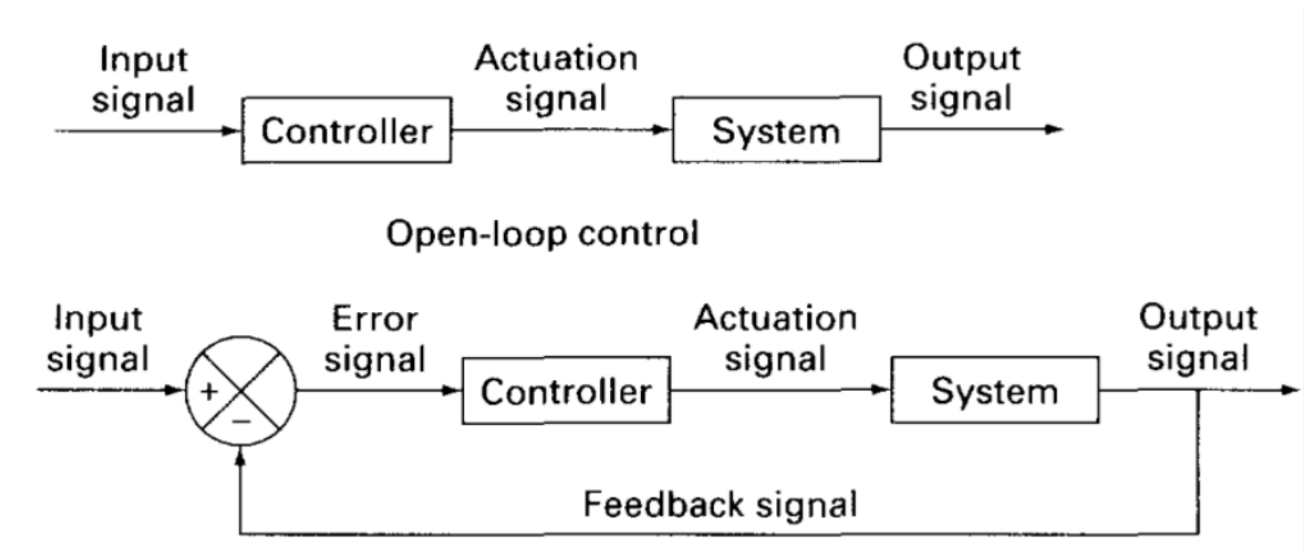
LEC. Six

Automatic Control Theory - The Classical Approach



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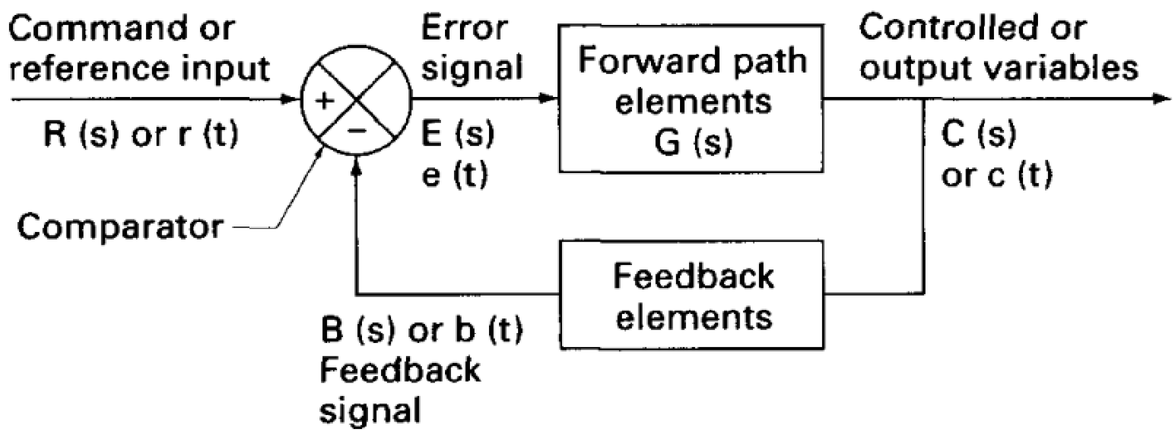
1. A feedback control system



$$\text{T.F.} = \frac{\text{Laplace transform of the output}}{\text{Laplace transform of the input}}$$

$R(s)$ reference input
$C(s)$ output signal
$B(s)$ feedback signal
$H(s)$ feedback transfer function
$E(s)$ error or actuating signal
$G(s)H(s)$ loop transfer function
$G(s)$ $C(s)/E(s)$ forward path or open-loop transfer function
$M(s)$ $C(s)/R(s)$ the closed-loop transfer function

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closed-loop transfer function C(s)/R(s):

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

ROUTH'S CRITERION

The feedback systems described here can be designed to control accurately the output to some desired tolerance.

The roots of the characteristic equation tell us whether or not the system is dynamically stable.

If all the roots of the characteristic equation have negative real parts the system will be dynamically stable. On the other hand,

If any root of the characteristic equation has a positive real part the system will be Unstable.

Consider the characteristic equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} \cdots a_1\lambda + a_0 = 0$$

The necessary but not sufficient conditions are that

1. All the coefficients of the characteristic equation must have the same sign.
2. All the coefficients must exist.

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The Routh stability criterion states:

1. If all the numbers of the first column have the same sign then the roots of the characteristic polynorninal have negative real parts. The system therefore is stable.

2. If the numbers in the first column change sign then the number of sign changes indicates the number of roots of the characteristic equation having positive real parts. Therefore, if there is a sign change in the first column the system will be unstable.

Definition of Routh array: Routh table

λ^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
λ^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
λ^{n-2}	b_1	b_2	b_3	$\cdots$
$\vdots$	c_1	c_2	c_3	$\cdots$

where $a_n, a_{n-1}, \ldots, a_0$ are the coefficients of the characteristic equation and the coefficients b_1, b_2, b_3, c_1, c_2, and so on are given by

$$b_1 \equiv \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 \equiv \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \qquad \text{and so forth}$$

$$c_1 \equiv \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \qquad c_2 \equiv \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \qquad \text{and so forth}$$

$$d_1 \equiv \frac{c_1 b_2 - c_2 b_1}{c_1} \qquad \text{and so forth}$$

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EXAMPLE 7.1. Determine whether the characteristic equations given below have stable or unstable roots.

(a) A' + 6A2 + 12A + 8 = 0 (b) 2~~ + 4A2 + 4A + 12 = 0

(c) ,4A4 + BA' + CA2 + DA + E = 0

Solution.

(a)

There are no sign changes in column 1; therefore, the system is stable

1	12	0
6	8	0
$\frac{64}{6}$	0	
8		

(b) there are two sign changes in column 1; therefore, the characteristic equation has two roots with positive real parts.

2	4	0
4	12	0
−2	0	
12		

(c)

A	C	E
B	D	0
$\frac{BC - AD}{B}$	E	0
$\frac{[D(BC - AC)/B] - BE}{(BC - AD)/B}$	0	
E		

For the airplane to be stable requires that:

$$A, B, C, D, E > 0$$
$$BC - AC > 0$$
$$D(BC - AD) - B^2E > 0$$

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Automatic Control Theoryt - The Classical Approach

If the first number in a row is 0 and the remaining elements of that row are nonzero, the Routh method breaks down. To overcome this problem the lead element that is 0 is replaced by a small positive number, E. With the substitution of E as the first element, the Routh array can be completed. After completing the Routh array we can examine the first column to determine whether there are any sign changes in the first column as E approaches 0.

EXAMPLE7.2 (a) $\lambda^5 + \lambda^4 + 3\lambda^3 + 3\lambda^2 + 4\lambda + 6 = 0$

(b) $\lambda^6 + 3\lambda^5 + 6\lambda^4 + 12\lambda^3 + 11\lambda^2 + 9\lambda + 6 = 0$

Now as E goes to 0 the sign of the first elements in rows 3 and 4 are positive. However, in row 5 the lead element goes to -2 as E goes to 0. We note two sign changes in the first column. which means it is unstable

1	3	4
1	3	6
ε	-2	
$\dfrac{3\varepsilon + 2}{\varepsilon}$	6	
$\dfrac{-6\varepsilon^2 - 6\varepsilon - 4}{3\varepsilon + 2}$	0	

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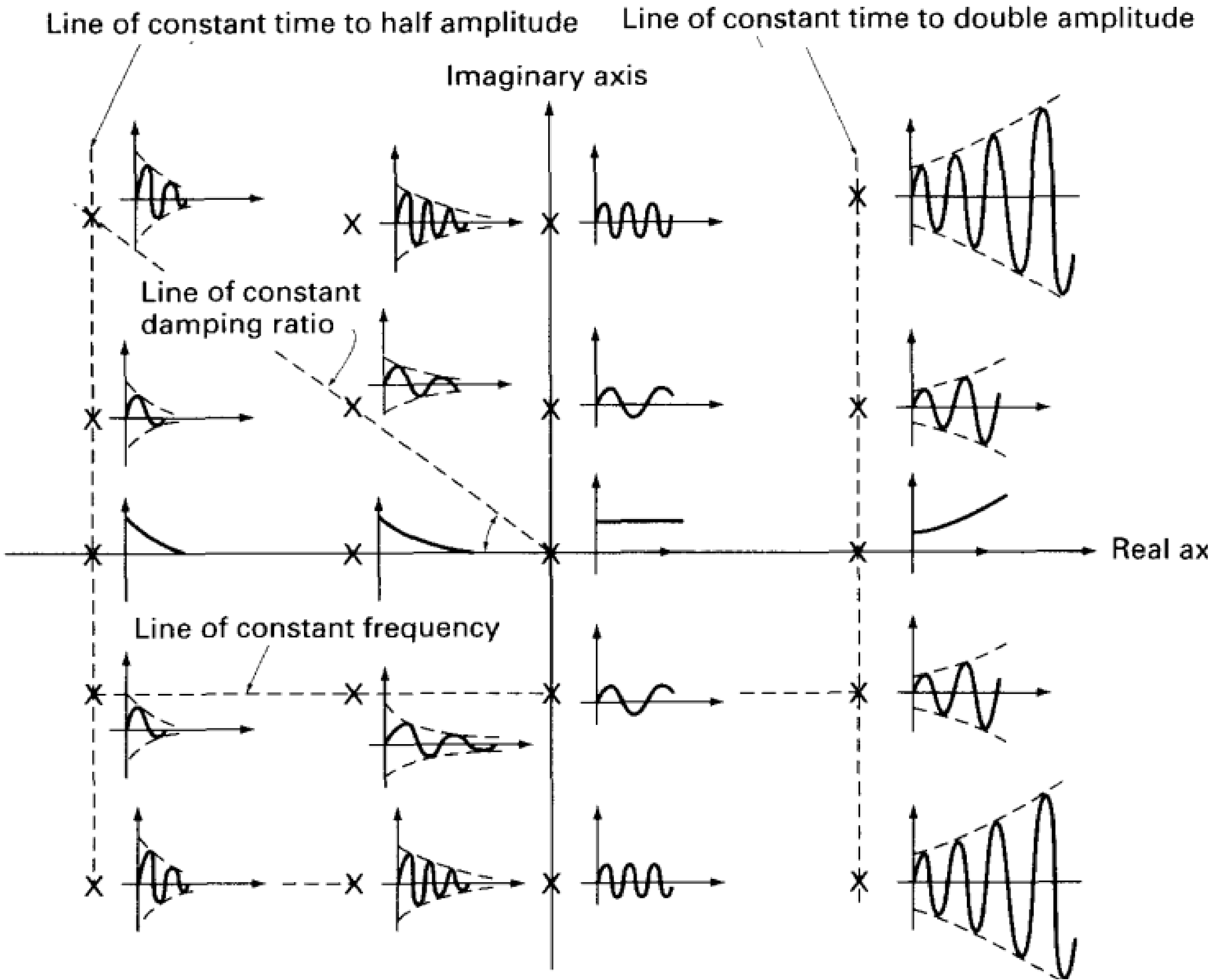
Root Locus:



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Root locus:

The characteristic equation of the closed loop system is given by the denominator:

$$1 + G(s)H(s) = 0$$

$$G(s)H(s) = -1$$

$$G(s)H(s) = \frac{k(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

where n > m and k is an unknown system parameter

$$\frac{k(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = -1 \quad \Rightarrow \quad \frac{|k|\,|s + z_1|\,|s + z_2| \cdots |s + z_m|}{|s + p_1|\,|s + p_2| \cdots |s + p_n|} = 1$$

$$\sum_{i=1}^{m} \angle(s + z_i) - \sum_{i=1}^{n} \angle(s + p_i) = (2q + 1)\pi$$

where q = 0, 1,2, . . . ,n-m-1

- It can be shown easily that the root locus contours start at the poles of transfer function, G(s)H(s) and end at the zeroes of the transfer function as k is varied from 0 to infinity. For example, if we rearrange the magnitude criteria in the following manner,
- Then as k goes to 0 the function becomes infinite

$$\frac{|s + z_1|\,|s + z_2| \cdots |s + z_m|}{|s + p_1|\,|s + p_2| \cdots |s + p_n|} = \frac{1}{|k|}$$

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Root locus:

Rules for graphical construction of the root locus plot

1. The root locus contours are symmetrical about the real axis.

2. The number of separate branches of the root locus plot is equal to the number of poles of the transfer function C(s)H(s). Branches of the root locus originate at the poles of G(s)H(s) for k = 0 and terminate at either the open-loop zeroes or at infinity for k = x. The number of branches that terminate at infinity is equal to the difference between the number of poles and zeroes of the transfer function G(s)H(s), where n = number of poles and m = number of zeros.

3. Segments of the real axis that are part of the root locus can be found in the following manner: Points on the real axis that have an odd number of poles and zeroes to their right are part of the real axis portion of the root locus.

4. The root locus branches that approach the open-loop zeroes at infinity do so along straight-line asymptotes that intersect the real axis at the center of gravity of the finite poles and zeroes. Mathematically this can be expressed as $\sigma = \left[\sum \text{Real parts of the poles} - \sum \text{Real parts of the zeroes} \right] / (n - m)$

5. The angle that the asymptotes make with the real axis is given by $\phi_a = \frac{180^\circ [2q + 1]}{n - m}$

For q = 0, 1.2,. ..,(n -m - 1)

6. The angle (ϕ) of departure of the root locus from a pole of G(s)H(s) can be found by the following expression: $\phi_p = \pm 180^\circ (2q + 1) + \phi \qquad q = 0, 1, 2, \ldots$

where (ϕ) is the net angle contribution at the pole of interest due to all other poles and zeroes of G(s)H(s). The arrival angle at a zero is given by a similar expression: $\phi_z = \pm 180^\circ (2q + 1) + \phi \qquad q = 0, 1, 2, \ldots$

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Root locus:

EXAMPLE 7.3. Sketch the root locus plot for the transfer function

$$G(s)H(s) = \frac{k(s+3)}{s(s+10)(s^2+8s+20)}$$

(m = 1) and four poles (n = 4):

zero: s = -3   poles: s = 0, s = -10, s = -4 ± 2i

The intersection of the asymptotes with the real axis and the angle of the asymptotes follow

$$\sigma = \frac{\Sigma \text{ real parts of the poles} - \Sigma \text{ real parts of the zero}}{[n-m]}$$

$$\sigma = \frac{(-0-10-4-4)-(-3)}{4-1} = \frac{-15}{3} = -5$$

$$\phi_A = \frac{180°[2q+1]}{n-m}$$

$$\phi_A = \frac{180°[2q+1]}{3} \qquad \text{and } q = 0, 1, \ldots, n-m-1,$$

where $n - m - 1 = 4 - 1 - 1 = 2$

$$\phi_A = 60°, \quad 180°, \quad 300°$$

The pole at the origin approaches zero at s = -3, the pole at s = - 10 goes to -oo, on the real axis, and the complex poles go to zeroes along asymptotes making an angle of 60" and 300" with the real axis as k goes from 0 to oo

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Root locus:

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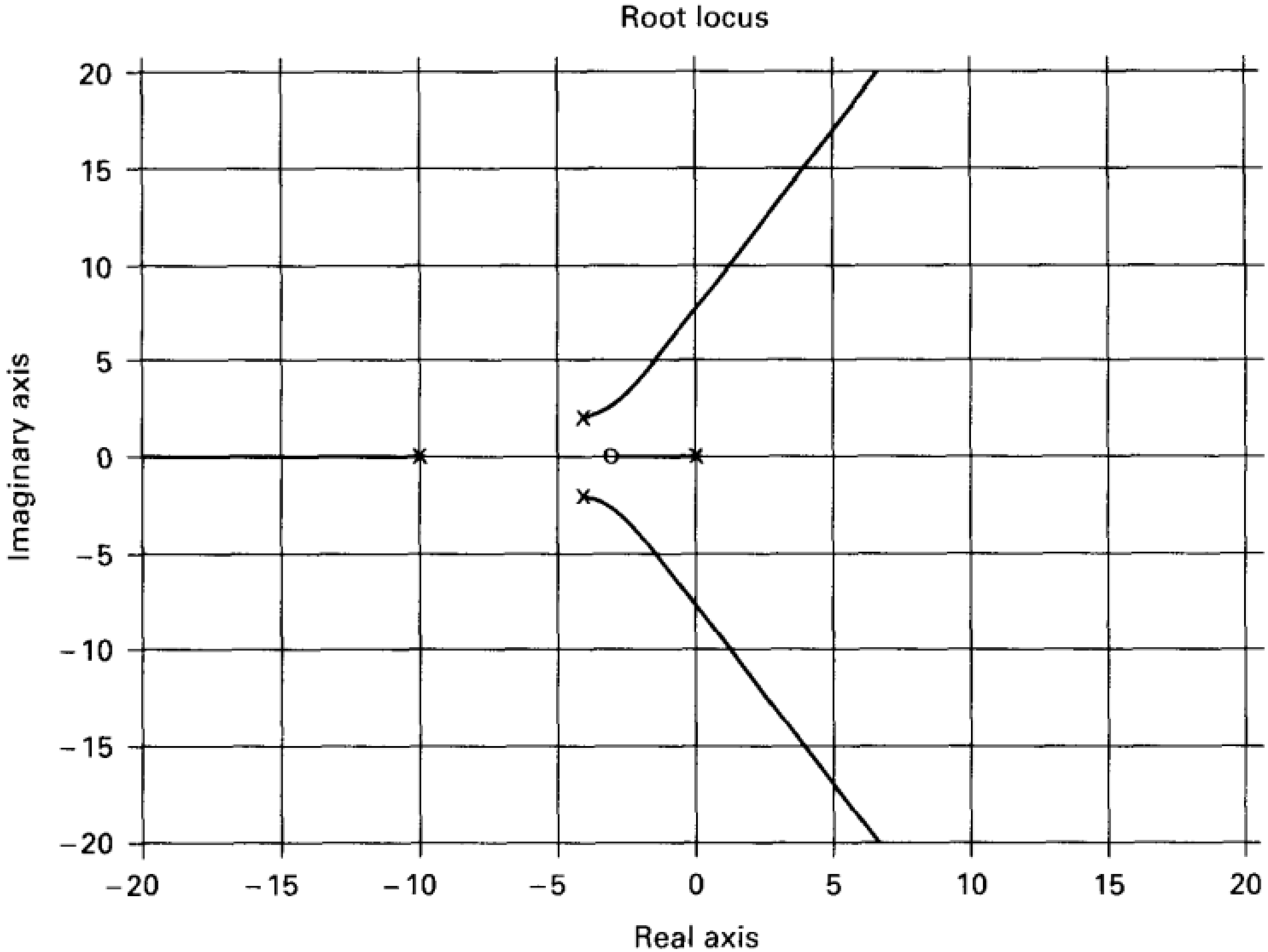

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FREQUENCY DOMAIN TECHNIQUES

The transfer function for a closed-loop feedback system can be written as

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

If we excite the system with a sinusoidal input such as $r(t) = A_I \sin(\omega t)$

the steady-state output of the system will have the form $c(t) = A_O \sin(\omega t + \phi)$

- The ratio of output to input for a Sinusoidal steady state can be obtained by replacing the Laplace transform variable s with iw:

$$M(i\omega) = \frac{G(i\omega)}{1 + G(i\omega)H(i\omega)} \qquad M(i\omega) = M(\omega)\underline{/\phi(\omega)}$$

$$M(\omega) = \left|\frac{G(i\omega)}{1 + G(i\omega)H(i\omega)}\right| \qquad \phi(\omega) = \underline{/G(i\omega)} - \underline{/[1 + G(i\omega)H(i\omega)]}$$

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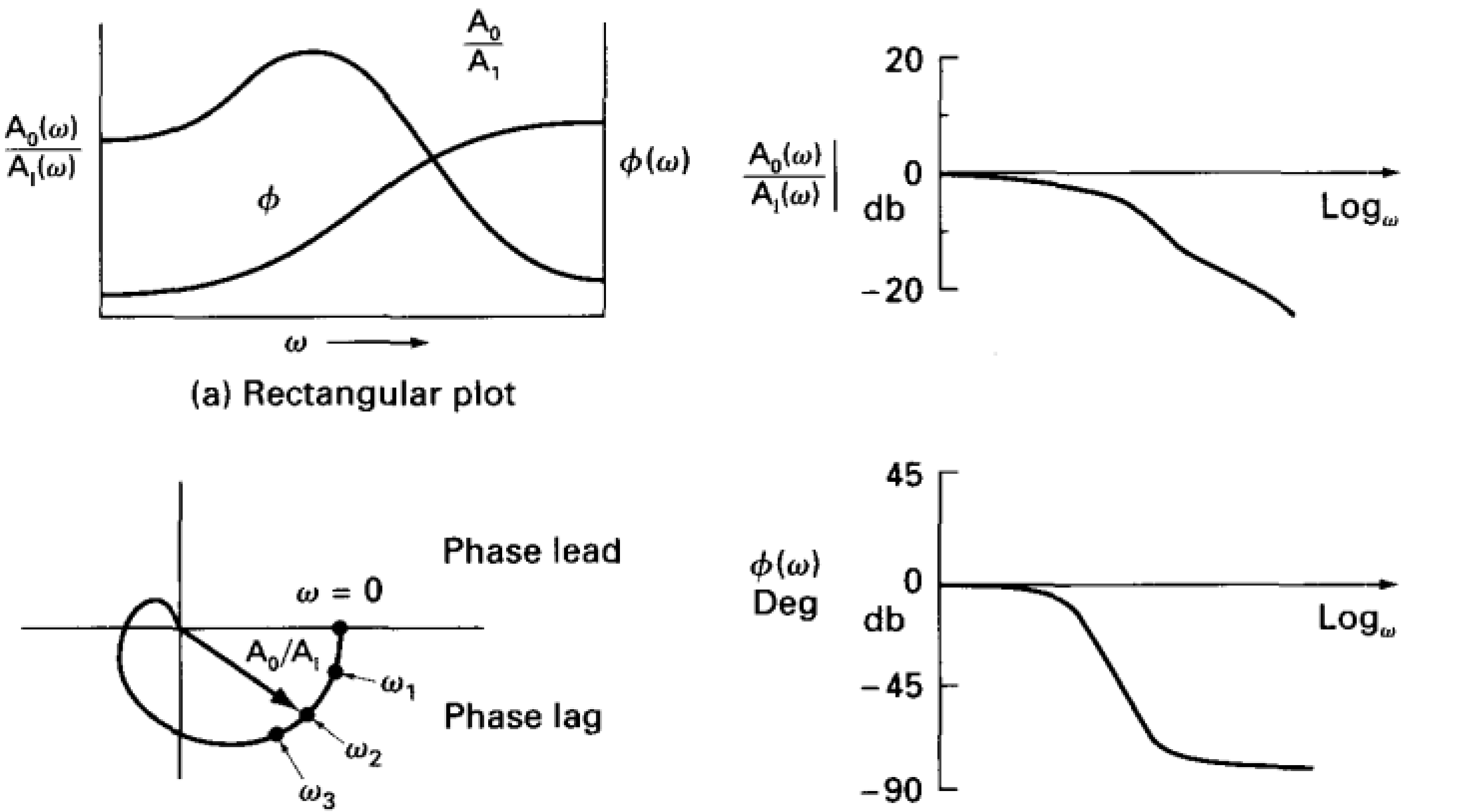
FREQUENCY DOMAIN TECHNIQUES

Various graphical ways of presenting frequency response data.



(a) Rectangular plot

(b) Polar plot

(c) Bode plot

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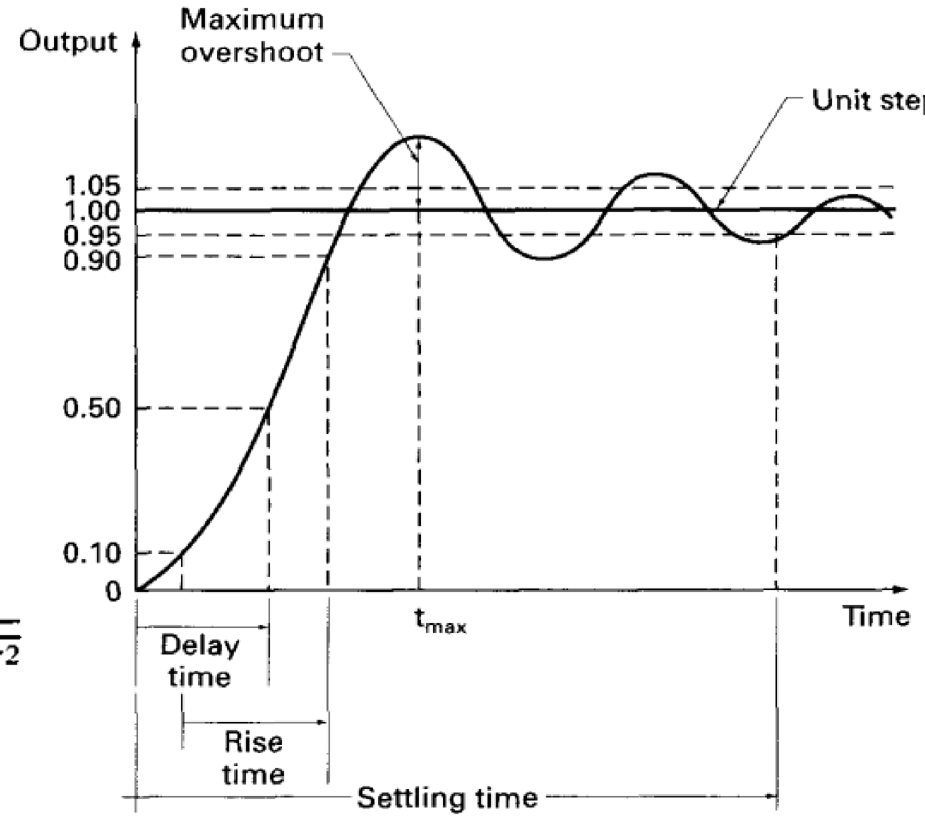
TIME-DOMAIN AND FREQUENCY-DOMAIN SPECIFICATIONS

The transfer function of a second-order system can be expressed as:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The time response of a second-order system to a step input for an underdamped system; that is, Ɛ < 1, is given by



$$c(t) = 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right)$$

$$t_d \approx \frac{1 + 0.6\zeta + 0.15\zeta^2}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$t_r \approx \frac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n}$$

$$t_s = \frac{3.0}{\omega_n \zeta}$$

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

For a unit step Percent maximum overshoot $= 100 \exp(-\pi\zeta/\sqrt{1-\zeta^2})$

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TIME-DOMAIN AND FREQUENCY-DOMAIN SPECIFICATIONS

. **In the frequency domain** the design specifications are given in terms of the response peak M,, the resonant frequency w,, the system bandwidth w,, and the gain and phase margins.

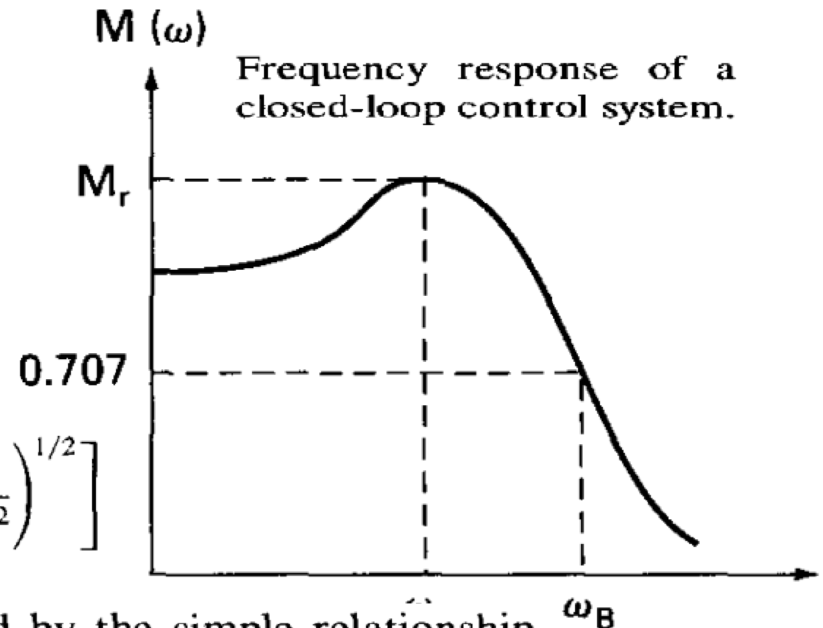
$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

$$\omega_B = \omega_n\left[(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}\right]^{1/2}$$

$$c(t)_{\max} \leq 1.17M_r$$

The phase margin $\phi = \tan^{-1}\left[2\zeta\left(\frac{1}{(4\zeta^4+1)^{1/2}-2\zeta^2}\right)^{1/2}\right]$



This very formidable equation can be approximated by the simple relationship

$\zeta \approx 0.01\phi$ for $\zeta \leq 0.7$ The phase margin ϕ is in degrees.

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From the preceding relationships developed for the second-order system the following observations can be made:

1. The maximum overshoot for a unit step in the time domain is a function of only ζ.
2. The resonance peak of the closed-loop system is a function of only ζ.
3. The maximum peak overshoot and resonance peak are related through the damping ratio.
4. The rise time increases while the bandwidth decreases for increases in system damping for a fixed ω_n, The bandwidth and rise time are inversely proportional to one another.
5. The bandwidth is directly proportional to ω_n.
6. The higher the bandwidth, the larger is the resonance peak.

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Gain and Phase Margin from Root Locus :

The gain margin can be estimated by taking the ratio of the gain when the locus crosses the imaginary axis to the gain selected for the system:

$$\text{Gain margin} = \frac{\text{Value of system gain } k \text{ when locus crosses the imaginary axis}}{\text{Selected value of system gain } k}$$

- The phase margin can be determined for the selected gain by estimating the frequency on the imaginary axis that satisfies the relationship

$$|G(i\omega_g)H(i\omega_g)| = 1$$

- The phase margin can be calculated from the equation

$$\phi_{PM} = 180^\circ + \arg G(i\omega_g)H(i\omega_g).$$

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EXAMPLE 7.5. The root locus plot for a system having the following transfer function is given in Figure :

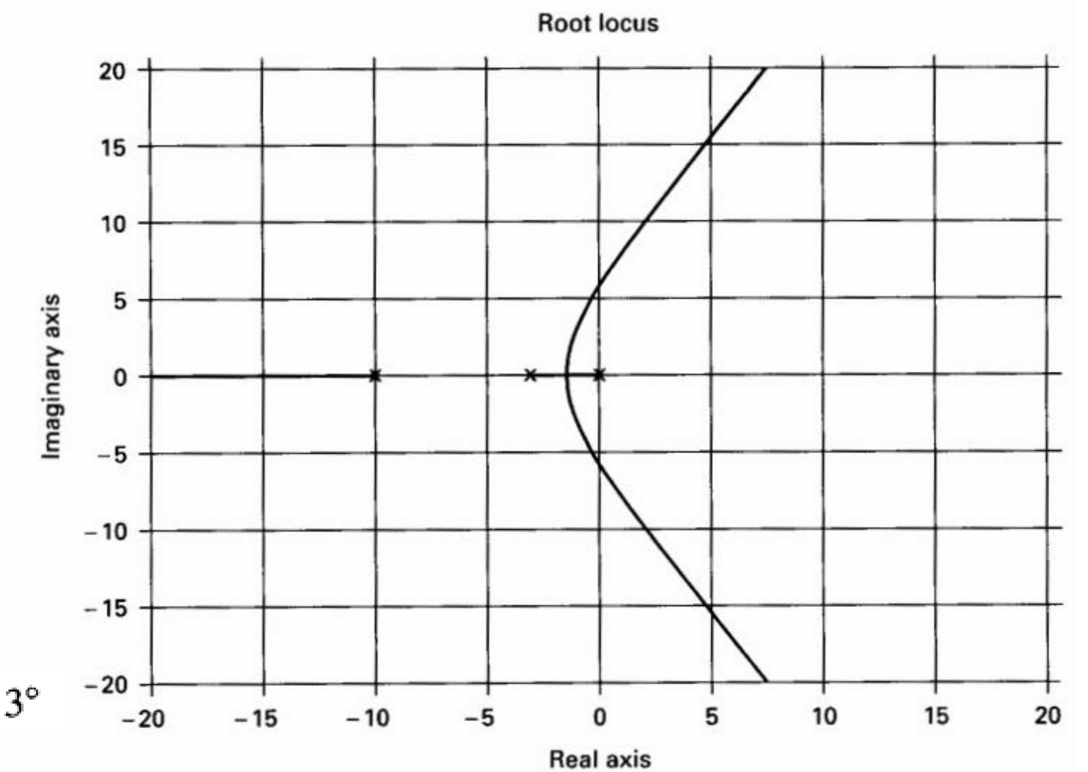
$$G(s)H(s) = \frac{k}{s(s+3)(s+10)}$$

Determine the following information:

(a) Select the system gain so that the dominant roots have a damping ratio, $\zeta = 0.6$.

(b) Estimate the settling time.

(c) Find the gain and phase margin for the gain selected in part (a).



Solution:

$$\zeta = \cos\theta$$

$$\theta = \cos^{-1}[\zeta] = \cos^{-1}[0.6] = 53°$$

The intersection of the line of constant damping ratio (8 = 53" , , ζ = 0.6) with the root locus occurs at s = - 1.2 + 1.65.

$$|G(s)H(s)| = 1 \qquad \frac{|k|}{|s|\,|s+3|\,|s+10|} = 1$$

$$\frac{|k|}{\left(\sqrt{(1.2)^2 + (1.65)^2}\right)\left(\sqrt{(1.8)^2 + (1.65)^2}\right)\left(\sqrt{(8.8)^2 + (1.65)^2}\right)} = 1 \qquad |k| = (2.04)\,(2.44)\,(8.95) = 44.55$$

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EXAMPLE 7.5.

The settling time t, can be estimated from the approximate formula

$$t_s = \frac{3.0}{\zeta\omega_n} \qquad \zeta\omega_n = 1.2 \qquad t_s = \frac{3.0}{\zeta\omega_n} = \frac{3.0}{1.2} = 2.5 \text{ s}$$

To determine the gain margin from the root locus plot we can use Equation

$$\text{Gain margin} = \frac{\text{Value of system gain } k \text{ when locus crosses the imaginary axis}}{\text{Selected value of system gain } k}$$

We need to determine the gain for the system when the root locus crosses the imaginary axis. From the root locus plot we can determine that s = +5.5i at the crossover point. The gain is determined from the magnitude criteria

$$\frac{|k|}{|s|\,|s+3|\,|s+10|} = 1$$

$$\frac{k}{(5.5)(6.26)(11.41)} = 1 \qquad \text{or } k = 393.$$

$$\text{Gain margin} = \frac{393}{44.55} = 8.82$$

The phase margin can be determined by finding the frequency w,, the gain crossover frequency, so that

$$|G(i\omega_g)H(i\omega_g)| = 1.0.$$

$$\frac{44.55}{\omega_g\sqrt{\omega_g^2 + 3^2}\sqrt{\omega_g^2 + 10^2}} = 1 \qquad \omega_g = 1.3.$$

$$\arg G(i\omega_g)H(i\omega_g) = -\angle i\omega_g - \angle(i\omega_g + 3) - \angle(i\omega_g + 10)$$

$$= -90^\circ - 23.4 - 7.4^\circ = 120.8$$

$$\phi_{PM} = 180^\circ - \arg G(i\omega_g)H(i\omega_g)$$

$$= 180^\circ - 120.8^\circ = 59.2^\circ$$

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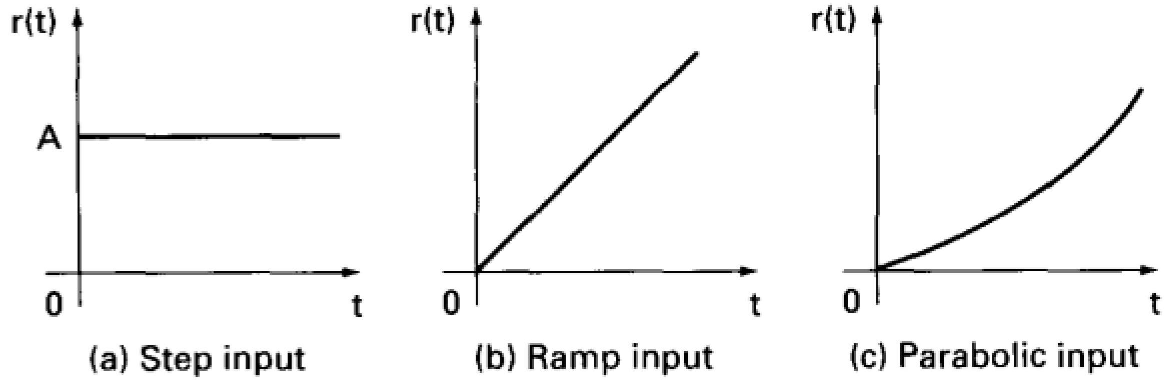
STEADY-STATE ERROR

An expression for the error signal can be developed. The error signal E(s) can be shown to be

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \underset{t\to\infty}{\text{Limit}}\ e(t) = \underset{s\to 0}{\text{Limit}}\ sE(s)$$



Typical input signals.

Steady-state errors

System type	Step input $r(t) = Au(t)$	Ramp input $r(t) = At$	Parabolic input $r(t) = At^2/2$
0	$\frac{A}{1 + K_p}$	∞	∞
1	0	$\frac{A}{K_v}$	∞
2	0	0	$\frac{A}{K_a}$

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STEADY-STATE ERROR

EXAMPLE 7.6. Given the following transfer function, determine the steady-state error of the system to unit step, ramp, and parabolic inputs:

$$G(s)H(s) = \frac{k(s+2)}{s(s+1)(s+4)(s+5)}$$

Solution. The transfer function G(s)H(s) is in the pole-zero form. Rewriting the transfer function in the time constant form yields

$$G(s)H(s) = \frac{2k(1+0.5s)}{20s(1+s)(1+0.25s)(1+0.2s)} = \frac{k}{10}\frac{(1+0.5s)}{s(1+s)(1+0.25s)(1+0.2s)}$$

$$= \frac{K(1+0.5s)}{s(1+s)(1+0.25s)(1+0.2s)} \qquad \text{where } K = k/10$$

From Table ,we see that the steady error is 0 for a step input, 1/Kv for the ramp input, and ∞ for the parabolic input.

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{K(1-0.5s)}{(1+s)(1+0.25s)(1+0.2s)} \qquad K_v = K = \frac{k}{10}$$

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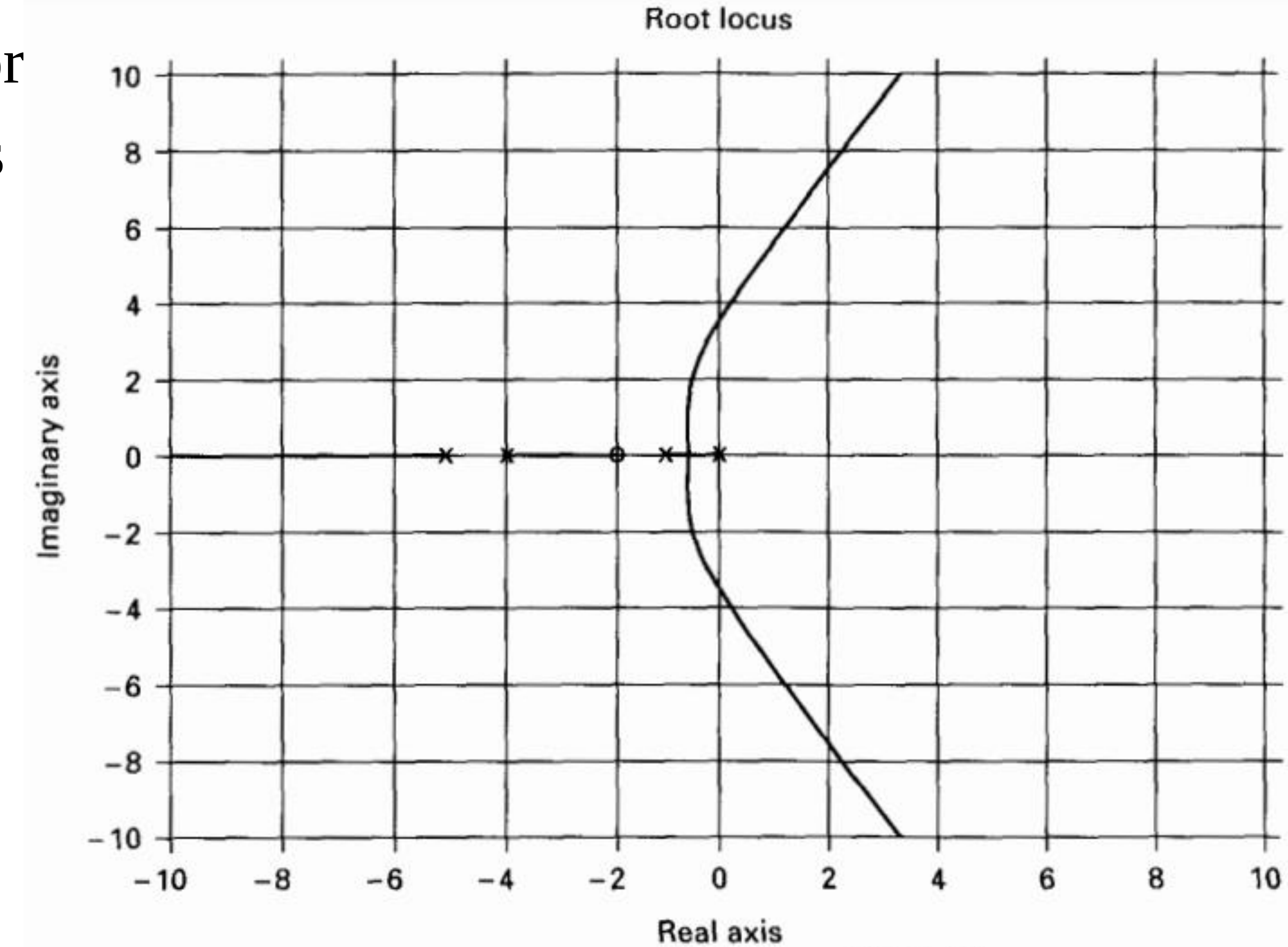
STEADY-STATE ERROR

The steady-state error for the ramp input is

$$e_{ss} = 10/k$$



Root locus plot of $G(s)\,H(s) = \dfrac{k(s+2)}{s(s+1)(s+4)(s+5)}$.

As the system gain is increased, the steady-state error will decrease. However, for this particular example, the system gain is limited because too large a gain will cause the system to be unstable.

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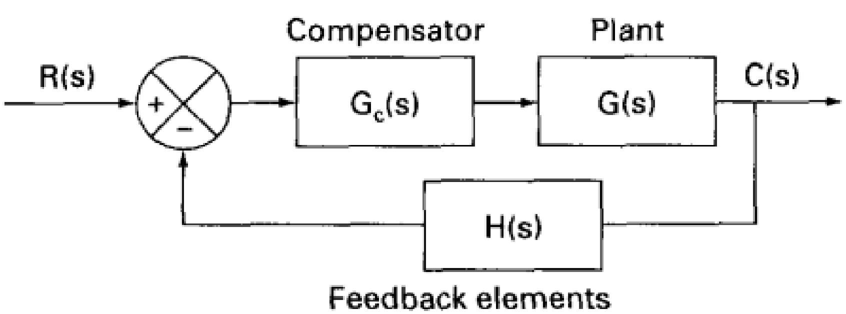
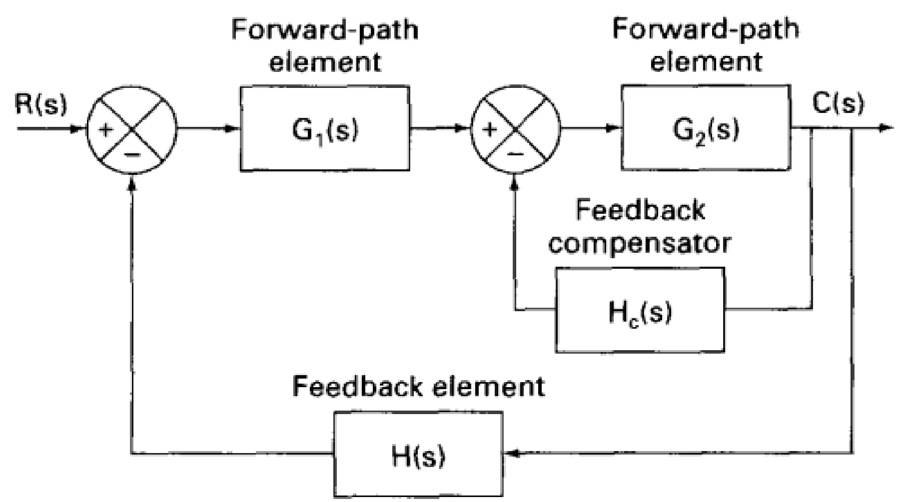
Compensation

The compensators can be thought of as an additional transfer function $G_c(s)$ that can be added to either the forward or feedback path of the control system. When the compensator is added to the forward path it is called a cascade or series compensator and when it is placed in the feedback path, it is called feedback or parallel compensator. In general, the compensators are electrical circuits or mechanical subsystems that provide the designer parameters that can be adjusted to improve the overall system performance.



(a) Forward-path compensation



(b) Inner-loop feedback compensation

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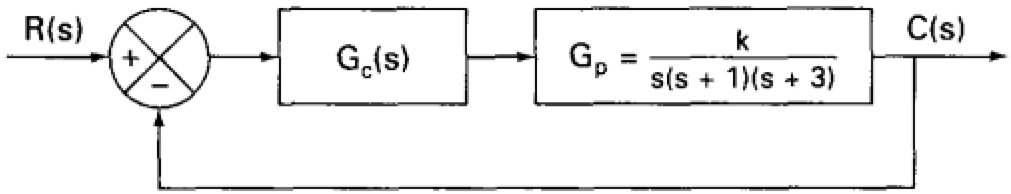
Compensation

Forward-Path Compensation: consider the simple control system shown in Figure 7.15. Suppose that the performance requirements are given in terms of the damping ratio and settling time as follows: $\zeta = 0.707 \qquad t_s < 3$ s.



Control system with a forward-path compensator.

From the root locus plot shown we can achieve the desired damping ratio by finding the gain for the point on the locus that intersects the radial line from the origin that makes an angle of 45" with respect to the negative real axis.

The undamped natural frequency ω_n, is the distance along the radial line of constant

ζ from the origin to the root locus. For this case $\omega_n = 0.5$ rad/s.

The settling time which can be estimated by $t_s = \dfrac{3.0}{\zeta\omega_n} = 3/0.35$

The settling time is not less than 3 s. . If the root locus plot could be made to intersect the $\zeta = 0.707$ line at a larger value of wn, the settling time constraint could be met.

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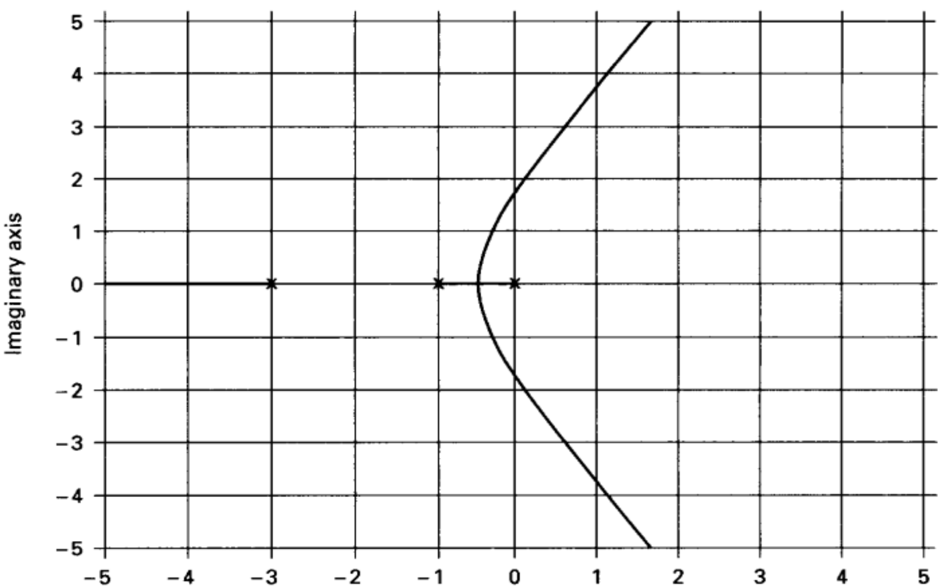
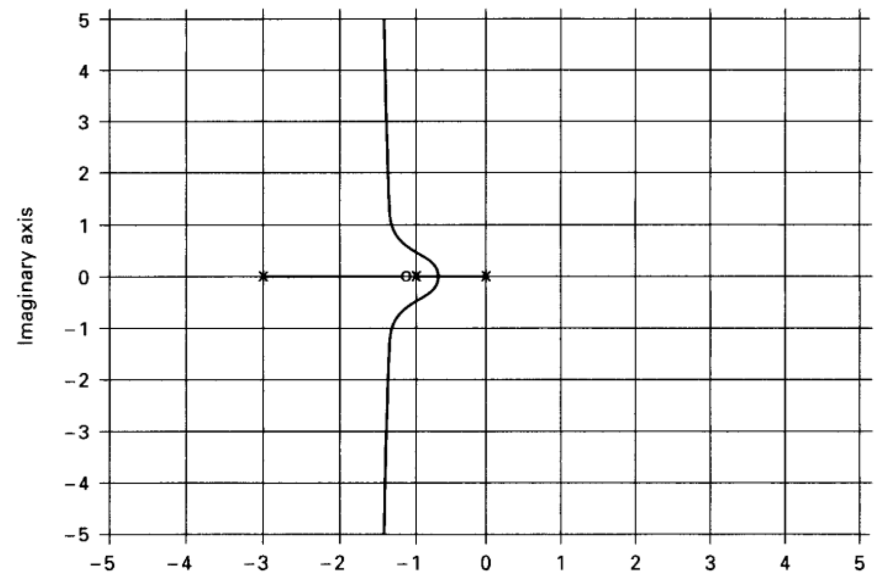
Compensation

If a simple zero, added to an open-loop transfer function G(s)H(s) causes the locus to bend more toward the left in the complex plane



Root locus plot of $G(s)H(s) = \dfrac{k}{s(s+1)(s+3)}$.



Root locus plot of $G(s)H(s) = \dfrac{k(s+1.1)}{s(s+1)(s+3)}$.

Figure (b) is a root locus plot with the addition of a zero as s = - 1.1. With the addition of the zero, the root locus plot bends toward the left. The value of on for the damping ratio of 0.707 is now 1.98 radls, which yields a settling time less than 3 s

LEC. Six

- **Compensation**

Lead compensator:

Unfortunately a simple zero is not very practical. In practice we add a transfer function of the form $G_c(s) = \frac{s + z_c}{s + p_c}$

where $z_c/p_c < 1$, or the compensator poles is located to the left of the compensator zero. Such a compensator is called a **lead compensator**.

- The lead compensator can be used to improve the transient response characteristics of the control system, by proper adjust the pole and zero location of the compensator to shape the root locus so that both the damping ratio and settling time specifications can be met
- It is possible to have a control system design with good transient characteristics but a a large steady-state error. When the steady-state error is large a lag compensator can be used to improve the steady-state error.
- where the compensator pole near the origin is located to the

$$G_c(s) = \frac{(s + z_c)}{(s + p_c)}$$

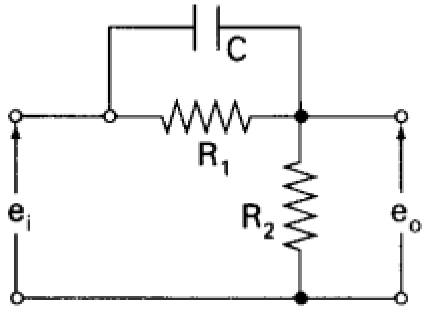
right of the compensator zero ($z_c/p_c > 1$).

For the case where both the transient and steady response are unsatisfactory a combination of a lag and lead compensator can be used. An example of a lag-lead compensator follows:

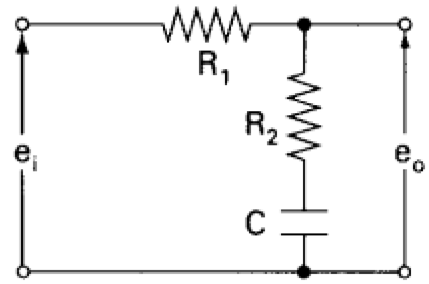
$$G_c(s) = \frac{(s + z_1)}{(s + p_1)}\frac{(s + z_2)}{(s + p_2)}$$

LEC. Six

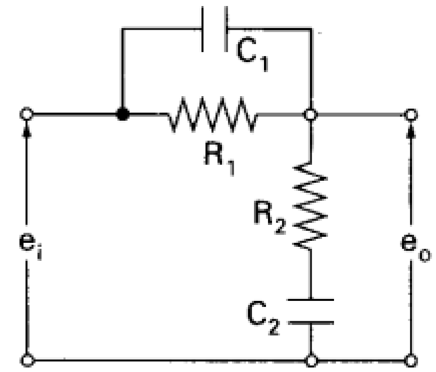
Compensation



(a) Lead circuit



(b) Lag circuit



(c) Lag-lead circuit

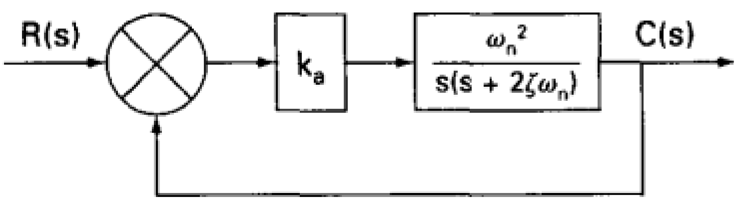
Electrical circuits used as a compensator.

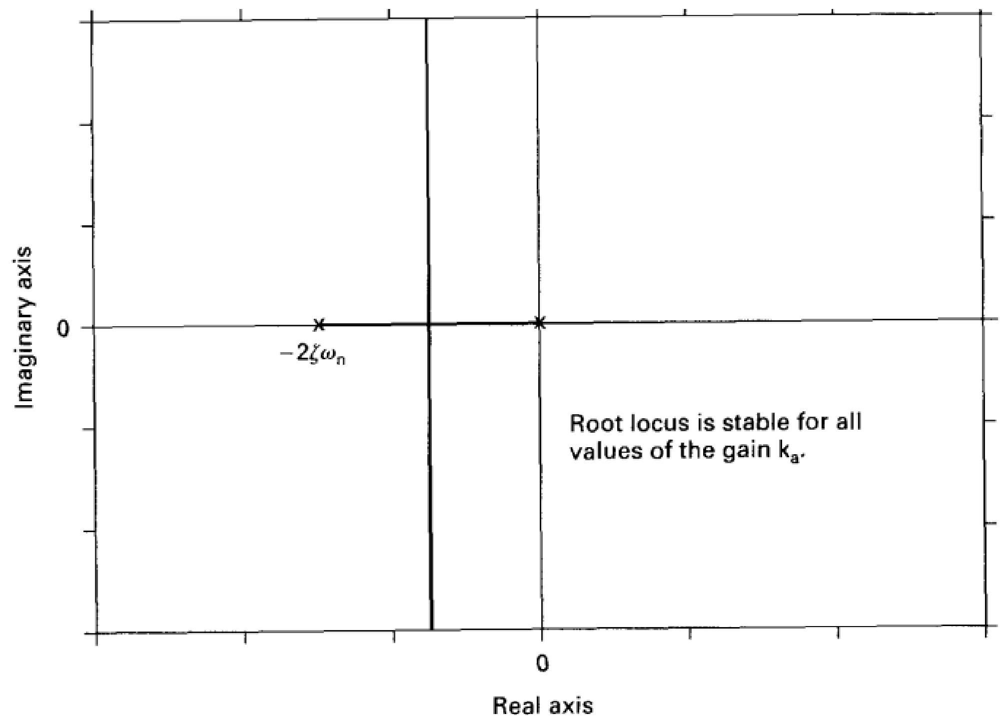
LEC. Six

Compensation

Feedback-Path Compensation

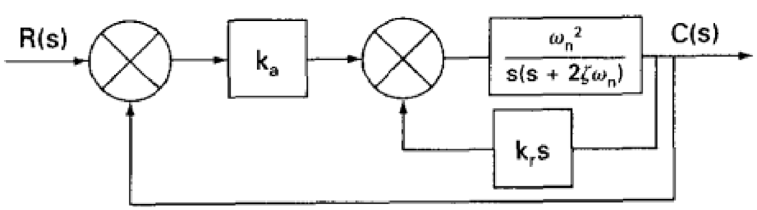
Feedback compensation can be used to improve the damping of the system by incorporating an inner rate feedback loop. The stabilizing effect of the inner loop rate feedback can be demonstrated by a simple example





The closed-loop transfer function for this system is given by

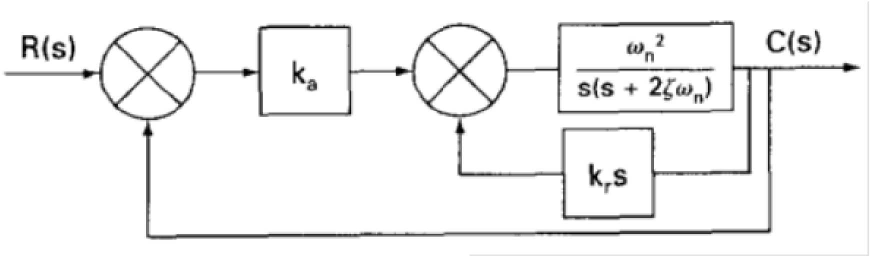
$$M(s) = \frac{k_a\omega_n}{s^2 + 2\zeta\omega_n s + k_a\omega_n^2}$$



LEC. Six

Compensation

Feedback-Path Compensation



The inner loop TF

$$G_1(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

$$H_1(s) = k_r s \qquad M(s)_{\text{I.L.}} = \frac{G_1(s)}{1 + G_1(s)H_1(s)} = \frac{\omega_n^2}{s^2 + (2\zeta\omega_n + k_r\omega_n^2)s}$$

The closed-loop transfer function can be obtained by letting

$$G(s)_2 = \frac{k_a\omega_n^2}{s^2 + (2\zeta\omega_n + k_r\omega_n^2)s} \qquad H_2(s) = 1$$

$$M(s)_{\text{O.L.}} = \frac{G_2(s)}{1 + G_2(s)H_2(s)} = \frac{k_a\omega_n^2}{s^2 + (2\zeta\omega_n + k_r\omega_n^2)s + k_a\omega_n^2}$$

If we compare the closed-loop transfer function for the cases with and without rate feedback we observe that in the closed-loop characteristic equation the damping term has been increased by $k\omega_n^2$. The gain k, can be used to increase the system damping.

LEC. Six

PID CONTROLLER

- We have shown examples of various kinds of control concepts. The simplest feedback controller is one for which the controller output is proportional to the error

signal. Such a controller is called **<u>Proportional controller</u>**. Obviously the controller's main **advantage is its simplicity**. It has the **disadvantage that there may be a steady-state error.**

- The steady-state error can be eliminated by using an **integral controller**

$$\eta(t) = k_i \int_0^t e(t)\,\mathrm{d}t \qquad \text{or} \qquad \eta(s) = \frac{k_i}{s} e(s)$$

where k, is the integral gain. **The advantage** of the integral controller is that the output is **proportional to the accumulated error**. **The disadvantage** of the integral controller is that we **make the system less stable** by adding the **pole at the origin**. Recall that the addition of a pole to the forward-path transfer function was shown to bend the root locus toward the right half plane.

- It is also possible to use a derivative controller defined as follows:

$$\eta(t) = k_d \frac{\mathrm{d}e}{\mathrm{d}t} \qquad \text{or} \qquad \eta(s) = k_d s e(s)$$

LEC. Six

PID CONTROLLER

- **The advantage** of the derivative controller is that the controller will **provide large corrections before the error becomes large**. The major **disadvantage of the derivative controller is that it will not produce a control output if the error is constant.**
- Each of the controllers-providing proportional, integral, and derivative

control-has its advantages and disadvantages. The disadvantages of each controller can be eliminated by combining all three controllers into a single PID Controller.

- ❑ The selection of the gains for the PID controller can be determined by a method developed by Ziegler and Nichols
- ❑ Based on their analysis they derived a set of rules for selecting the PID gains. The gains k_p, k_i, and k_d are determined in terms of two parameters, k_{Pu}, called the ultimate gain, and T_u, the period of the oscillation that occurs at the ultimate gain.

LEC. Six

PID CONTROLLER

Gains for P, PI, and PID controllers

Type of controller	k_p	k_i	k_d
P (proportional controller)	$k_p = 0.5k_{p_u}$		
PI (proportional-integral controller)	$k_p = 0.45k_{p_u}$	$k_i = 0.45k_{p_u}/(0.83T_u)$	
PID (proportional-integral-derivative controller)	$k_p = 0.6k_{p_u}$	$k_i = 0.6k_{p_u}/(0.5T_u)$	$k_d = 0.6k_{p_u}(0.125T_u)$

To apply this technique the root locus plot for the control system with the integral and derivative gains set to 0 must become marginally stable. That is, as

the proportional gain is increased the locus must intersect the imaginary axis. The proportional gain, **k_P** for which this occurs is called the ultimate The purely imaginary roots, $\lambda = \pm i\omega$, determine the value of T_u:

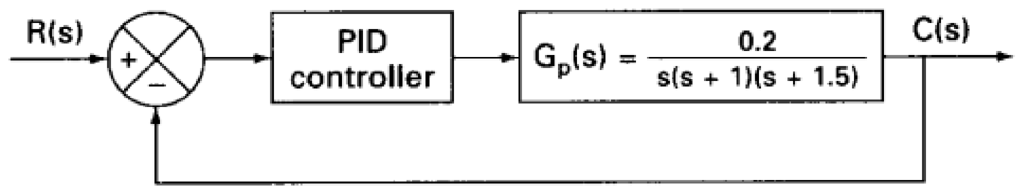
$$T_u = \frac{2\pi}{\omega}$$

- All other roots of the system must have negative real parts; that is, they must be in the left-hand portion of the complex s plane. If these restrictions are satisfied the P, PI, or PID gains easily can be determined

LEC. Six

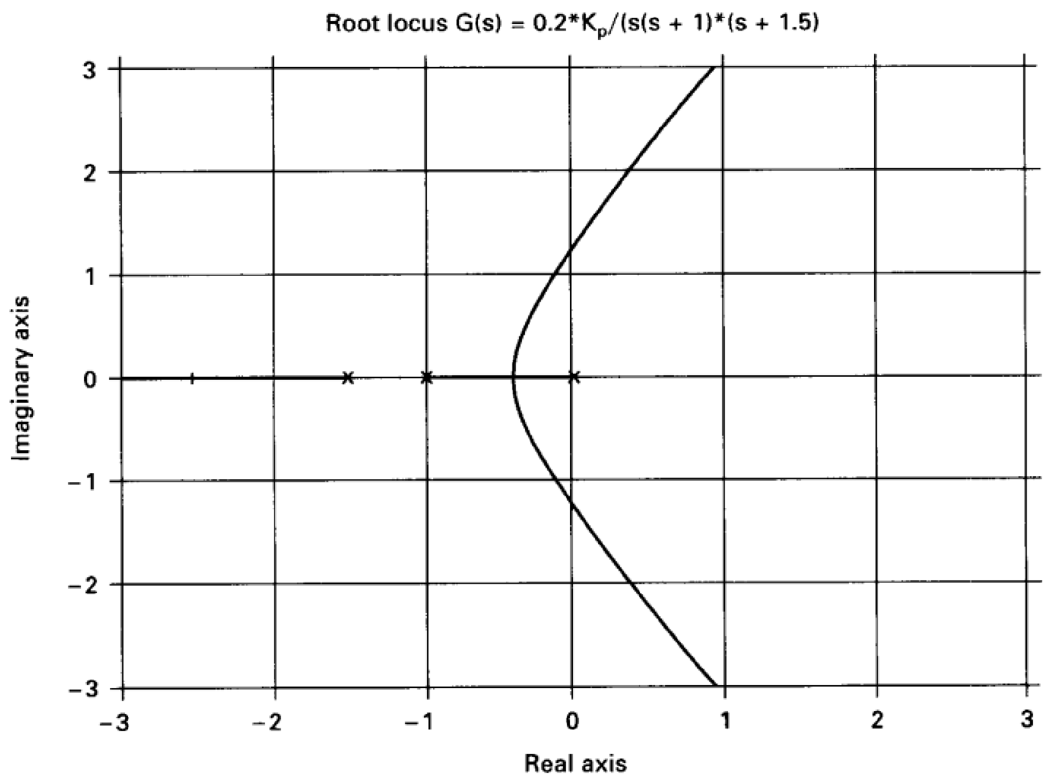
PID CONTROLLER

EXAMPLE 7.7. : Design a PID controller for the control system shown



. Two branches of the locus cross the imaginary axis and all other roots lie in the left half plane

$$G(s) = \frac{0.2k_p}{s(s + 1)(s + 1.5)}$$



The ultimate gain $\mathbf{k_{pu}}$ is found by finding the gain when the root locus intersects the imaginary axis

- The locus intersects the imaginary

+ 1.25. The gain at the crossover point can be estimated from the magnitude criteria:

$$\frac{|0.2|\, k_{p_u}}{|s|\,\|s + 1\|\,|s + 1.5|} = 1$$

PID CONTROLLER

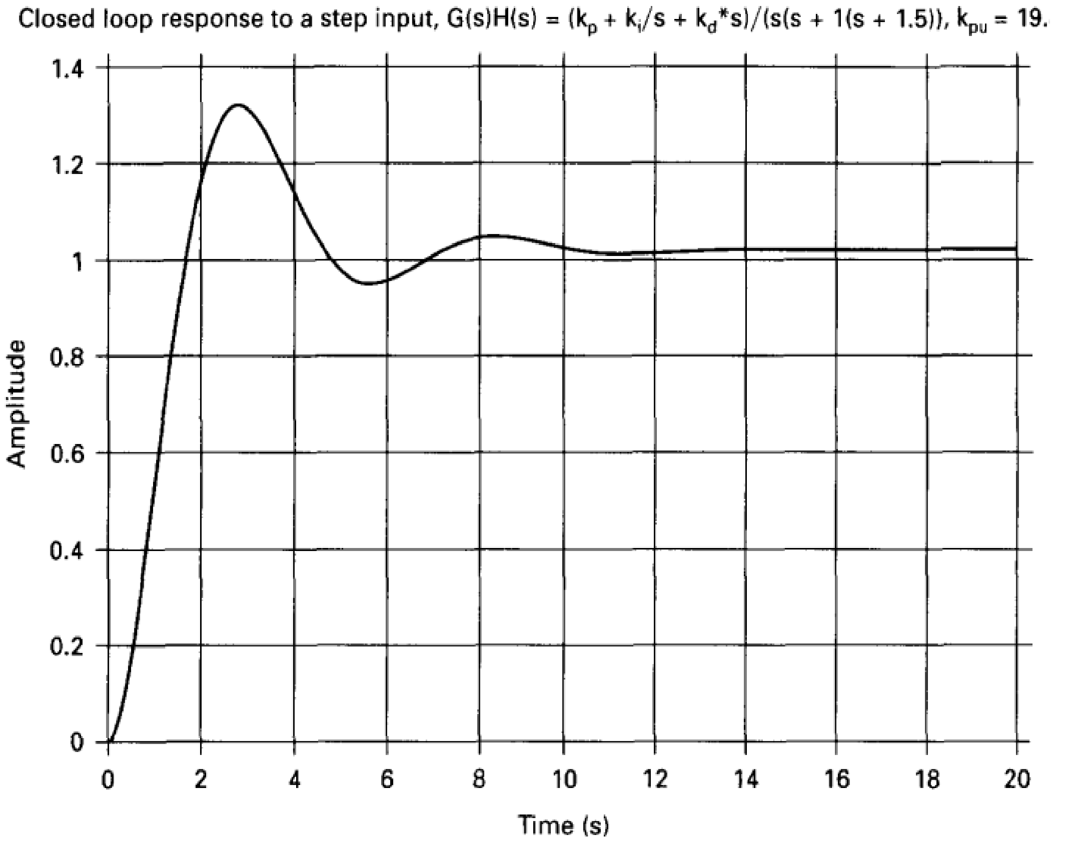
- Substituting s = 1.25 into the magnitude criteria yields $\mathbf{k_{Pu}}$ = 19.8
- The period of the undamped oscillation $T_u = \frac{2\pi}{\omega} = \frac{2\pi}{1.25} = 5.03$
- T_u is obtained as follows:
- Knowing $\mathbf{k_{pu}}$ and $\mathbf{T_u}$ the proportional, integral, and derivative gains k_p, k_i, and k_d can be evaluated:

$$k_p = 0.6\ k_{p_u} = (0.6)(19.8) = 11.88$$

$$k_1 = 0.6\ k_{p_u}/(0.5T_u)$$
$$= (0.6)(11.88)/[(0.5)(19.8)] = 0.72$$

$$k_d = 0.6\ k_{p_u}\ (0.125T_u)$$
$$= (0.6)(19.8)(0.125)(5.03) = 7.47$$

- The response of control system to a step input is given in Figure



LEC. Six

PROBLEMS

7.1. Given the characteristic equation $\lambda^3 + 3\lambda^2 + 3\lambda + 1 + k = 0$

find the range of values of k for which the system is stable.

7.2. Given the fourth-order characteristic equation $\lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda + k = 0$
for what values of k will the system be stable?

7.3. Given the following characteristic equation determine the stability of the system using the Routh criterion. If the system is unstable determine the number of roots lying in the left portion of the complex plane.

$$\lambda^6 + 3\lambda^5 + 5\lambda^4 + 9\lambda^3 + 8\lambda^2 + 6\lambda + 4 = 0$$

7.8. The single degree of freedom pitching motion of an airplane was shown to be represented by a second-order differential equation. If the equation is given as

$$\ddot{\theta} + 0.5\dot{\theta} + 2\theta = \delta_e$$

where the θ and δ_e are in radians, estimate the rise time, peak overshoot, and settling time for step input of the elevator angle of 0.10 rad.

LEC. Six

PROBLEMS

7.9. Determine the frequency domain characteristic for Problem 7.8. In particular estimate the resonance peak, M_r, resonant frequency, ω_r, bandwidth, ω_B, and the phase margin.

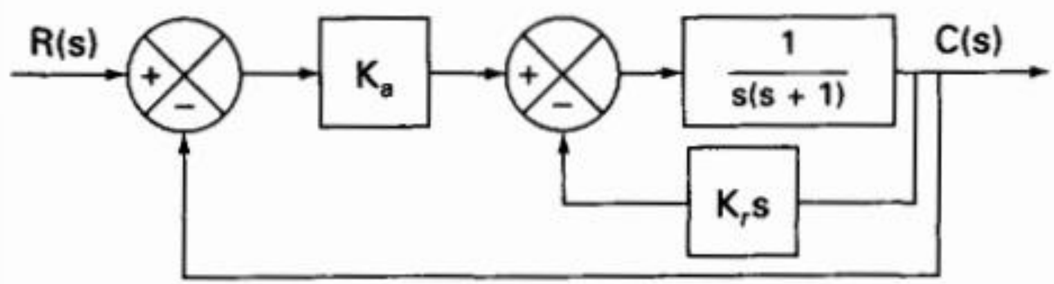
7.11. Calculate the position, velocity, and acceleration error constants K_p, K_v, and K_a for the loop transfer function $G(s)H(s)$ that follows:

(a) $\dfrac{10}{s(s+1)(s+10)}$ (b) $\dfrac{k}{s(1+0.1s)(1+s)}$ (c) $\dfrac{k}{s(s^2+4s+100)}$

(d) $\dfrac{s+2}{s(s^2+4s+6)}$ (e) $\dfrac{15(s+2)}{s^2(s+5)(s+3)}$

7.15. In the control system shown in Figure P7.15 rate feedback is to be used to increase the system damping. Estimate the gains k_a and k_r so that the system meets the following performance specifications:

Damping ratio, $\zeta = 0.7$
Settling time, ≤ 3.0 s



LEC. Six

PROBLEMS

7.16(C). Given the control system shown in Figure P7.16 where the plant transfer function $G(s)$ is given by

$$G(s) = \frac{2.0}{s(s+1)(s+3)}$$

design a PID controller for this system.

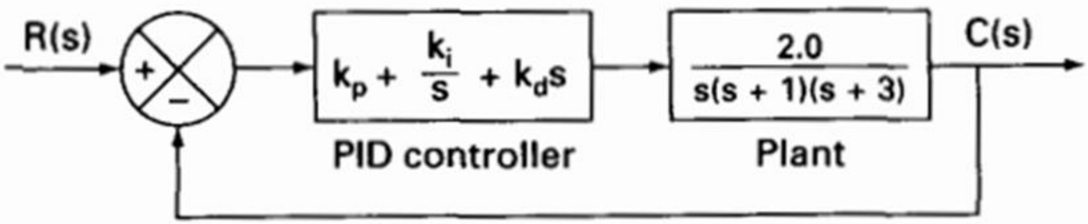


FIGURE P7.16

7.17(C). If the plant transfer function for Problem 7.16 is changed to

$$G(s) = \frac{7.0}{(s+5)(s^2+2s+5)}$$

design a PID controller for this system.

LEC. Six