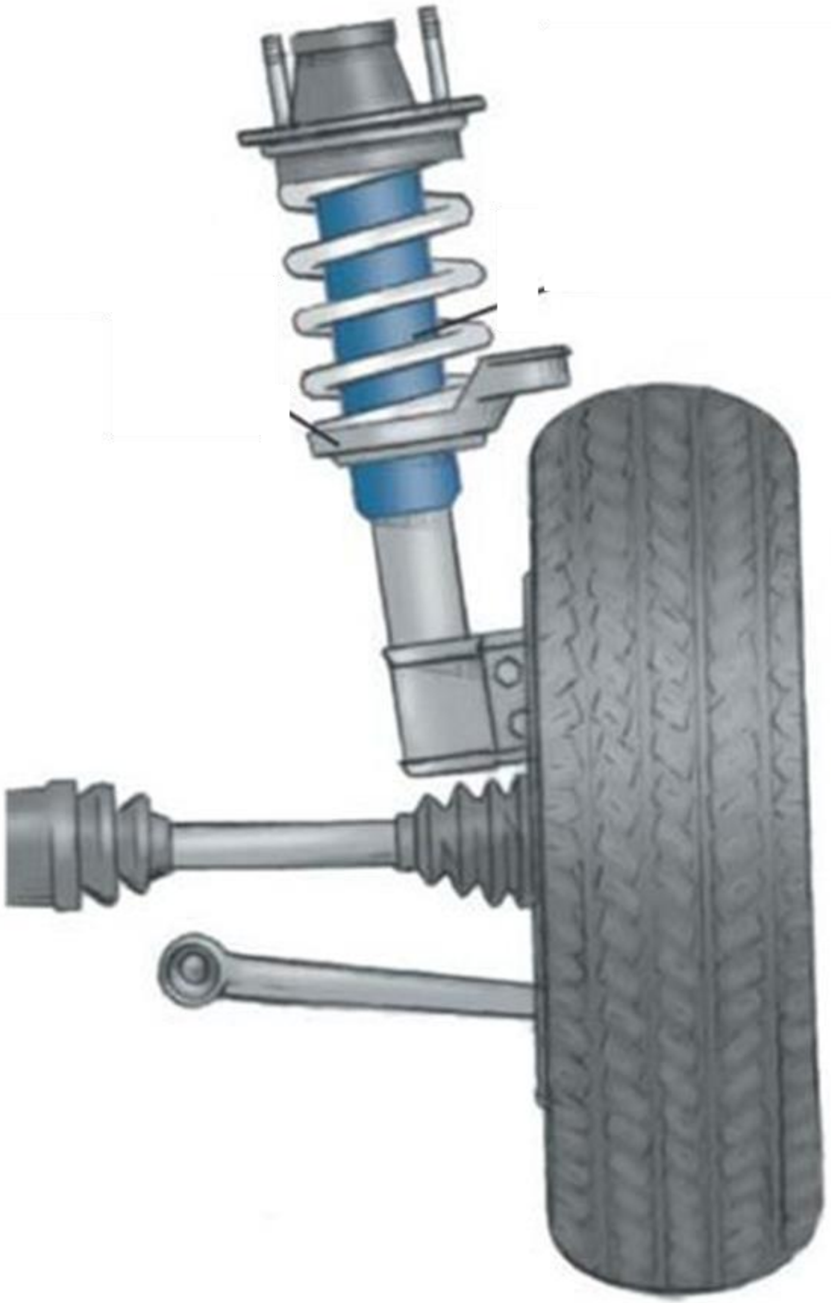


Tishk International University

Engineering Faculty -Aviation Department

Mechanical Vibration - Second Grade

Lecture 6: Free Vibration with Viscous Damping

Lecturer: Jwan Khaleel Mohammed

Email: jwan.khaleel@tiu.edu.iq

➢ Lecture Content:

- **Damping elements**
- **Equation of Motion**
- **The solution**

➢ Lecture Outcome:

After completing this lecture, you will be able to do the following:

- Derive the equation of motion of a damped free vibration for single-degree-of-freedom system using a suitable technique.
- Find the solution of the damped free vibration for a SDOF systems.
- Finding Natural frequency and period of frequency
- Compute Displacement, Velocity and Acceleration.

➢ Damping elements

➢ Damping elements are assumed to have neither inertia nor the mean to store or release potential energy.

➢ The mechanical motion imparted to these elements is converted to hear or sound and, hence, they are called non-conservative or dissipative because this energy is not recoverable by the mechanical system. **The following are three common types of damping mechanisms:**

- Viscous damping
- Coulomb or dry friction damping
- Material or solid or hysteretic damping

In these cases, the damping force is usually expressed as a function of velocity.

➢ Viscous damping

Viscous damping is the most used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, or oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include

1. Fluid film between sliding surfaces.
2. Fluid flow around a piston in a cylinder.
3. Fluid flow through an orifice.
4. Fluid film around a journal in a bearing.

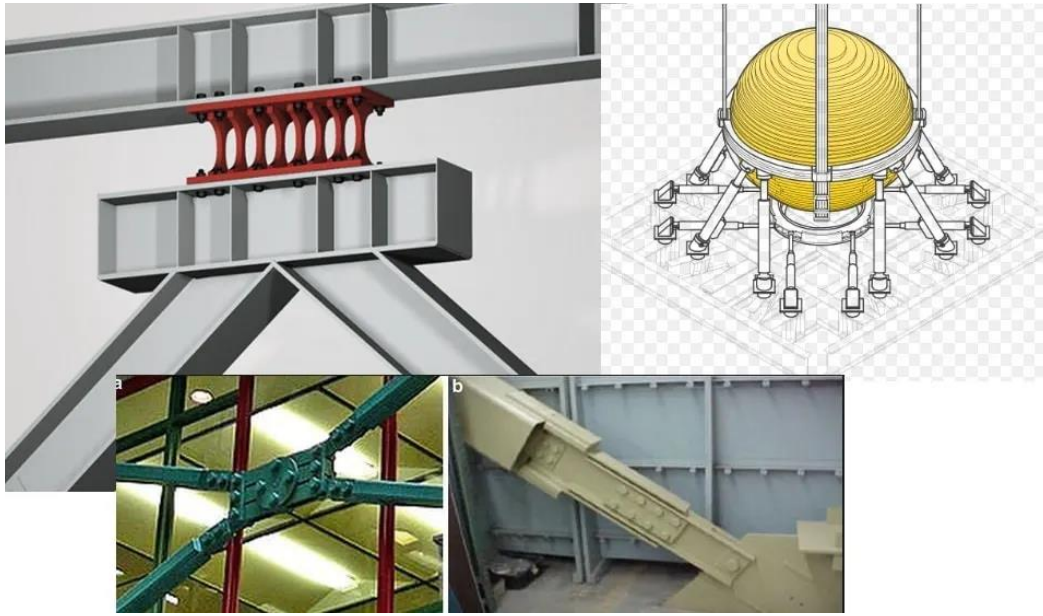
Coulomb or Dry-Friction Damping.

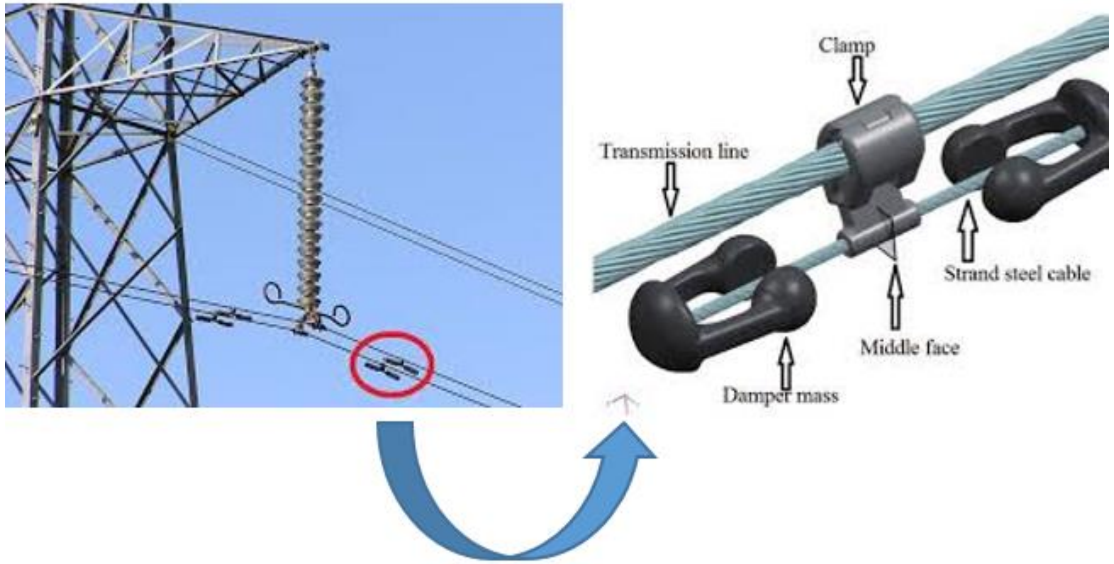
Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication.

Material or Solid or Hysteretic Damping.

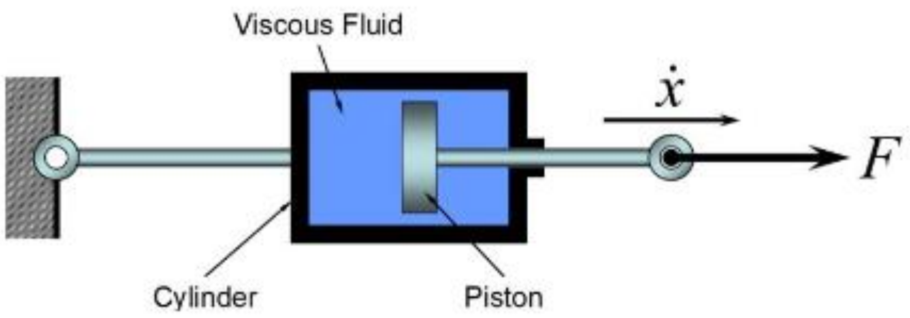
When a material is deformed, energy is absorbed and dissipated by the material. The effect is due to friction between the internal planes, which slip or slide as the deformations take place. When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop. The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping.

SEISMIC VIBRATION DAMPING

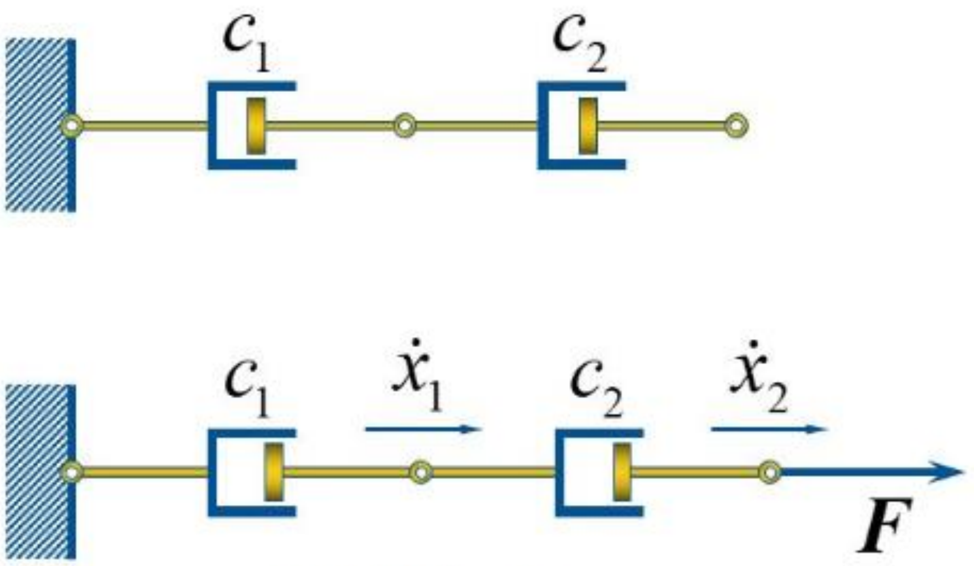




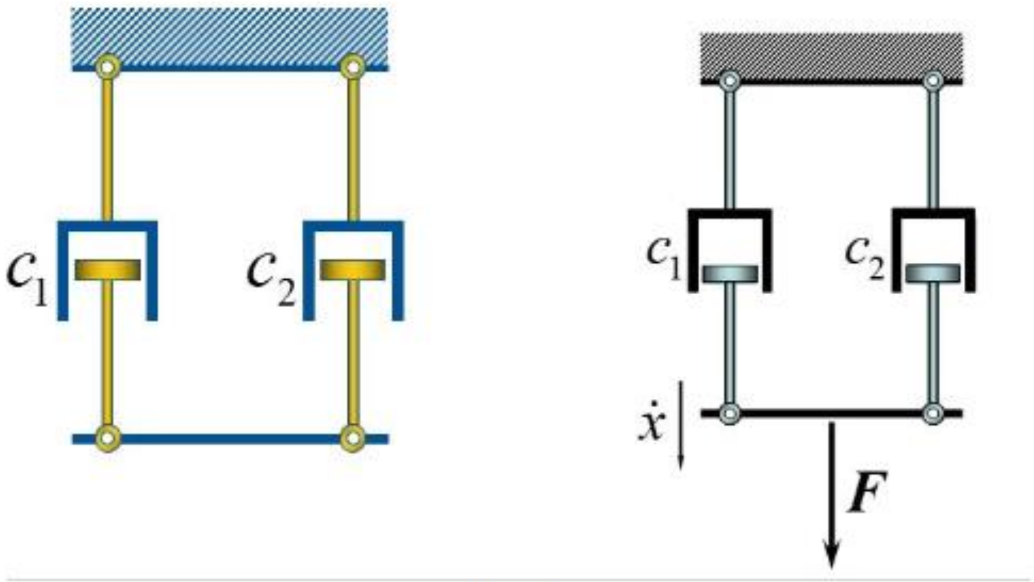
➢Working principle of damper | How do damper works?



Viscous Damping



Dampers in Series



Dampers in Parallel

Parallel dampers: $c_{eq} = c_1 + c_2$

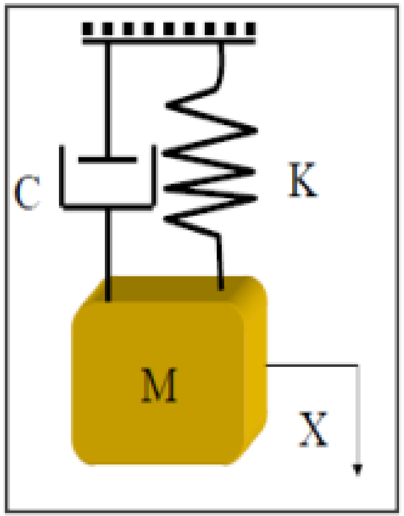
Series dampers: $\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2}$

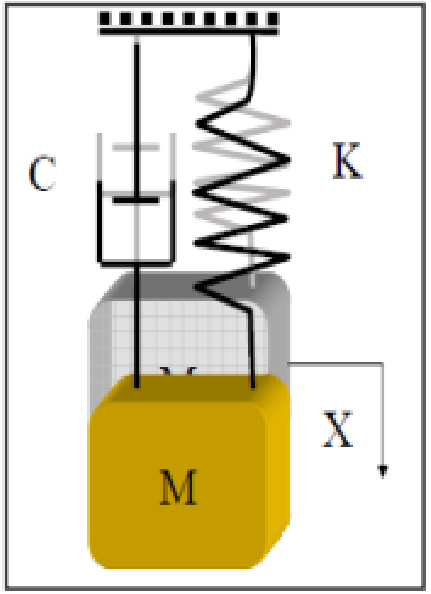
- **In the context of damping, particularly in physics and engineering, these terms refer to different behaviors exhibited by damped oscillatory systems:**
  - **Damped Oscillation:** When an oscillating system is subjected to a damping force (such as friction or air resistance), its amplitude gradually decreases over time. The rate at which the amplitude decreases depends on the damping coefficient and the system's characteristics.
  - **Underdamped:** This refers to a situation where the damping is relatively weak compared to the natural frequency of the system. In underdamped oscillation, the system oscillates with decreasing amplitude, but it oscillates around a mean position. The motion is characterized by oscillations that gradually decrease in amplitude.
  - **Overdamped:** In an overdamped system, the damping is so strong that the system returns to its equilibrium position without oscillating. The motion is sluggish, and there are no oscillations observed. The system reaches equilibrium in the slowest possible manner without oscillating back and forth.
  - **Critically Damped:** This is a special case where the damping is adjusted precisely to the point where the system returns to its equilibrium position as quickly as possible without oscillating. Critically damped systems reach equilibrium in the shortest time without overshooting the equilibrium position. This is often desirable in engineering applications where rapid settling time is important.

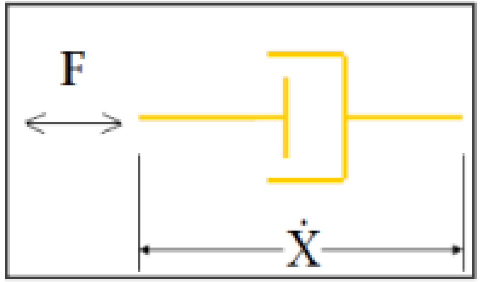
➢ Equation of Motion:

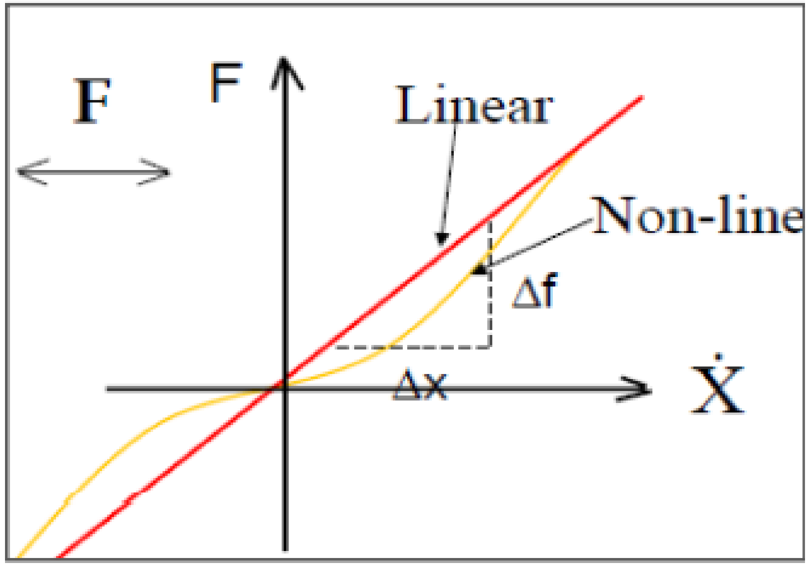
- As stated in previous lectures, the viscous damping force F is proportional to the velocity $\dot{x}$ or v and can be expressed as:

$$F = -c\dot{x}$$





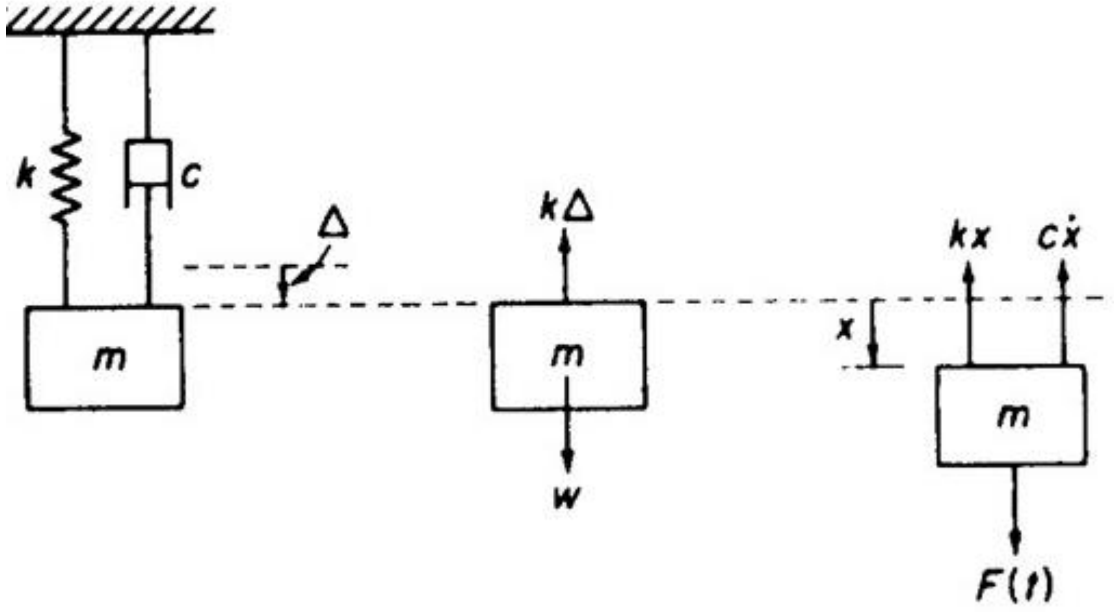




- where c is the damping constant or coefficient of viscous damping $(N.\sec/m \; or \; lbf.\sec/ft)$ $(kg/\sec)$.
- The negative sign indicates that the damping force is opposite to the direction of velocity.
- A single-degree-of-freedom system with a viscous damper is shown in Fig. If x is measured from the equilibrium position of the mass m, the application of Newton's law yields the equation of motion:

Apply Newton's 2nd Law

$$\sum F_{x\downarrow+} = m\ddot{x} \qquad (mg = k\delta) \text{ (static equilibrium)}$$



$$mg - (kx + k\delta) - c\dot{x} = m\ddot{x}$$

$$mg - kx - k\delta - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Second order differential equation, homogenous

➢ Solution:

- To solve Equation of motion, we assume a solution in the form

$$x(t) = Ce^{st}$$

- where C and s are undetermined constants. Inserting this function into Eq. leads to the characteristic equation

$$ms^2 + cs + k = 0$$

- The roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- These roots give two solutions to Equation:

$$x_1(t) = C_1e^{s_1t} \quad \text{and} \quad x_2(t) = C_2e^{s_2t}$$

EOM of viscously damped free vibration system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Assume $x(t) = Ce^{st}$

$$\dot{x}(t) = sCe^{st}$$

$$\ddot{x}(t) = s^2Ce^{st}$$

$$ms^2Ce^{st} + csCe^{st} + kCe^{st} = 0$$

$$(ms^2 + cs + k)Ce^{st} = 0$$

$c^2 - 4mk > 0$	$\rightarrow$	$s_1, s_2 (different\ real\ numbers)$
$c^2 - 4mk = 0$	$\rightarrow$	$r = -c/2m$
$c^2 - 4mk < 0$	$\rightarrow$	$s_1, s_2 (complex\ numbers)$

- Thus, the general solution of EOM is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$= C_1 e^{\left\{-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m}-\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right\}t}$$

- where C_1 and C_2 are arbitrary constants to be determined from the initial conditions of the system.

Critical Damping Constant and the Damping Ratio:

- The **critical damping** is defined as the value of the damping constant c for which the radical in Eq. becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad \text{or} \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

- For any damped system, the **damping ratio** ξ is defined as the ratio of the damping constant to the critical damping constant: $\zeta = c/c_c$ we can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n$$

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

the solution, can be written as

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

- Hence, we assume that $.\xi \neq 0$ and consider the following three cases:

$\zeta^2 - 1 < 0$	$\Rrightarrow$	$0 < \zeta < 1$	$\Rrightarrow$	*Underdamped motion*
$\zeta^2 - 1 > 0$	$\Rrightarrow$	$\zeta > 1$	$\Rrightarrow$	*Overdamped motion*
$\zeta^2 - 1 = 0$	$\Rrightarrow$	$\zeta = 1$	$\Rrightarrow$	*Critically damped motion*

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = \left(-\zeta + i\sqrt{1 - \zeta^2}\right)\omega_n$$

$$s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right)\omega_n$$

$$x(t) = C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left\{C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}\right\}$$

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

$$= e^{-\zeta\omega_n t}\left\{(C_1 + C_2)\cos\sqrt{1-\zeta^2}\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\omega_n t\right\}$$

$$= e^{-\zeta\omega_n t}\left\{C_1' \cos\sqrt{1-\zeta^2}\omega_n t + C_2' \sin\sqrt{1-\zeta^2}\omega_n t\right\}$$

$$= X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\omega_n t + \phi_0\right)$$

$$= X e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\omega_n t - \phi\right)$$

where (C_1', C_2'), (X, ϕ), and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

To determine the constants C_1' and C_2':

$$x(t) = e^{-\zeta\omega_n t}\left\{C_1'\cos\sqrt{1-\zeta^2}\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\omega_n t\right\}$$

Eq. is differentiated to find the velocity, $\dot{x}(t)$, as

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}\{C_1'\cos\sqrt{1-\zeta^2}\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\omega_n t\}$$
$$+ e^{-\zeta\omega_n t}\{-\sqrt{1-\zeta^2}\omega_n C_1'\sin\sqrt{1-\zeta^2}\omega_n t$$
$$+ \sqrt{1-\zeta^2}\omega_n C_2'\cos\sqrt{1-\zeta^2}\omega_n t\}$$

For the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$, C_1' and C_2' can be found:

$$C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos \sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\omega_n} \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

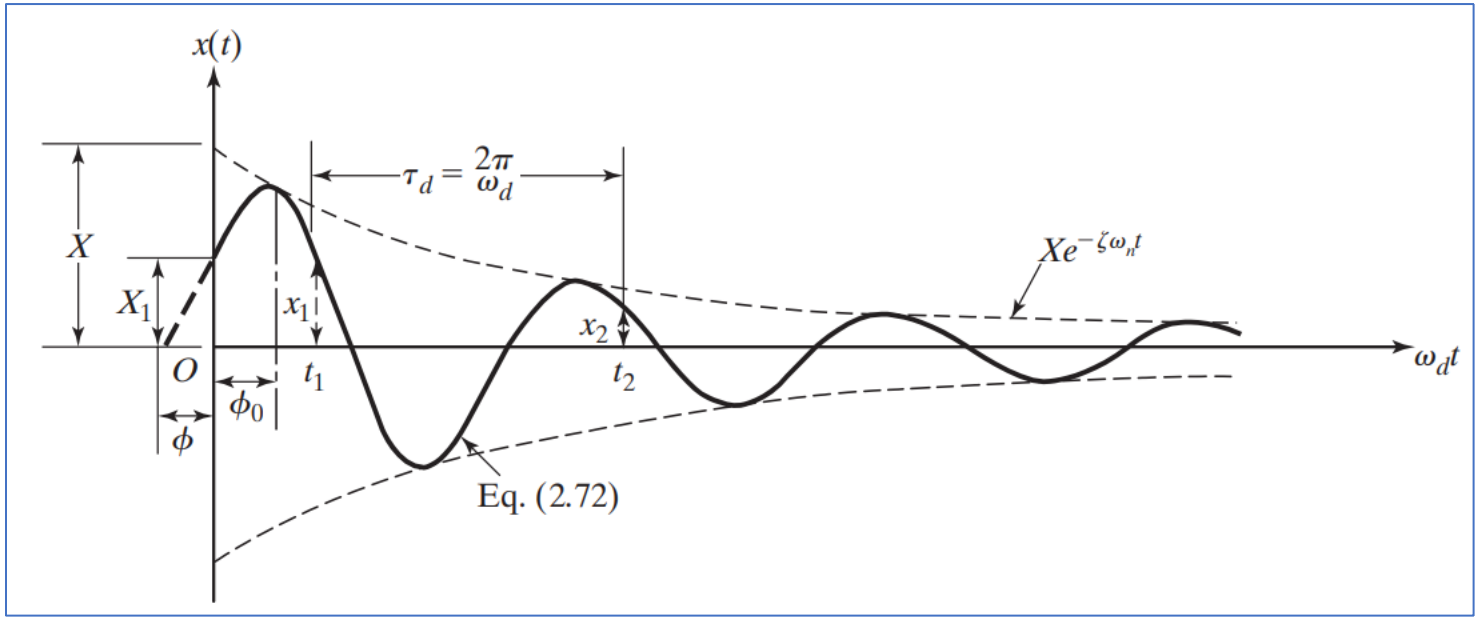
The constants (X, ϕ) and (X_0, ϕ_0) can be expressed as

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\zeta\omega_n}}{\sqrt{1-\zeta^2}\,\omega_n}$$

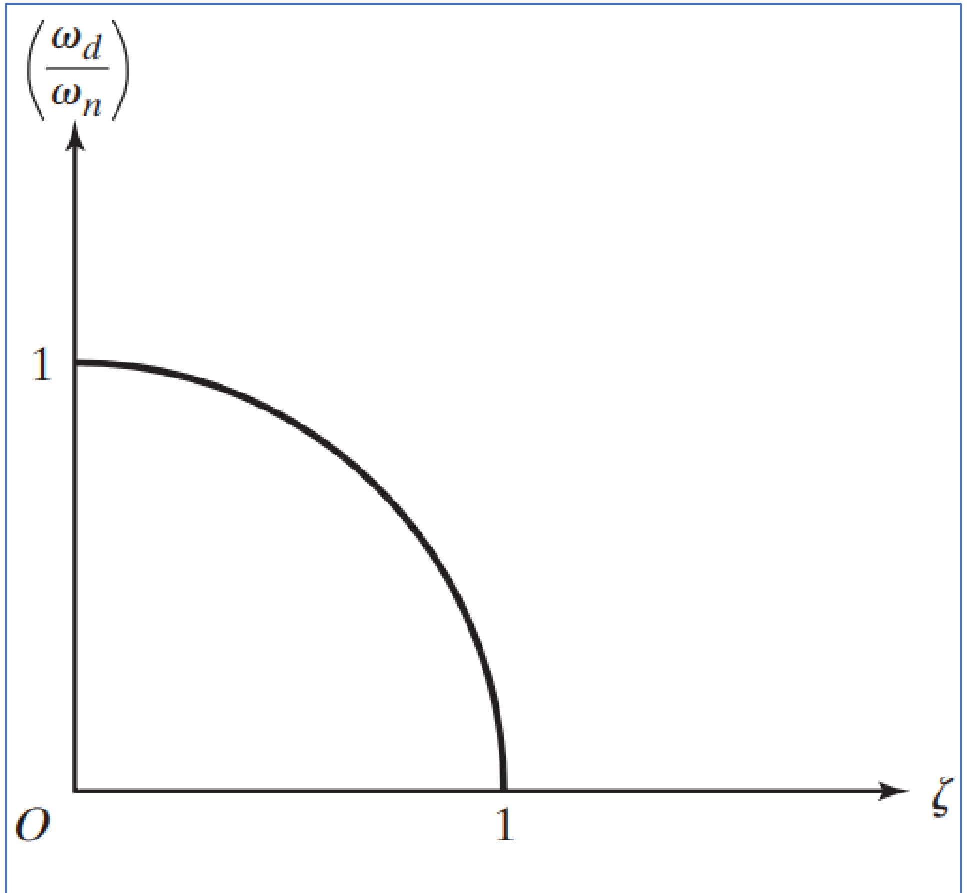
$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right)$$

$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_n\sqrt{1-\zeta^2}}\right)$$

NOTE: The motion described by Eq. is a damped harmonic motion of angular frequency $\sqrt{1-\zeta^2}\,\omega_n$ but because of the factor $e^{-\zeta\omega_n t}$ the amplitude decreases exponentially with time, as shown in Fig. The quantity $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ is called the **frequency of damped vibration.** The frequency of damped vibration ω_d is always less than the undamped natural frequency ω_n.



Underdamped solution.



Variation of ω_d with damping.

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots s_1 and s_2 in Eq. are equal:

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

Because of the repeated roots, the solution is given by

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

The application of the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$ for this case gives

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t}$$

It can be seen that the motion represented is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \to 0$ as $t \to \infty$, the motion will eventually diminish to zero,

Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, shows that the roots s_1 and s_2 are real and distinct and are given by

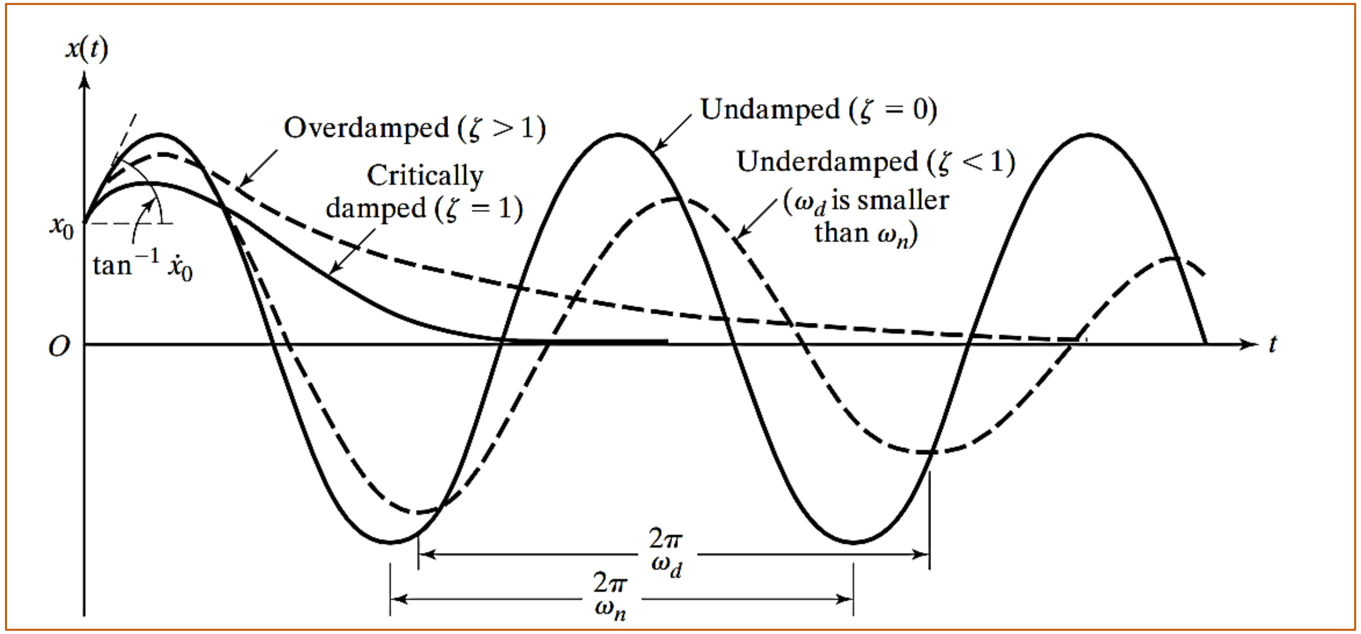
$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$
$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

with $s_2 \ll s_1$. In this case, the solution, can be expressed as

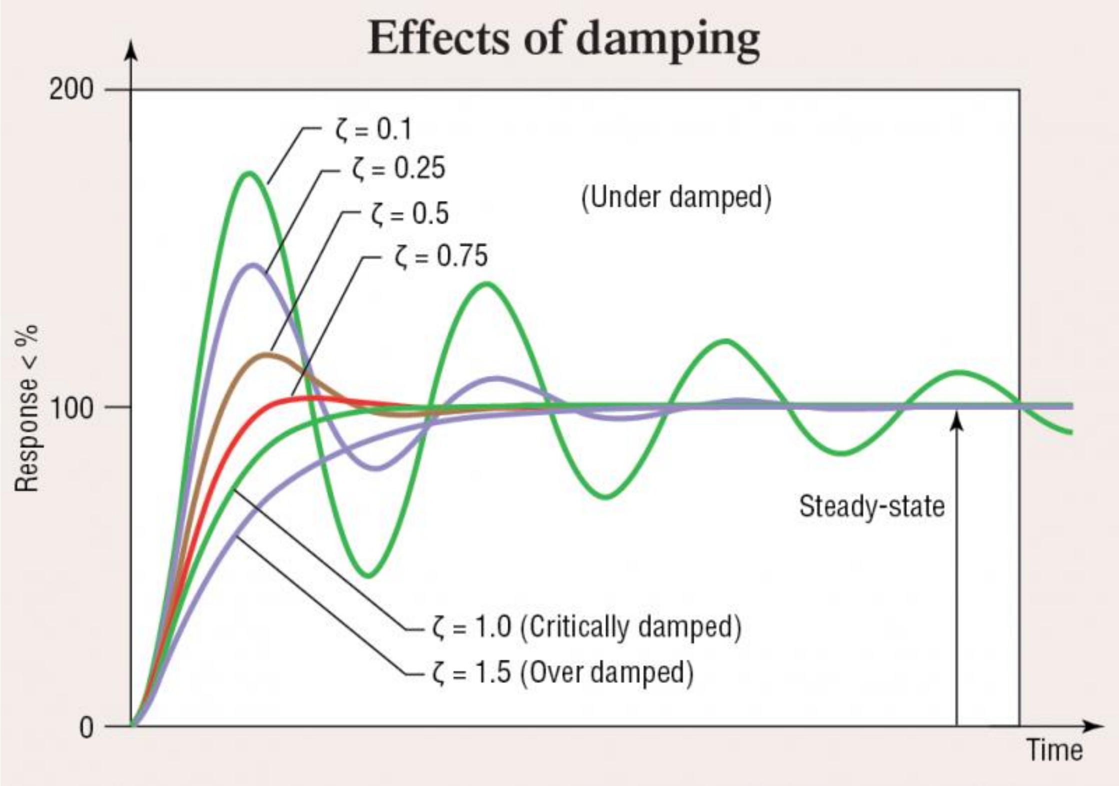
$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, the constants C_1 and C_2 can be obtained:

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$
$$C_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$



Comparison of motions with different types of damping.



➢ Torsional Systems with Viscous Damping

- The methods presented previously was for linear vibrations with viscous damping, these methods can be extended directly to viscously damped torsional (angular) vibrations.

$$T = -c_t\dot{\theta}$$

$$J_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$$

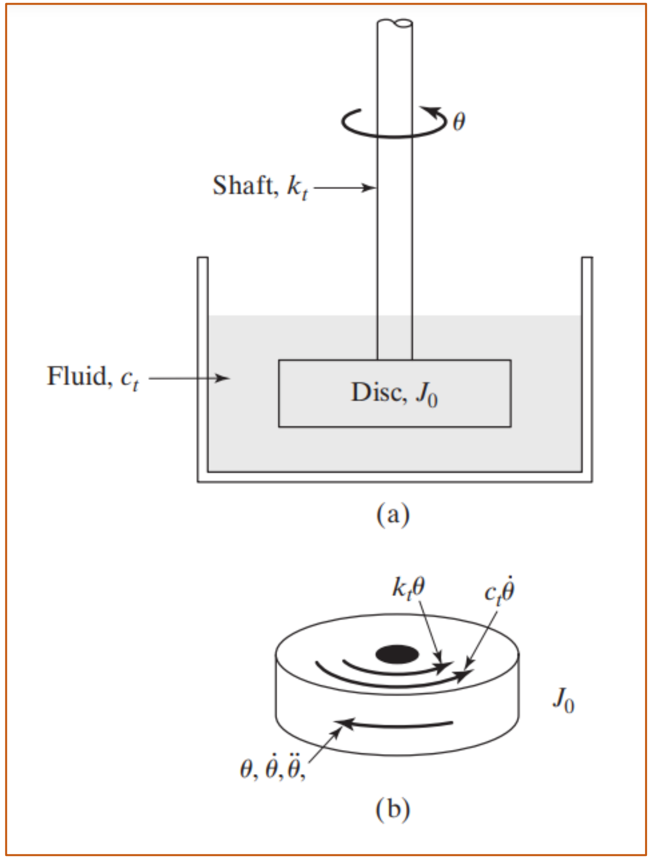
$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}}$$

and

$$\zeta = \frac{c_t}{c_{tc}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_t J_0}}$$

where c_{tc} is the critical torsional damping constant.



Example 1:

Consider a small spring about $30\,mm$ (or $1.18\,in$) long, welded to a stationary table (ground) so that it is fixed at the point of contact, with a $12\,mm\ (or\ 0.47in)$ bolt welded to the other end, which is free to move. The mass of this system is about $49.2\ \times 10^{-3}\ kg$ (equivalent to about 1.73 ounces). The spring stiffness constant of $k\ =\ 857.8\,N/m$ and the damping rate of the spring is measured to be $0.11\,kg/s$. Calculate the damping ratio and determine if the free motion of the spring–bolt system is overdamped, underdamped, or critically damped

Solution:

From equation natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8\ \text{N/m}}{49.2 \times 10^{-3}\ \text{kg}}} = 132\ \text{rad/s}$$

definition of the critical damping coefficient and these values for m and k yields

$$c_{cr} = 2\sqrt{km} = 2\sqrt{(857.8\ N/m)(49.2 \times 10^{-3}\ kg)}$$
$$= 12.993\ \text{kg/s}$$

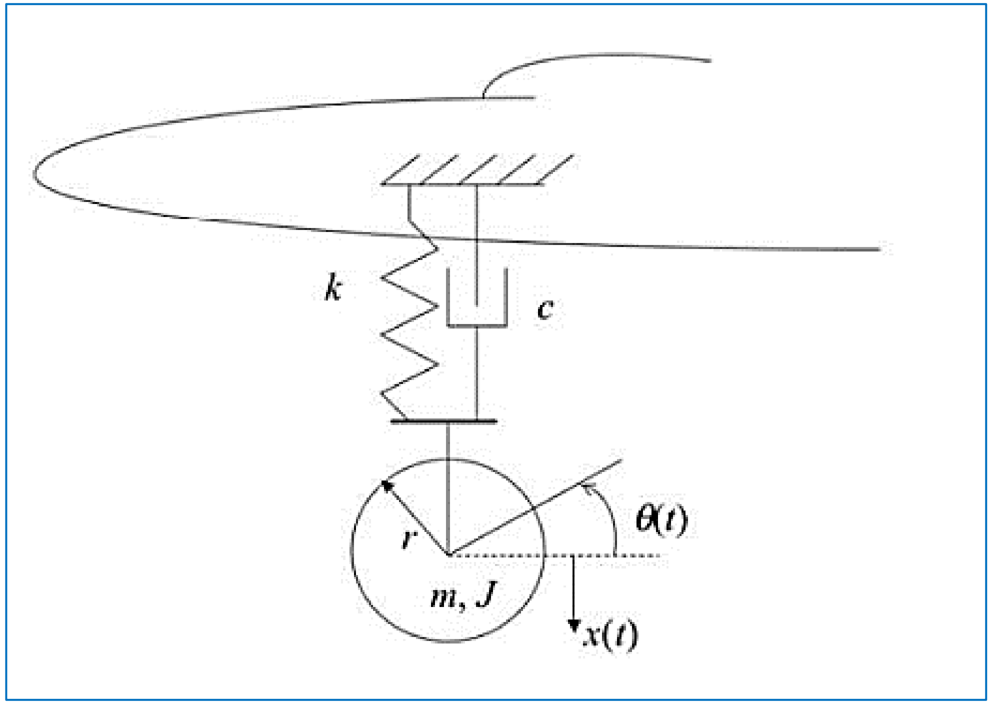
If c is measured to be 0.11 kg/s, the critical damping ratio becomes

$$\zeta = \frac{c}{c_{cr}} = \frac{0.11(\text{kg/s})}{12.993(\text{kg/s})} = 0.0085$$

or 0.85% damping. Since ζ is less than 1, the system is underdamped. The motion resulting from giving the spring–bolt system a small displacement will be oscillatory.

Example 2:

Consider the system in figure shown, which represents a simple model of an aircraft landing system. Assume, $x = r\theta$ What is the damped natural frequency?



Solution:

the undamped equation of motion is

$$\left(m+\frac{J}{r^2}\right)\ddot{x}+kx=0$$

From examining the equation of motion the natural frequency is:

$$\omega_n=\sqrt{\frac{k}{m_{eq}}}=\sqrt{\frac{k}{m+\frac{J}{r^2}}}$$

An add hoc way do to this is to add the damping force to get the damped equation of motion:

$$\left(m+\frac{J}{r^2}\right)\ddot{x}+c\dot{x}+kx=0$$

The value of ζ is determined by examining the velocity term:

$$\frac{c}{m+\frac{J}{r^2}}=2\zeta\omega_n\Rightarrow\zeta=\frac{c}{m+\frac{J}{r^2}}\frac{1}{2\sqrt{\frac{k}{m+\frac{J}{r^2}}}}$$

$$\Rightarrow\zeta=\frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}}$$

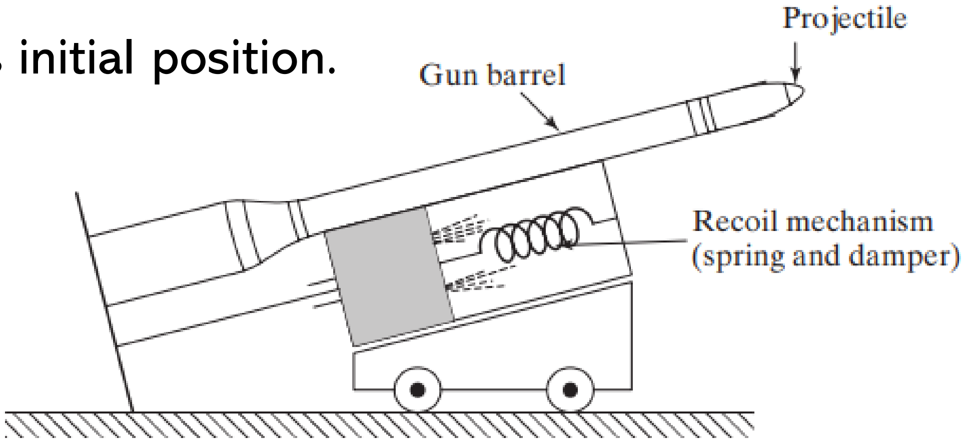
Thus the damped natural frequency is

$$\omega_d=\omega_n\sqrt{1-\zeta^2}=\sqrt{\frac{k}{m+\frac{J}{r^2}}}\sqrt{1-\left(\frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}}\right)^2}$$

$$\Rightarrow\omega_d=\sqrt{\frac{k}{m+\frac{J}{r^2}}-\frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}}=\frac{r}{2(mr^2+J)}\sqrt{4(kmr^2+kJ)-c^2r^2}$$

Example 3:

The schematic diagram of a large cannon is shown in Fig. When the gun is fired, high pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the direction opposite that of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the recoil mechanism. In a particular case, the gun barrel and the recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10,000 N/m. The gun recoils 0.4 m upon firing. Find (1) the critical damping coefficient of the damper, (2) the initial recoil velocity of the gun, and (3) the time taken by the gun to return to a position 0.1 m from its initial position.



Solution:

1. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{500}} = 4.4721 \text{ rad/s}$$

and the critical damping coefficient of the damper is

$$c_c = 2m\omega_n = 2(500)(4.4721) = 4472.1 \text{ N-s/m}$$

2. The response of a critically damped system is given by

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{E.1}$$

where $C_1 = x_0$ and $C_2 = \dot{x}_0 + \omega_n x_0$. The time t_1 at which $x(t)$ reaches a maximum value can be obtained by setting $\dot{x}(t) = 0$. The differentiation of Eq. (E.1) gives

$$\dot{x}(t) = C_2 e^{-\omega_n t} - \omega_n (C_1 + C_2 t)e^{-\omega_n t}$$

Hence $\dot{x}(t) = 0$ yields

$$t_1 = \left(\frac{1}{\omega_n} - \frac{C_1}{C_2}\right) \tag{E.2}$$

Solution (cont.):

In this case, $x_0 = C_1 = 0$; hence Eq. (E.2) leads to $t_1 = 1/\omega_n$. Since the maximum value of $x(t)$ or the recoil distance is given to be $x_{max} = 0.4$ m, we have

$$x_{max} = x(t = t_1) = C_2 t_1 e^{-\omega_n t_1} = \frac{\dot{x}_0}{\omega_n} e^{-1} = \frac{\dot{x}_0}{e\omega_n}$$

or

$$\dot{x}_0 = x_{max}\omega_n e = (0.4)(4.4721)(2.7183) = 4.8626 \text{ m/s}$$

3. If t_2 denotes the time taken by the gun to return to a position 0.1 m from its initial position, we have

$$0.1 = C_2 t_2 e^{-\omega_n t_2} = 4.8626 t_2 e^{-4.4721 t_2} \qquad \text{(E.3)}$$

The solution of Eq. (E.3) gives $t_2 = 0.8258$ s.

Solve the following assignments and upload your answer on google classroom

1. A simple pendulum is found to vibrate at a frequency of 0.5 Hz in a vacuum and 0.45 Hz in a viscous fluid medium. Find the damping constant, assuming the mass of the bob of the pendulum is 1 kg

2. For a spring-mass-damper system, m = 50 kg and k = 5,000 N/m Find the following: (a) critical damping constant c_c (b) damped natural frequency when c = c_c /2 and (c) logarithmic decrement.

3. The maximum permissible recoil distance of a gun is specified as 0.5 m. If the initial recoil velocity is to be between 8 m/s and 10 m/s, find the mass of the gun and the spring stiffness of the recoil mechanism. Assume that a critically damped dashpot is used in the recoil mechanism and the mass of the gun must be at least 500 kg.

4. Determine the value of damping ratio and damping frequency for the following viscously damped systems:

a. $m = 10$ kg, $c = 150$ N-s/m, $k = 1000$ N/m

b. $m = 10$ kg, $c = 200$ N-s/m, $k = 1000$ N/m

c. $m = 10$ kg, $c = 250$ N-s/m, $k = 1000$ N/m

The End of the Lecture

Enjoy Your Time