Aircraft Transfer Functions

First, we shall assume that the aircraft's motion consists of small deviations from its equilibrium flight condition.

Second, we shall assume that the motion of the airplane can be analyzed by separating the equations into two groups.

The X-force, Z-force, and pitching moment equations. And the Y-force, rolling, and yawing moment equations form the lateral equations.

The longitudinal motion of an airplane (controls fixed) disturbed from its equilibrium flight condition is characterized by two oscillatory modes of motion, Figure1 illustrates these basic modes.

We see that one mode is lightly damped and has a long period. This motion is called the long-period or phugoid mode.

The second basic motion is heavily damped and has a very short period; it is appropriately called the short-period mode.

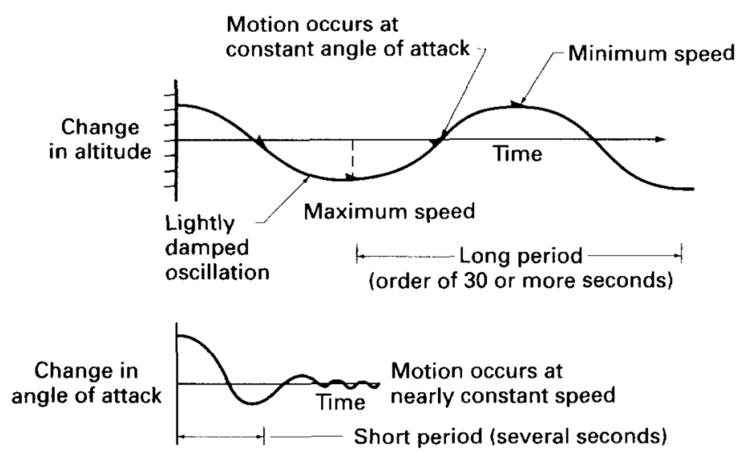

FIGURE 1 The phugoid and short-period motions.

1. **State Variable Representation of the Equations of Motion**
   The linearized longitudinal equations developed earlier are simple, ordinary linear differential equations with constant coefficients. The coefficients in the differential equations are made up of the aerodynamic stability derivatives, mass, and inertia characteristics of the airplane. These equations can be written as a set of first-order differential equations, called the state-space or state variable equations and represented mathematically as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\eta} \qquad \text{……….(1)}$$

where x is the state vector, q is the control vector, and the matrices A and B contain the aircraft's dimensional stability derivatives.

The linearized longitudinal set of equations developed earlier are repeated here:

Longitudinal equations

$$\left(\frac{d}{dt} - X_u\right) \Delta u - X_w \,\Delta w + (g \cos \theta_0)\, \Delta\theta = X_{\delta_e}\, \Delta\delta_e + X_{\delta_T}\, \Delta\delta_T$$

$$-Z_u\, \Delta u + \left[(1 - Z_{\dot w})\frac{d}{dt} - Z_w\right] \Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g \sin \theta_0\right] \Delta\theta = Z_{\delta_e}\, \Delta\delta_e + Z_{\delta_T}\, \Delta\delta_T$$

$$-M_u\, \Delta u - \left(M_{\dot w}\frac{d}{dt} + M_w\right) \Delta w + \left(\frac{d^2}{dt^2} - M_q \frac{d}{dt}\right) \Delta\theta = M_{\delta_e}\, \Delta\delta_e + M_{\delta_T}\, \Delta\delta_T$$

In practice, the force derivatives Z_q and $Z_{\dot w}$ usually are neglected because they contribute very little to the aircraft response. Therefore, to simplify our presentation of the equations of motion in the state-space form we will neglect both Z_q and $Z_{\dot w}$. Rewriting the equations in the state-space form yields

$$\begin{bmatrix} \Delta\dot u \\ \Delta\dot w \\ \Delta\dot q \\ \Delta\dot\theta \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot w} Z_u & M_w + M_{\dot w} Z_w & M_q + M_{\dot w} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta + M_{\dot w} Z_\delta & M_{\delta_T} + M_{\dot w} Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix} \qquad (2)$$

where the state vector **x** and control vector **η** are given by

$$\mathbf{x} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix} \qquad (3)$$

and the matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot w} Z_u & M_w + M_{\dot w} Z_w & M_q + M_{\dot w} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & X_{\delta_T} \\ M_\delta + M_{\dot w} Z_\delta & M_{\delta_T} + M_{\dot w} Z_{\delta_T} \\ 0 & 0 \end{bmatrix}$$

Equation (1) can be obtained by assuming a solution of the form

$$\mathbf{x} = \mathbf{x}_r e^{\lambda_r t} \qquad (5)$$

Substituting Equation (5) into Equation (1) yields

$$[\lambda_r \mathbf{I} - \mathbf{A}]\mathbf{x}_r = 0 \qquad (6)$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7)$$

For a nontrivial solution to exist, the determinant

$$|\lambda_i \mathbf{I} - \mathbf{A}| = 0$$

EXAMPLE PROBLEM 4.2. Given the differential equations that follow

$$\dot{x}_1 + 0.5x_1 - 10x_2 = -1\delta$$

$$\dot{x}_2 - x_2 + x_1 = 2\delta$$

where x_1 and x_2 are the state variables and δ is the forcing input to the system:

(a) Rewrite these equations in state space form; that is,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\eta}$$

(b) Find the free response eigenvalues.
(c) What do these eigenvalues tell us about the response of this system?

Solution. Solving the differential equations for the highest order derivative yields

$$\dot{x}_1 = -0.5x_1 + 10x_2 - \delta$$

$$\dot{x}_2 = -x_1 + x_2 + 2\delta$$

or in matrix form
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 10 \\ -1.0 & 1.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \delta$$

which is the state space formulation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\eta}$$

where $\mathbf{A} = \begin{bmatrix} -0.5 & 10 \\ -1.0 & 1.0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

The eigenvalues of the system can be determined by solving the equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

Substituting the A matrix into the preceding equation yields

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -0.5 & 10 \\ -1.0 & 1.0 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -0.5 & 10 \\ -1.0 & 1.0 \end{bmatrix} \right| = \begin{vmatrix} \lambda + 0.5 & -10 \\ 1.0 & \lambda - 1.0 \end{vmatrix} = 0$$

Expanding the determinant yields the characteristic equation

$$(\lambda + 0.5)(\lambda - 1.0) + 10 = 0 \quad \text{or} \quad \lambda^2 - 0.5\lambda + 9.5 = 0$$

$$\lambda_{1,2} = 0.25 \pm 3.07i$$

The eigenvalues are complex and the real part of the root is positive. This means that the system is dynamically unstable. If the system were given an initial disturbance, the motion would grow sinusoidally and the frequency of the oscillation would be governed by the imaginary part of the complex eigenvalue. The time to double amplitude can be calculated.

$$t_{\text{double}} = \frac{0.693}{|\eta|} = \frac{0.693}{0.25} = 2.77 \text{ s}$$

The period of the sinusoidal motion can be calculated from Equation

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{3.07} = 2.05 \text{ s}$$

2. LONGITUDINAL APPROXIMATIONS

We can think of the long-period or phugoid mode as a gradual interchange of potential and kinetic energy about the equilibrium altitude and airspeed. This is illustrated in Figure 1.

Long Period Approximation:

Here we see that the long-period mode is characterized by changes in pitch attitude, altitude, and velocity at a nearly constant angle of attack. An approximation to the long-period mode can be obtained by neglecting the pitching moment equation and assuming that the change in angle of attack is 0; that is,

$$\Delta\alpha = \frac{\Delta w}{u_0} \qquad \Delta\alpha = 0 \rightarrow \Delta w = 0 \qquad \textbf{(8)}$$

Making these assumptions, the homogeneous longitudinal state equations reduce to the following:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ \dfrac{-Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}$$

The eigenvalues of the long-period approximation are obtained by solving the equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0 \qquad \text{or} \qquad \begin{vmatrix} \lambda - X_u & g \\ \dfrac{Z_u}{u_0} & \lambda \end{vmatrix} = 0$$

$$\omega_{n_p} = \sqrt{\frac{-Z_u g}{u_0}} \qquad \zeta_p = \frac{-X_u}{2\omega_{n_p}}$$

If we neglect compressibility effects, the frequency and damping ratios for the long-period motion can be approximated by the following equations

$$\omega_{n_p} = \sqrt{2}\,\frac{g}{u_0} \qquad \zeta_p = \frac{1}{\sqrt{2}}\frac{1}{L/D}$$

3. Short-Period Approximation

An approximation to the short-period mode of motion can be obtained by assuming $\Delta u = 0$ and dropping the X-force equation. The longitudinal state-space equations reduce to the following:

$$\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

This equation can be written in terms of the angle of attack by using the relationship

$$M_\alpha = \frac{1}{I_y}\frac{\partial M}{\partial \alpha} = \frac{1}{I_y}\frac{\partial M}{\partial(\Delta w/u_0)} = \frac{u_0}{I_y}\frac{\partial M}{\partial w} = u_0 M_w$$

In addition, one can replace the derivatives due to w and w with derivatives due to α and (**α'**) by using the following equations. The definition of the derivative Ma is

$$Z_\alpha = u_0 Z_w \qquad \text{and} \qquad M_{\dot{\alpha}} = u_0 M_{\dot{w}}$$

Using these expressions, the state equations for the short-period approximation can be rewritten as

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_\alpha}{u_0} & 1 \\ M_\alpha + M_{\dot{\alpha}}\dfrac{Z_\alpha}{u_0} & M_q + M_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix}$$

The eigenvalues of the state equation can again be determined by solving the equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

which yields

$$\begin{vmatrix} \lambda - \dfrac{Z_\alpha}{u_0} & -1 \\ -M_\alpha - M_{\dot{\alpha}}\dfrac{Z_\alpha}{u_0} & \lambda - (M_q + M_{\dot{\alpha}}) \end{vmatrix} = 0$$

The characteristic equation for this determinant is

$$\lambda^2 - \left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)\lambda + M_q\frac{Z_\alpha}{u_0} - M_\alpha = 0$$

The approximate short-period roots can be obtained easily from the characteristic equation,

$$\lambda_{sp} = \left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)\Big/2 \pm \left[\left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)^2 - 4\left(M_q\frac{Z_\alpha}{u_0} - M_\alpha\right)\right]^{1/2}\Big/2$$

or in terms of the damping and frequency

$$\omega_{n_{sp}} = \left[\left(M_q\frac{Z_\alpha}{u_0} - M_\alpha\right)\right]^{1/2}$$

$$\zeta_{sp} = -\left[M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right]\Big/(2\omega_{n_{sp}})$$

Summary of longitudinal approximations

	Long period (phugoid)	Short period
Frequency	$\omega_{n_p} = \sqrt{\frac{-Z_u g}{u_0}}$	$\omega_{nsp} = \sqrt{\frac{Z_\alpha M_q}{u_0} - M_\alpha}$
Damping ratio	$\zeta_p = \frac{-X_u}{2\omega_{n_p}}$	$\zeta_{sp} = -\frac{M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}}{2\omega_{n\,sp}}$

4. **Short-Period Dynamics**

The equation with control input from the elevator in state space form can be written as:

$$\begin{bmatrix}\Delta\dot{\alpha}\\ \Delta\dot{q}\end{bmatrix} = \begin{bmatrix}Z_\alpha/u_0 & 1\\ M_\alpha + M_{\dot{\alpha}}Z_\alpha/u_0 & M_q + M_{\dot{\alpha}}\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta q\end{bmatrix} + \begin{bmatrix}Z_{\delta_e}/u_0\\ M_{\delta_e} + M_{\dot{\alpha}}Z_{\delta_e}/u_0\end{bmatrix}[\Delta\delta_e] \quad (8.1)$$

The control due to the propulsion system is neglected here for simplicity. Taking the Laplace transform of this equation yields

$$(s - Z_\alpha/u_0)\,\Delta\alpha(s) - \Delta q(s) = Z_{\delta_e}/u_0\,\Delta\delta_e(s) \qquad (9)$$

$$-(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0)\,\Delta\alpha(s) + [s - (M_q + M_{\dot\alpha})]\,\Delta q(s) = (M_{\delta_e} + M_{\dot\alpha} Z_{\delta_e}/u_0)\,\Delta\delta_e \qquad (10)$$

If we divide these equations by $\Delta\delta_e(s)$ we obtain a set of algebraic equations in terms of the transfer functions $\Delta\alpha(s)/\Delta\delta_e(s)$ and $\Delta q(s)/\Delta\delta_e(s)$:

$$(s - Z_\alpha/u_0)\frac{\Delta\alpha(s)}{\Delta\delta_e(s)} - \frac{\Delta q(s)}{\Delta\delta_e(s)} = Z_{\delta_e}/u_0 \qquad (11)$$

$$-(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0)\frac{\Delta\alpha(s)}{\Delta\delta_e(s)} + [s - (M_q + M_{\dot\alpha})]\frac{\Delta q(s)}{\Delta\delta_e(s)} = M_{\delta_e} + M_{\dot\alpha}\frac{Z_{\delta_e}}{u_0} \qquad (12)$$

Solving for $\Delta\alpha(s)/\Delta\delta_e(s)$ and $\Delta q(s)/\Delta\delta_e(s)$ by Cramer's rule yields

$$\frac{\Delta\alpha(s)}{\Delta\delta_e(s)} = \frac{N^{\alpha}_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{\begin{vmatrix} Z_{\delta_e}/u_0 & -1 \\ M_{\delta_e} + M_{\dot\alpha}\dfrac{Z_{\delta_e}}{u_0} & s - (M_q + M_{\dot\alpha}) \end{vmatrix}}{\begin{vmatrix} s - Z_\alpha/u_0 & -1 \\ -(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0) & s - (M_q + M_{\dot\alpha}) \end{vmatrix}} \qquad (13)$$

Expanded, the numerator and denominator are polynomials in the Laplace variable s. The coefficients of the polynomials are a function of the stability derivatives

$$\frac{\Delta\alpha(s)}{\Delta\delta_e(s)} = \frac{N^{\alpha}_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{A_\alpha s + B_\alpha}{As^2 + Bs + C} \qquad (14)$$

The coefficients in the numerator and denominator are given in Table 8.2. The transfer function for the change in pitch rate to the change in elevator angle can be shown to be:

$$\frac{\Delta q(s)}{\Delta\delta_e(s)} = \frac{N^{q}_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{\begin{vmatrix} s - Z_\alpha/u_0 & Z_{\delta_e}/u_0 \\ -(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0) & M_{\delta_e} + M_{\dot\alpha}\dfrac{Z_{\delta_e}}{u_0} \end{vmatrix}}{\begin{vmatrix} s - Z_\alpha/u_0 & -1 \\ -(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0) & s - (M_q + M_{\dot\alpha}) \end{vmatrix}} \qquad (15)$$

$$\frac{\Delta q(s)}{\Delta\delta_e(s)} = \frac{N^{q}_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{A_q s + B_q}{As^2 + Bs + C} \qquad (16)$$

TABLE 2
Short-period transfer function approximations

	A, A_α, or A_q	B, B_α, or B_q	C
$\Delta_{sp}(s)$	1	$-(M_q + M_{\dot\alpha} + Z_\alpha/u_0)$	$Z_\alpha M_q/u_0 - M_\alpha$
$N^{\alpha}_{\delta_e}(s)$	Z_{δ_e}/u_0	$M_{\delta_e} - M_q Z_{\delta_e}/u_0$	
$N^{q}_{\delta_e}(s)$	$M_{\delta_e} + M_{\dot\alpha} Z_{\delta_e}/u_0$	$M_\alpha Z_{\delta_e}/u_0 - M_{\delta_e} Z_\alpha/u_0$	

5. **Long-Period or Phugoid Dynamics**

The state-space equation for the long period or phugoid approximation are as follows:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ -\dfrac{Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ -\dfrac{Z_{\delta_e}}{u_0} & -\dfrac{Z_{\delta_T}}{u_0} \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_T \end{bmatrix} \qquad (17)$$

The Laplace transformation of the approximate equations for the long period are

$$(s - X_u)\,\Delta u(s) + g\,\Delta\theta(s) = X_{\delta_e}\,\Delta\delta_e(s) + X_{\delta_T}\,\Delta\delta_T(s) \qquad (18)$$

$$\frac{Z_u}{u_0}\,\Delta u(s) + s\,\Delta\theta(s) = -\frac{Z_{\delta_e}}{u_0}\,\Delta\delta_e(s) - \frac{Z_{\delta_T}}{u_0}\,\Delta\delta_T(s) \qquad (19)$$

The transfer function $\Delta u(s)/\Delta\delta_e(s)$ and $\Delta\theta(s)/\Delta\delta_e(s)$ can be found by setting $\Delta\delta_T(s)$ to 0 and solving for the appropriate transfer function as follows:

$$(s - X_u)\frac{\Delta u(s)}{\Delta\delta_e(s)} + g\,\frac{\Delta\theta(s)}{\Delta\theta_e(s)} = X_{\delta_e} \qquad (20)$$

$$\frac{Z_u}{u_0}\frac{\Delta u(s)}{\Delta\delta_e(s)} + s\,\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = -\frac{Z_{\delta_e}}{u_0} \qquad (21)$$

These equations can be solved to yield the transfer functions

$$\frac{\Delta u(s)}{\Delta\delta_e(s)} = \frac{\begin{vmatrix} X_{\delta_e} & g \\ \dfrac{-Z_{\delta_e}}{u_0} & s \end{vmatrix}}{\begin{vmatrix} s - X_u & g \\ \dfrac{Z_u}{u_0} & s \end{vmatrix}} \qquad \frac{\Delta u(s)}{\Delta\delta_e(s)} = \frac{X_{\delta_e}s + gZ_{\delta_e}/u_0}{s^2 + X_u s - \dfrac{Z_u g}{u_0}} \qquad (22)$$

In a similar manner $\Delta\theta(s)/\Delta\delta(s)$ can be shown to be

$$\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{-\frac{Z_{\delta e}}{u_0}s + \left(\frac{X_u Z_{\delta e}}{u_0} - \frac{Z_u X_{\delta e}}{u_0}\right)}{s^2 - X_u s - \frac{Z_u g}{u_0}} \qquad (23)$$

The transfer functions can be written in a symbolic form in the following manner:

$$\frac{\Delta u(s)}{\Delta\delta_e(s)} = \frac{N^u_{\delta_e}(s)}{\Delta_p(s)} = \frac{A_u s + B_u}{As^2 + Bs + C} \qquad (24)$$

$$\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{N^\theta_{\delta_e}}{\Delta_p(s)} = \frac{A_\theta s + B_\theta}{As^2 + Bs + C} \qquad (25)$$

Long-period transfer function approximations

	A, A_u, or A_θ	B, B_u, or B_θ	C
$\Delta_p(s)$	1	$-X_u$	$-Z_u g/u_0$
$N^u_{\delta_e}(s)$	X_{δ_e}	gZ_{δ_e}/u_0	
$N^\theta_{\delta_e}(s)$	$-Z_{\delta_e}/u_0$	$X_u Z_{\delta_e}/u_0 - Z_u X_{\delta_e}/u_0$	

where A_u, B_u, and so forth are defined in Table 8.3. The transfer functions for the propulsive control, that is, $\Delta u(s)/\Delta\delta_T(s)$ and $\Delta\theta(s)/\Delta\delta_T(s)$, have the same form except that the derivatives with respect to δ_e are replaced by derivatives with respect to δ_T. Therefore, Table 8.3 can be used for both aerodynamic and propulsive control transfer functions provided that the appropriate control derivatives are used.

6. **Roll Dynamics:** The equation of motion for a pure rolling motion,

$$\Delta\dot{p} - L_p\,\Delta p = L_{\delta_a}\,\Delta\delta_a \qquad (25)$$

The transfer function $\Delta p(s)/\delta_a(s)$ and $\Delta\phi(s)/\Delta\delta_a(s)$ can be obtained by taking the Laplace transform of the roll equation:

$$(s - L_p)\,\Delta p(s) = L_{\delta_a}\,\Delta\delta_a(s) \qquad \frac{\Delta p(s)}{\Delta\delta_a(s)} = \frac{L_{\delta a}}{s - L_p} \qquad (26)$$

But the roll rate Δp is defined as $\Delta\dot{\phi}$; therefore, $\Delta p(s) = s\Delta\phi(s)$ (27)

$$\frac{\Delta\phi(s)}{\Delta\delta_a(s)} = \frac{L_{\delta_a}}{s(s - L_p)} \qquad (28)$$

7. **Dutch Roll Approximation:** The final simplified transfer function we will develop is for the Dutch roll motion. The approximate equations can be shown to be

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta/u_0 & -(1 - Y_r/u_0) \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_r}/u_0 & 0 \\ N_{\delta_r} & N_{\delta_a} \end{bmatrix} \begin{bmatrix} \Delta\delta_r \\ \Delta\delta_a \end{bmatrix}$$

Taking the Laplace transform and rearranging yields

$$(s - Y_\beta/u_0)\,\Delta\beta(s) + (1 - Y_r/u_0)\,\Delta r(s) = Y_{\delta_r}/u_0\,\Delta\delta_r(s)$$

$$-N_\beta\,\Delta\beta(s) + (s - N_r)\,\Delta r(s) = N_{\delta_a}\,\Delta\delta_a(s) + N_{\delta_r}\,\Delta\delta_r(s)$$

The transfer functions $\Delta\beta(s)/\Delta\delta_r(s)$, $\Delta r(s)/\Delta\delta_r(s)$, $\Delta\beta(s)/\Delta\delta_a(s)$, and $\Delta r(s)/\Delta\delta_a(s)$ can be obtained by setting $\Delta\delta_a(s)$ to 0 and solving for $\Delta\beta(s)/\Delta\delta_r(s)$ and $\Delta r(s)/\Delta\delta_r(s)$. Next set $\Delta\delta_r(s)$ equal to 0 and solve for $\Delta\beta(s)/\Delta\delta_a(s)$ and $\Delta r(s)/\Delta\delta_a(s)$. The transfer functions $\Delta\beta(s)/\Delta\delta_r(s)$ and $\Delta r(s)/\Delta\delta_r(s)$ are obtained as follows:

$$(s - Y_\beta/u_0)\frac{\Delta\beta(s)}{\Delta\delta_r(s)} + (1 - Y_r/u_0)\frac{\Delta r(s)}{\Delta\delta_r(s)} = Y_{\delta_r}/u_0 \qquad (28)$$

$$-N_\beta\frac{\Delta\beta(s)}{\Delta\delta_r(s)} + (s - N_r)\frac{\Delta r(s)}{\Delta\delta_r(s)} = N_{\delta_r}$$

Solving for the transfer function yields

$$\frac{\Delta\beta(s)}{\Delta\delta_r(s)} = \frac{\begin{vmatrix} Y_{\delta_r}/u_0 & 1 - Y_r/u_0 \\ N_{\delta_r} & s - N_r \end{vmatrix}}{\begin{vmatrix} s - Y_\beta/u_0 & 1 - Y_r/u_0 \\ -N_\beta & s - N_r \end{vmatrix}} \qquad \frac{\Delta r(s)}{\Delta\delta_r(s)} = \frac{\begin{vmatrix} s - Y_\beta/u_0 & Y_{\delta_r}/u_0 \\ -N_\beta & N_{\delta_r} \end{vmatrix}}{\begin{vmatrix} s - Y_\beta/u_0 & 1 - Y_r/u_0 \\ -N_\beta & s - N_r \end{vmatrix}}$$

$$\frac{\Delta\beta(s)}{\Delta\delta_r(s)} = \frac{N^\beta_{\delta_r}(s)}{\Delta_{DR}(s)} = \frac{A_\beta s + B_\beta}{As^2 + Bs + C} \qquad \frac{\Delta r(s)}{\Delta\delta_r(s)} = \frac{N^\beta_{\delta_r}(s)}{\Delta_{DR}(s)} = \frac{A_r s + B_r}{As^2 + Bs + C}$$

The coefficients of the polynomials in the Dutch roll transfer functions are included in following Table 3. The denominator coefficients are in the first row and the numerator coefficients are defined for each transfer function in the subsequent rows.

Dutch roll transfer function approximations

	A, A_β, or A_r	B, B_β, or B_r	C
$\Delta_{DR}(s)$	1	$-(Y_\beta + u_0 N_r)/u_0$	$(Y_\beta N_r - N_\beta Y_r + N_\beta u_0)/u_0$
$N^{\beta}_{\delta_r}(s)$	Y_r/u_0	$(Y_r N_{\delta_r} - Y_{\delta_r} N_r - N_{\delta_r} u_0)/u_0$	
$N^{r}_{\delta_r}(s)$	N_{δ_r}	$(N_\beta Y_{\delta_r} - Y_\beta N_{\delta_r})/u_0$	
$N^{\beta}_{\delta_a}(s)$	0	$(Y_r N_{\delta_a} - u_0 N_{\delta_a})/u_0$	
$N^{r}_{\delta_a}(s)$	N_{δ_a}	$-Y_\beta N_{\delta_a}/u_0$	

8. CONTROL SURFACE ACTUATOR:

In addition to the various transfer functions that represent the aircraft dynamics, we need to develop the transfer functions for the other elements that make up the control system. This would include the servo actuators to deflect the aerodynamic control surfaces as well as the transfer function for any sensors in the control loop; for example, an attitude gyro, rate gyro, altimeter, or velocity sensor. The transfer functions for most sensors can be approximated by a gain, k.

Control surface servo actuators can be either electrical, hydraulic, pneumatic, or some combination of the three. The transfer function is similar for each type. We will develop the control surface servo actuator transfer function for a servo based on an electric motor.

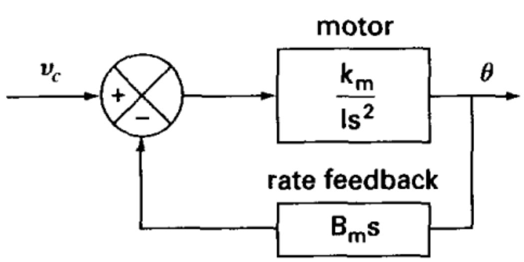

FIGURE 1
Motor with rate feedback.

$$T_m = k_m v_c \qquad I\ddot{\theta} = T_m \qquad \frac{\theta}{v_c} = \frac{k_m}{Is^2}$$

$$\frac{\theta}{v_c} = \frac{k}{s(\tau_m s + 1)} \quad \text{where} \quad \tau_m = \frac{I}{k_m B_m} \quad \text{and} \quad k = \frac{1}{B_m}$$

If τ_m (time constant), is small, the motor responds rapidly and the transfer function of the motor with rate feedback can be approximated as:

$$\frac{\theta}{v_c} = \frac{k}{s}$$

A simple position control servo system can be developed from the control diagram shown in Figure 2. The motor shaft angle, θ, can be replaced by the flap angle, δf, of the control surface. For the positional feedback system the closed loop transfer function can be shown to have the following form:

$$\frac{\delta_f}{v_c} = \frac{k}{\tau s + 1}$$ where $$k = 1/k_f \quad \text{and} \quad \tau = \frac{B_m}{k_f k_a}$$

The time constant of the control surface servo is typical of the order of 0.1 s

9. DISPLACEMENT AUTOPILOT

One of the earliest autopilots to be used for aircraft control is the so-called displacement autopilot. A displacement type autopilot can be used to control the angular orientation of the airplane. Conceptually, the displacement autopilot works in the following manner. In a pitch attitude displacement autopilot, the pitch angle is sensed by a vertical gyro and compared with the desired pitch angle to create an error angle. The difference or error in pitch attitude is used to produce proportional displacements of the elevator so that the error signal is reduced. Figure 3 is a block diagram of either a pitch or roll angle displacement autopilot.

The heading angle of the airplane also can be controlled using a similar scheme. The heading angle is sensed by a directional gyro and the error signal is used to displace the rudder to reduce the error signal. A displacement heading autopilot also is shown in Figure 4.

In practice, the displacement autopilot is engaged once the airplane has been trimmed in straight and level flight.

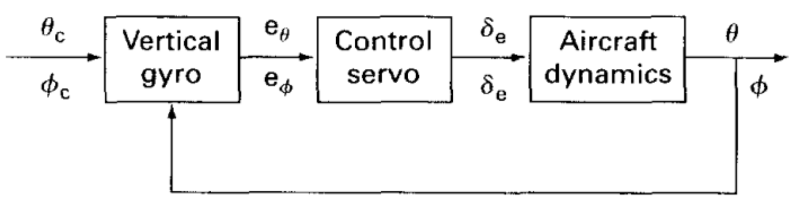

FIGURE 3
A roll or pitch displacement autopilot.

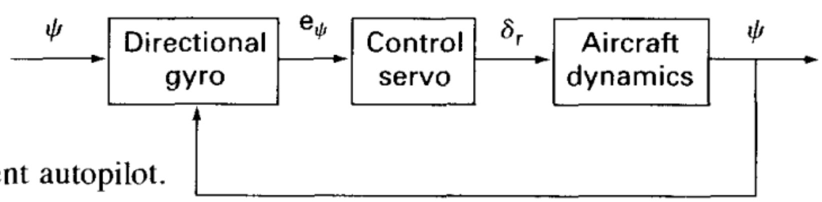

FIGURE 5
A heading displacement autopilot.

To maneuver the airplane while the autopilot is engaged, the pilot must adjust the commanded signals. For example, the airplane can be made to climb or descend by changing the pitch command. Turns can be achieved by introducing the desired bank angle while simultaneously changing the heading command.

10. SUMMERy

Longitudinal equations

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\,\Delta\delta_e + X_{\delta_T}\,\Delta\delta_T$$

$$-Z_u\,\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\,\Delta\delta_e + Z_{\delta_T}\,\Delta\delta_T$$

$$-M_u\,\Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta_e}\,\Delta\delta_e + M_{\delta_T}\,\Delta\delta_T$$

Combined Terms

$$\Delta\dot{u} + \Delta\theta g\cos\theta_0 = X_u\Delta u + X_w\Delta w + X_{\delta_e}\delta_e + X_{\delta_T}\delta_T$$

$$\Delta\dot{w} + \Delta\theta g\sin\theta_0 - u_0\Delta\dot{\theta} = Z_u\Delta u + Z_w\Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta\dot{\theta} + Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T$$

$$\Delta\ddot{\theta} = M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta\dot{\theta} + M_{\delta_e}\delta_e + M_{\delta_T}\delta_T$$

Longitudinal equations

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\,\Delta\delta_e + X_{\delta_T}\,\Delta\delta_T$$

$$-Z_u\,\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\,\Delta\delta_e + Z_{\delta_T}\,\Delta\delta_T$$

$$-M_u\,\Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta_e}\,\Delta\delta_e + M_{\delta_T}\,\Delta\delta_T$$

Rewriting in state space

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g\\ Z_u & Z_w & u_0 & 0\\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T}\\ Z_\delta & Z_{\delta_T}\\ M_\delta + M_{\dot{w}}Z_\delta & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta\\ \Delta\delta_T\end{bmatrix} \quad (2)$$

where the state vector **x** and control vector **η** are given by

$$\mathbf{x} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix}$$

and the matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot{w}} Z_u & M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Characteristic equation $\quad |\lambda \mathbf{I} - \mathbf{A}| = 0$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\eta}$$

Long Period Approximation:

An approximation to the long-period mode can be obtained by neglecting the pitching moment equation and assuming that the change in angle of attack is 0;

the homogeneous longitudinal state equations reduce to the following:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ \dfrac{-Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix} \qquad |\lambda \mathbf{I} - \mathbf{A}| = 0 \quad \text{or} \quad \begin{vmatrix} \lambda - X_u & g \\ \dfrac{Z_u}{u_0} & \lambda \end{vmatrix} = 0$$

$$\omega_{n_p} = \sqrt{\frac{-Z_u g}{u_0}} \qquad \zeta_p = \frac{-X_u}{2\omega_{n_p}}$$

Short-Period Approximation: An approximation to the short-period mode of motion can be obtained by assuming Δu = 0 and dropping the X-force equation

$$\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix} \qquad |\lambda \mathbf{I} - \mathbf{A}| = 0$$

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_\alpha}{u_0} & 1 \\ M_\alpha + M_{\dot{\alpha}} \dfrac{Z_\alpha}{u_0} & M_q + M_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix}$$

Summary of longitudinal approximations

	Long period (phugoid)	Short period
Frequency	$\omega_{n_p} = \sqrt{\dfrac{-Z_u g}{u_0}}$	$\omega_{nsp} = \sqrt{\dfrac{Z_\alpha M_q}{u_0} - M_\alpha}$
Damping ratio	$\zeta_p = \dfrac{-X_u}{2\omega_{n_p}}$	$\zeta_{sp} = -\dfrac{M_q + M_{\dot{\alpha}} + \dfrac{Z_\alpha}{u_0}}{2\omega_{n\,sp}}$

1. Short-Period Dynamics

$$\frac{\Delta q(s)}{\Delta \delta_e(s)} = \frac{N^q_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{\begin{vmatrix} s - Z_\alpha/u_0 & Z_{\delta_e}/u_0 \\ -(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0) & M_{\delta_e} + M_{\dot\alpha}\dfrac{Z_{\delta_e}}{u_0} \end{vmatrix}}{\begin{vmatrix} s - Z_\alpha/u_0 & -1 \\ -(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0) & s - (M_q + M_{\dot\alpha}) \end{vmatrix}} \quad (15)$$

$$\frac{\Delta q(s)}{\Delta \delta_e(s)} = \frac{N^q_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{A_q s + B_q}{As^2 + Bs + C} \quad (16)$$

$$\frac{\Delta \alpha(s)}{\Delta \delta_e(s)} = \frac{N^\alpha_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{\begin{vmatrix} Z_{\delta_e}/u_0 & -1 \\ M_{\delta_e} + M_{\dot\alpha}\dfrac{Z_{\delta_e}}{u_0} & s - (M_q + M_{\dot\alpha}) \end{vmatrix}}{\begin{vmatrix} s - Z_\alpha/u_0 & -1 \\ -(M_\alpha + M_{\dot\alpha} Z_\alpha/u_0) & s - (M_q + M_{\dot\alpha}) \end{vmatrix}} \quad (13)$$

$$\frac{\Delta \alpha(s)}{\Delta \delta_e(s)} = \frac{N^\alpha_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{A_\alpha s + B_\alpha}{As^2 + Bs + C} \quad (14)$$

TABLE 2
Short-period transfer function approximations

	A, A_α, or A_q	B, B_α, or B_q	C
$\Delta_{sp}(s)$	1	$-(M_q + M_{\dot\alpha} + Z_\alpha/u_0)$	$Z_\alpha M_q/u_0 - M_\alpha$
$N^\alpha_{\delta_e}(s)$	Z_{δ_e}/u_0	$M_{\delta_e} - M_q Z_{\delta_e}/u_0$	
$N^q_{\delta_e}(s)$	$M_{\delta_e} + M_{\dot\alpha} Z_{\delta_e}/u_0$	$M_\alpha Z_{\delta_e}/u_0 - M_{\delta_e} Z_\alpha/u_0$	

Long-Period or Phugoid Dynamics

$$\frac{\Delta u(s)}{\Delta \delta_e(s)} = \frac{\begin{vmatrix} X_{\delta_e} & g \\ \dfrac{-Z_{\delta_e}}{u_0} & s \end{vmatrix}}{\begin{vmatrix} s - X_u & g \\ \dfrac{Z_u}{u_0} & s \end{vmatrix}}$$

$$\frac{\Delta u(s)}{\Delta \delta_e(s)} = \frac{X_{\delta_e} s + g Z_{\delta_e}/u_0}{s^2 + X_u s - \dfrac{Z_u g}{u_0}}$$

$$\frac{\Delta u(s)}{\Delta \delta_e(s)} = \frac{N^u_{\delta_e}(s)}{\Delta_p(s)} = \frac{A_u s + B_u}{As^2 + Bs + C}$$

$$\frac{\Delta \theta(s)}{\Delta \delta_e(s)} = \frac{-\dfrac{Z_{\delta_e}}{u_0} s + \left(\dfrac{X_u Z_{\delta_e}}{u_0} - \dfrac{Z_u X_{\delta_e}}{u_0}\right)}{s^2 - X_u s - \dfrac{Z_u g}{u_0}}$$

$$\frac{\Delta \theta(s)}{\Delta \delta_e(s)} = \frac{N^\theta_{\delta_e}}{\Delta_p(s)} = \frac{A_\theta s + B_\theta}{As^2 + Bs + C}$$

TABLE 2
Short-period transfer function approximations

	A, A_α, or A_q	B, B_α, or B_q	C
$\Delta_{sp}(s)$	1	$-(M_q + M_{\dot\alpha} + Z_\alpha/u_0)$	$Z_\alpha M_q/u_0 - M_\alpha$
$N^\alpha_{\delta_e}(s)$	Z_{δ_e}/u_0	$M_{\delta_e} - M_q Z_{\delta_e}/u_0$	
$N^q_{\delta_e}(s)$	$M_{\delta_e} + M_{\dot\alpha} Z_{\delta_e}/u_0$	$M_\alpha Z_{\delta_e}/u_0 - M_{\delta_e} Z_\alpha/u_0$	

Long-period transfer function approximations

	A, A_u, or A_θ	B, B_u, or B_θ	C
$\Delta_p(s)$	1	$-X_u$	$-Z_u g/u_0$
$N^u_{\delta_e}(s)$	X_{δ_e}	gZ_{δ_e}/u_0	
$N^\theta_{\delta_e}(s)$	$-Z_{\delta_e}/u_0$	$X_u Z_{\delta_e}/u_0 - Z_u X_{\delta_e}/u_0$	

Roll Dynamics

Put the roll rate Δp is defined as $\Delta\dot{\phi}$; therefore, $\Delta p(s) = s\Delta\phi(s)$ (27)

$$\frac{\Delta\phi(s)}{\Delta\delta_a(s)} = \frac{L_{\delta_a}}{s(s - L_p)} \quad (28)$$

Dutch Roll Approximation

$$\frac{\Delta\beta(s)}{\Delta\delta_r(s)} = \frac{N^\beta_{\delta_r}(s)}{\Delta_{DR}(s)} = \frac{A_\beta s + B_\beta}{As^2 + Bs + C} \qquad \frac{\Delta r(s)}{\Delta\delta_r(s)} = \frac{N^r_{\delta_r}(s)}{\Delta_{DR}(s)} = \frac{A_r s + B_r}{As^2 + Bs + C}$$

Dutch roll transfer function approximations

	A, A_β, or A_r	B, B_β, or B_r	C
$\Delta_{DR}(s)$	1	$-(Y_\beta + u_0 N_r)/u_0$	$(Y_\beta N_r - N_\beta Y_r + N_\beta u_0)/u_0$
$N^\beta_{\delta_r}(s)$	Y_{δ_r}/u_0	$(Y_r N_{\delta_r} - Y_{\delta_r} N_r - N_{\delta_r} u_0)/u_0$	
$N^r_{\delta_r}(s)$	N_{δ_r}	$(N_\beta Y_{\delta_r} - Y_\beta N_{\delta_r})/u_0$	
$N^\beta_{\delta_a}(s)$	0	$(Y_r N_{\delta_a} - u_0 N_{\delta_a})/u_0$	
$N^r_{\delta_a}(s)$	N_{δ_a}	$-Y_\beta N_{\delta_a}/u_0$	

11.