CHAPTER 7

Automatic Control Theory—The Classical Approach

"The transport aircraft of the future may well be on automatic control from the moment of take-off to the automatic landing at its destination."

William Bollay, 14th Wright Brothers Lecture, 1950

7.1 INTRODUCTION

Control theory deals with the analysis and synthesis of logic for the control of a system. In the broadest sense, a system can be thought of as a collection of components or parts that work together to perform a particular function. The airplane is an example of a complex system designed to transport people and cargo.

What today we call control theory developed along two different analytical approaches. The first approach was based on frequency response methods, the root locus technique, transfer functions, and Laplace transforms. It had its beginning in the late 1930s. This approach to control theory is sometimes called classical or conventional control theory. A major feature of these analysis methods was their adaptability to simple graphical procedures, which was particularly important during this time period because computers were not available. Analysis techniques had to be suitable for calculations made without computers. The analysis tools, based upon the work of Bode, Nyquist, and Evans, form the foundation of "classical" control theory. To apply classical control theory to the design of a control system one needs to understand Laplace transforms and the concept of a transfer function.

With the advent of high-speed digital computers, control system analysis methods were developed based on the state-space formulation of the system. These analysis techniques, developed since the 1960s, are commonly called modern control theory. To understand modern control methods one must understand matrix algebra and the state-space concept of representing a system of governing equations. The selection of the names classical and modern is somewhat unfortunate in that it seems to relegate the classical approach to a lesser status when this is not the case. A control system designer needs to know both the classical and modern control approaches. In this and the next three chapters we divide control theory into

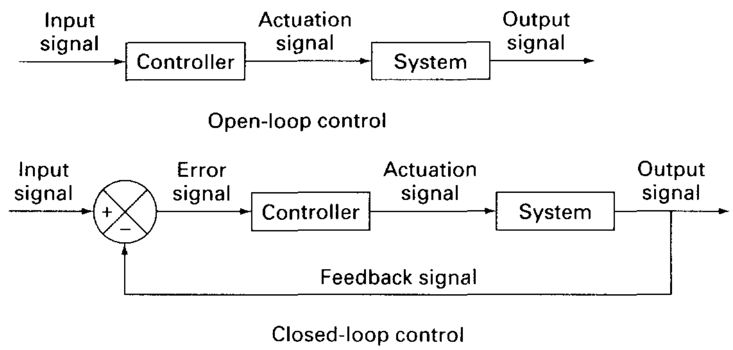

FIGURE 7.1
Examples of open-loop and closed-loop control systems.

the two categories, classical and modern, for convenience. Both approaches have their strengths and weaknesses and find wide acceptance and use by control system designers.

It is not possible to cover all aspects of control theory approach in just four chapters. Therefore, it has been assumed that the reader has had an undergraduate course in control theory. Chapters 7 and 9 provide a brief review of some of the theoretical aspects of the classical and the modern control; Chapters 8 and 10 apply the techniques to the design of simple airplane autopilots.

In each chapter we provide simple examples of the control analysis techniques that one can do readily with a simple pocket calculator. Once the theoretical basis of these techniques is understood more complicated problems can be attempted. A number of software packages are available for control system analysis and design. We have found the software package MATLAB* to be quite useful and used it in developing problems and examples for these chapters. Readers are encouraged to use whatever control software is available at their university or company to help them with the problems at the end of the chapters.

Before discussing control system design, a review of some of the basic concepts of control theory will be presented. Control systems can be classified as either open-loop or closed-loop systems, as illustrated in Figure 7.1. An open-loop control system is the simplest and least complex of all control devices. In the open-loop system the control action is independent of the output. In closed-loop system the control action depends on the output of the system. Closed-loop control systems are called feedback control systems. The advantage of the closed-loop system is its accuracy.

To obtain a more accurate control system, some form of feedback between the output and input must be established. This can be accomplished by comparing the controlled signal (output) with the commanded or reference input. In a feedback system one or more feedback loops are used to compare the controlled signal with the command signal to generate an error signal. The error signal then is used to

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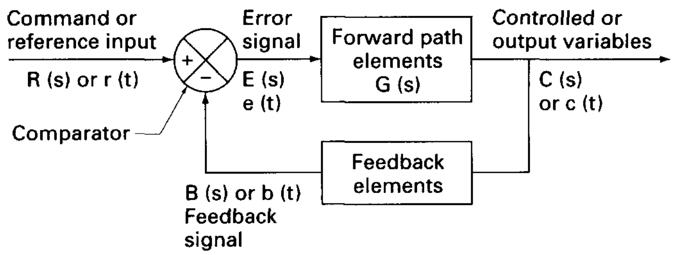

FIGURE 7.2
A feedback control system.

drive the output signal into agreement with the desired input signal. The typical closed-loop feedback system shown in Figure 7.2 is composed of a forward path, a feedback path, and an error-detection device called a comparator. Each component of the control system is defined in terms of its transfer function. The transfer function, T.F., is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input where the initial conditions are assumed to be 0:

$$\text{T.F.} = \frac{\text{Laplace transform of the output}}{\text{Laplace transform of the input}} \tag{7.1}$$

The transfer function of each element of the control system can be determined from the equations that govern the dynamic characteristics of the element. The aircraft transfer functions are developed in Chapter 8 from the equations of motion.

The closed-loop transfer function for the feedback control system shown in Figure 7.2 can be developed from the block diagram. The symbols used in the block diagram are defined as follows:

$R(s)$	reference input
$C(s)$	output signal (variable to be controlled)
$B(s)$	feedback signal
$E(s)$	error or actuating signal
$G(s)$	$C(s)/E(s)$ forward path or open-loop transfer function
$M(s)$	$C(s)/R(s)$ the closed-loop transfer function
$H(s)$	feedback transfer function
$G(s)H(s)$	loop transfer function

The closed-loop transter function, $C(s)/R(s)$, can be obtained by simple algebraic manipulation of the block diagram. The actuating or error signal is the difference between the input and feedback signals:

$$E(s) = R(s) - B(s) \tag{7.2}$$

The feedback signal $B(s)$ can be expressed in terms of the feedback transfer function and the output signal:

$$B(s) = H(s)C(s) \tag{7.3}$$

and the output signal $C(s)$ is related to the error signal and forward path transfer function in the following manner:

$$C(s) = G(s)E(s) \tag{7.4}$$

Substituting Equations (7.2) and (7.3) into 7.4 yields

$$C(s) = G(s)R(s) - G(s)H(s)C(s) \tag{7.5}$$

Equation (7.5) can be solved for the closed-loop transfer function $C(s)/R(s)$:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{7.6}$$

which is the ratio of the system output to the input. Most control systems are much more complex than the one shown in Figure 7.2. However, theoretically the more complex control systems consisting of many feedback elements can be reduced to the simple form just described.

The feedback systems described here can be designed to control accurately the output to some desired tolerance. However, feedback in itself does not ensure that the system will be stable. Therefore, to design a feedback control system one needs analysis tools that allow the designer to select system parameters so that the system will be stable. In addition to determining the absolute stability, the relative stability of the control system also must be determined. A system that is stable in the absolute sense may not be a satisfactory control system. For example, if the system damping is too low the output will be characterized by large amplitude oscillations about the desired output. The large overshooting of the response may make the system unacceptable.

Autopilots can be designed using either frequency- or time-domain methods developed from servomechanism theory or by time-domain analysis using state feedback design. In this chapter the techniques from servomechanism theory will be discussed and several simple applications of the design techniques will be demonstrated by applying the techniques to the design of autopilots.

The servomechanism design techniques include the Routh criterion, root locus, Bode, and Nyquist methods. A brief description of these techniques is presented either in the following sections or in the appendices at the end of this book. For a more rigorous treatment of this material, the reader is referred to [7.2–7.5].

7.2 ROUTH'S CRITERION

As noted earlier, the roots of the characteristic equation tell us whether or not the system is dynamically stable. If all the roots of the characteristic equation have negative real parts the system will be dynamically stable. On the other hand, if any root of the characteristic equation has a positive real part the system will be unstable. The system is considered to be marginally stable if one or more of the roots is a pure imaginary number. The marginally stable system represents the boundary between a dynamically stable or unstable system. For a closed-loop control system the denominator of Equation (7.6) is the characteristic equation.

A simple means of determining the absolute stability of a system can be obtained by the Routh stability criterion. The method allows us to determine whether any of the roots of the characteristic equation have positive real parts, without actually solving for the roots. Consider the characteristic equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} \cdots a_1\lambda + a_0 = 0 \tag{7.7}$$

So that no roots of Equation (7.7) have positive real parts the necessary but not sufficient conditions are that

1. All the coefficients of the characteristic equation must have the same sign.
2. All the coefficients must exist.

To apply the Routh criterion, we must first define the Routh array as in Table 7.1. The Routh array is continued horizontally and vertically until only zeros are obtained. The last step is to investigate the signs of the numbers in the first column of the Routh table. The Routh stability criterion states

1. If all the numbers of the first column have the same sign then the roots of the characteristic polynominal have negative real parts. The system therefore is stable.
2. If the numbers in the first column change sign then the number of sign changes indicates the number of roots of the characteristic equation having positive real parts. Therefore, if there is a sign change in the first column the system will be unstable.

When developing the Routh array, several difficulties may occur. For example, the first number in one of the rows may be 0, but the other numbers in the row may not be. Obviously, if 0 appears in the first position of a row, the elements in the following row will be infinite. In this case, the Routh test breaks down. Another

TABLE 7.1
Definition of Routh array: Routh table

λ^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
λ^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
λ^{n-2}	b_1	b_2	b_3	$\cdots$
$\vdots$	c_1	c_2	c_3	$\cdots$

where $a_n, a_{n-1}, \ldots, a_0$ are the coefficients of the characteristic equation and the coefficients b_1, b_2, b_3, c_1, c_2, and so on are given by

$$b_1 \equiv \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 \equiv \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \qquad \text{and so forth}$$

$$c_1 \equiv \frac{b_1 a_{n-3} - a_{n-1}b_2}{b_1} \qquad c_2 \equiv \frac{b_1 a_{n-5} - a_{n-1}b_3}{b_1} \qquad \text{and so forth}$$

$$d_1 \equiv \frac{c_1 b_2 - c_2 b_1}{c_1} \qquad \text{and so forth}$$

possibility is that all the numbers in a row are 0. Methods for handling these special cases can be found in most textbooks on automatic control theory.

Several examples of applying the Routh stability criterion are shown in Example Problem 7.1.

EXAMPLE PROBLEM 7.1. Determine whether the characteristic equations given below have stable or unstable roots.

(a) $\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$

(b) $2\lambda^3 + 4\lambda^2 + 4\lambda + 12 = 0$

(c) $A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$

Solution. The first two rows of the array are written down by inspection and the succeeding rows are obtained by using the relationship for each row element as presented previously:

$$\begin{array}{ccc} 1 & 12 & 0 \\ 6 & 8 & 0 \\ \dfrac{64}{6} & 0 & \\ 8 & & \end{array}$$

There are no sign changes in column 1; therefore, the system is stable. The Routh array for the second characteristic equation is as follows:

$$\begin{array}{ccc} 2 & 4 & 0 \\ 4 & 12 & 0 \\ -2 & 0 & \\ 12 & & \end{array}$$

Note that there are two sign changes in column 1; therefore, the characteristic equation has two roots with positive real parts. The system in unstable.

The Routh stability criterion can be applied to the quartic characteristic equation that describes either the longitudinal or lateral motion of an airplane. The quartic characteristic equation for either the longitudinal or lateral equation of motion is given in part c of this problem where A, B, C, D, and E are functions of the longitudinal or lateral stability derivatives. Forming the Routh array from the characteristic equation yields

$$\begin{array}{ccc} A & C & E \\ B & D & 0 \\ \dfrac{BC - AD}{B} & E & 0 \\ \dfrac{[D(BC - AC)/B] - BE}{(BC - AD)/B} & 0 & \\ E & & \end{array}$$

For the airplane to be stable requires that

$$
\begin{aligned}
&A, B, C, D, E && > 0 \\
&BC - AC && > 0 \\
&D(BC - AD) - B^2E && > 0
\end{aligned}
$$

The last two inequalities were obtained by inspection of the first column of the Routh array.

If the first number in a row is 0 and the remaining elements of that row are nonzero, the Routh method breaks down. To overcome this problem the lead element that is 0 is replaced by a small positive number, ε. With the substitution of ε as the first element, the Routh array can be completed. After completing the Routh array we can examine the first column to determine whether there are any sign changes in the first column as ε approaches 0.

The other potential difficulty occurs when a complete row of the Routh array is 0. Again the Routh method breaks down. When this condition occurs it means that there are symmetrically located roots in the s plane. The roots may be real with opposite sign or complex conjugate roots. The polynomial formed by the coefficient of the first row just above the row of zeroes is called the auxiliary polynomial. The roots of the auxiliary polynomial are symmetrical roots of the characteristic equation. The situation can be overcome by replacing the row of zeroes by the coefficients of the polynomial obtained by taking the derivative of the auxiliary polynomial. These exceptions to the Routh method are illustrated by way of example problems.

EXAMPLE PROBLEM 7.2. In this example we will examine the two potential cases where the Routh method breaks down. The two characteristic equations are as follows:

(a) $\lambda^5 + \lambda^4 + 3\lambda^3 + 3\lambda^2 + 4\lambda + 6 = 0$

(b) $\lambda^6 + 3\lambda^5 + 6\lambda^4 + 12\lambda^3 + 11\lambda^2 + 9\lambda + 6 = 0$

For equation a, the lead element of the third row of the Routh table is 0 which prevents us from completing the table. This difficulty is avoided by replacing the lead element 0 in the third row by a small positive values ε. With the 0 removed and replaced by ε the Routh table can be completed as follows:

$$
\begin{array}{ccc}
1 & 3 & 4 \\
1 & 3 & 6 \\
\varepsilon & -2 & \\
\dfrac{3\varepsilon + 2}{\varepsilon} & 6 & \\
\dfrac{-6\varepsilon^2 - 6\varepsilon - 4}{3\varepsilon + 2} & 0 & \\
6 & &
\end{array}
$$

Now as ε goes to 0 the sign of the first elements in rows 3 and 4 are positive. However, in row 5 the lead element goes to -2 as ε goes to 0. We note two sign changes in the first column of the Routh tables; therefore, the system has two roots with positive real parts, which means it is unstable.

The second difficulty that can cause a problem with the Routh method is a complete row of the Routh table being zeroes. This difficulty is illustrated by the Routh table for equation b.

The Routh table can be constructed as follows:

1	6	11	6
3	12	9	
2	8	6	
0	0		

Note that the fourth row of the Routh table is all zeroes. The auxiliary equation is formed from the coefficients in the row just above the row of zeroes. For this example the auxiliary equation is

$$2\lambda^4 + 8\lambda^2 + 6 = 0$$

Taking the derivative of the auxiliary equation yields

$$8\lambda^3 + 16\lambda = 0$$

The row of zeroes in the fourth row is replaced by the coefficients 8 and 16. The Routh table now can be completed.

1	6	11	6
3	12	9	
2	8	6	
8	16		
4	6		
4	0		
6			

The auxiliary equation can also be solved to determine the symmetric roots,

$$\lambda^4 + 4\lambda^2 + 3 = 0$$

which can be factored as follows:

$$(\lambda^2 + 1)(\lambda^2 + 3) = 0$$

or $$\lambda = \pm i \qquad \text{and} \qquad \lambda = \pm\sqrt{3}\, i$$

If we examine column 1 of the Routh table we conclude that there are no roots with positive real parts. However, solution of the auxiliary equations reveals that we have two pairs of complex roots lying on the imaginary axis. The purely imaginary roots lead to undamped oscillatory motions. In the absolute sense, the system is stable; that is, no part of the motion is growing with time. However, the purely oscillatory motions would be unacceptable for a control system.

Even though the method developed by Routh provides an easy way of assessing the absolute stability, it gives us no indication of the relative stability of the system. To assess the relative stability requires another analysis tool such as the root locus technique.

7.3 ROOT LOCUS TECHNIQUE

In designing a control system, it is desirable to be able to investigate the performance of the control system when one or more parameters of the system are varied. As has been shown repeatedly, the characteristic equation plays an important role in the dynamic behavior of aircraft motions. The same is true for linear control systems. In control system design, a powerful tool for analyzing the performance of a system is the root locus technique. Basically, the technique provides graphical information in the s plane on the trajectory of the roots of the characteristic equation for variations in one or more of the system parameters. Typically, most root locus plots consist of only one parametric variation. The control system designer can use the root locus method to obtain accurate time-domain response and frequency response information on a closed-loop control system.

The root locus technique was introduced by W. R. Evans in 1949. He developed a series of rules that allow the control systems engineer to quickly draw the root locus diagram. Although many software packages are available for accurately determining the root locus plots, the graphical rules remain important. They provide the control systems engineer a valuable tool to assessing system changes. With Evans's technique one can sketch a root locus plot in several minutes. The rules for constructing a root locus plot are presented later in this section.

The transfer function was described earlier as the ratio of the output to the input. On examining a transfer function we note that the denominator is the characteristic equation of the system. The roots of the denominator are the eigenvalues that describe the free response of the system, where the free response is the solution to the homogeneous equation. In controls terminology the characteristic roots are called the poles of the transfer function. The numerator of the transfer function governs the particular solution and the roots of the numerator are called zeros.

As was noted earlier in Chapters 4 and 5 the roots of the characteristic equation (or poles) must have negative real parts if the system is to be stable. In control system design the location of the poles of the closed-loop transfer function allows the designer to predict the time-domain performance of the system.

However, in designing a control system the designer typically will have a number of system parameters unspecified. The root locus technique permits the designer to view the movement of the poles of the closed-loop transfer functic ı as one or more unknown system parameters are varied.

Before describing the root locus technique it would be helpful to examine he significance of the root placement in the complex plane and the type of respoı ;e

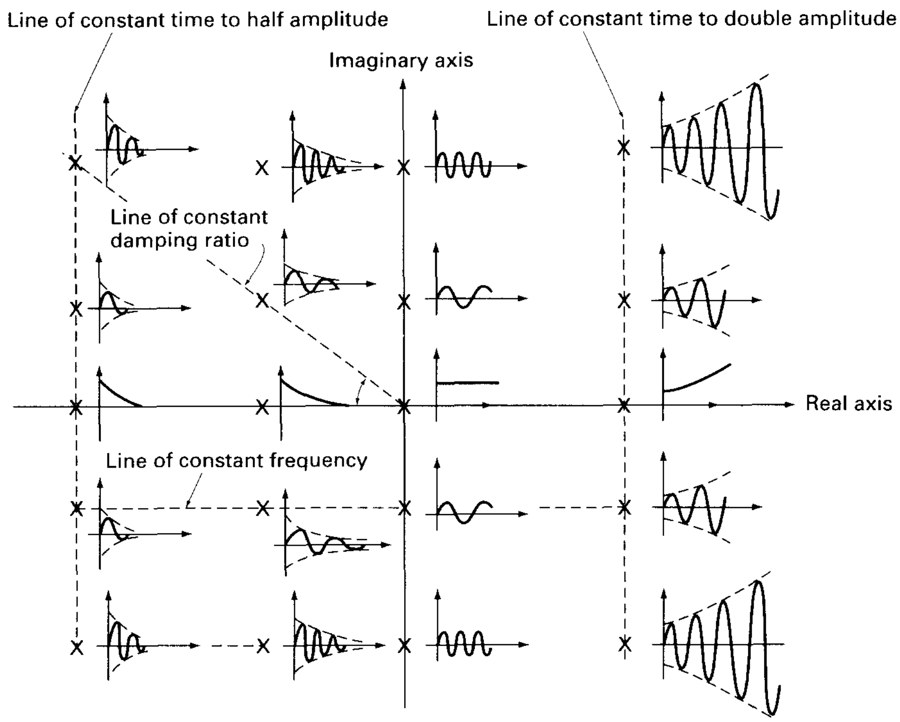

FIGURE 7.3
Impulse response as a function of the pole location in the complex s plane.

that can be expected to occur. Figure 7.3 illustrates some of the important features of pole location. First we note that any pole lying in the left half portion of the complex plane is stable; that is, the response decays with time. Any pole in the right half plane leads to a response that grows with time, which will result in an unstable system. The farther the root is to the left of the imaginary axis, the faster the response decays. All poles lying along a particular vertical line will have the same time to half amplitude. Poles lying along the same horizontal line have the same damped frequency, ω, and period. The farther the pole is from the real axis, the higher the frequency of the response will be. Poles lying along a radial line through the origin have the same damping ratio, ζ, and roots lying on the same circular arc around the origin will have the same undamped natural frequency. Finally, some comments must be made about the poles lying on the imaginary axis. Poles of the order 1 on the imaginary axis lead to undamped oscillations; however, multiple order poles result in responses that grow with time.

The closed-loop transfer function was shown earlier to be

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{7.8}$$

The characteristic equation of the closed loop system is given by the denominator of equation (7.8):

$$1 + G(s)H(s) = 0 \tag{7.9}$$

or

$$G(s)H(s) = -1 \tag{7.10}$$

The transfer function $G(s)H(s)$ can be expressed in factored form as follows:

$$G(s)H(s) = \frac{k(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \tag{7.11}$$

where $n > m$ and k is an unknown system parameter. Substituting this equation into the characteristic equation yields.

$$\frac{k(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = -1 \tag{7.12}$$

The characteristic equation is complex and can be written in terms of a magnitude and angle as follows:

$$\frac{|k|\,|s + z_1|\,|s + z_2| \cdots |s + z_m|}{|s + p_1|\,|s + p_2| \cdots |s + p_n|} = 1 \tag{7.13}$$

$$\sum_{i=1}^{m} \angle(s + z_i) - \sum_{i=1}^{n} \angle(s + p_i) = (2q + 1)\pi \tag{7.14}$$

where $q = 0, 1, 2, \ldots, n - m - 1$. Solution of these equations yields the movement of the roots as a function of the unknown system parameter, k. These equations can be solved on the computer to determine the root locus contours. However, a simple graphical technique developed by W. R. Evans can be used to rapidly sketch a root locus plot. This graphical procedure is presented in the next section.

It can be shown easily that the root locus contours start at the poles of transfer function, $G(s)H(s)$ and end at the zeroes of the transfer function as k is varied from 0 to infinity. For example, if we rearrange the magnitude criteria in the following manner,

$$\frac{|s + z_1|\,|s + z_2| \cdots |s + z_m|}{|s + p_1|\,|s + p_2| \cdots |s + p_n|} = \frac{1}{|k|} \tag{7.15}$$

then as k goes to 0 the function becomes infinite. This implies that the roots approach the poles as k goes to 0. On the other hand, as k goes to infinity the function goes to 0, which implies that the roots are at the transfer function zeros. Therefore, the root locus plot of the closed-loop system starts with a plot of the poles and zeros of the transfer function, $G(s)H(s)$. Evans developed a series of rules based on the magnitude and angle criteria for rapidly sketching the root locus branches on a pole zero map. A proof of these rules can be found in most control textbooks and will not be presented here. Table 7.2 is a summary of the rules for constructing a root locus contour.

TABLE 7.2
Rules for graphical construction of the root locus plot

1. The root locus contours are symmetrical about the real axis.
2. The number of separate branches of the root locus plot is equal to the number of poles of the transfer function $G(s)H(s)$. Branches of the root locus originate at the poles of $G(s)H(s)$ for $k = 0$ and terminate at either the open-loop zeroes or at infinity for $k = \infty$. The number of branches that terminate at infinity is equal to the difference between the number of poles and zeroes of the transfer function $G(s)H(s)$, where n = number of poles and m = number of zeros.
3. Segments of the real axis that are part of the root locus can be found in the following manner: Points on the real axis that have an odd number of poles and zeroes to their right are part of the real axis portion of the root locus.
4. The root locus branches that approach the open-loop zeroes at infinity do so along straight-line asymptotes that intersect the real axis at the center of gravity of the finite poles and zeroes. Mathematically this can be expressed as

$$\sigma = \left[\sum \text{Real parts of the poles} - \sum \text{Real parts of the zeroes}\right] \Big/ (n - m)$$

   where n is the number of poles and m is the number of finite zeroes.
5. The angle that the asymptotes make with the real axis is given by

$$\phi_a = \frac{180°[2q + 1]}{n - m}$$

   for $q = 0, 1, 2, \ldots, (n - m - 1)$
6. The angle of departure of the root locus from a pole of $G(s)H(s)$ can be found by the following expression:

$$\phi_p = \pm 180°(2q + 1) + \phi \qquad q = 0, 1, 2, \ldots$$

   where ϕ is the net angle contribution at the pole of interest due to all other poles and zeroes of $G(s)H(s)$. The arrival angle at a zero is given by a similar expression:

$$\phi_z = \pm 180°(2q + 1) + \phi \qquad q = 0, 1, 2, \ldots$$

   The angle ϕ is determined by drawing straight lines from all the poles and zeroes to the pole or zero of interest and then summing the angles made by these lines.
7. If a portion of the real axis is part of the root locus and a branch is between two poles, the branch must break away from the real axis so that the locus ends on a zero as k approaches infinity. The breakaway points on the real axis are determined by solving

$$1 + GH = 0$$

   for k and then finding the roots of the equation $dk/ds = 0$. Only roots that lie on a branch of the locus are of interest.

The root locus technique discussed in this chapter provides the analyst or designer a convenient method for assessing the absolute and relative stability of a control system. In terms of the root locus diagram, if any of the roots of the characteristic equation of the closed-loop system lie in the right half plane the system is unstable. On the other hand, if all the roots lie in the left half plane the system is stable. Complex roots lying on the imaginary axis yield constant amplitude oscillations. Repeated roots on the imaginary axis result in unstable behavior.

For roots lying in the left side of the root locus plot the question becomes one of determining the relative stability of the system. A system that is stable in the absolute sense may not be a very useful control system. We need to know more about the relative stability of the system. Relative stability deals with how fast the system responds to control input and how fast disturbances are suppressed. The relative stability of the control system is measured by various performance indices such as time to half amplitude, percent over shoot, rise time, or settling time. These concepts will be discussed in the next section.

EXAMPLE PROBLEM 7.3. Sketch the root locus plot for the transfer function

$$G(s)H(s) = \frac{k(s+3)}{s(s+10)(s^2+8s+20)}$$

Solution. This transfer function has one finite zero ($m = 1$) and four poles ($n = 4$):

zero: $s = -3$
poles: $s = 0, s = -10, s = -4 \pm 2i$

The poles and zeroes of the transfer function can be plotted on the root locus diagram. The poles and zeroes of $G(s)H(s)$ are denoted by a small x or 0, respectively, on the root locus plot. Using rule 3 from Table 7.2 we observe that the portion of the real axis that is part of the locus lies between $s = 0$ and -3 and from -10 to $-\infty$.

The number of branches of the root locus that terminate at a zero at infinity is equal to the difference between the number of poles (n) and the number of zeroes (m) of the transfer function (rule 2). In this case we have four poles and one zero; therefore, we have three branches of the locus going to zeroes at infinity.

The branches of the locus that go to a zero at infinity do so along straight-line asymptotes. The intersection of the asymptotes with the real axis and the angle of the asymptotes follows (see rules 4 and 5 of Table 7.2):

$$\sigma = \frac{\Sigma \text{ real parts of the poles} - \Sigma \text{ real parts of the zero}}{[n-m]}$$

$$\sigma = \frac{(-0-10-4-4)-(-3)}{4-1} = \frac{-15}{3} = -5$$

and $$\phi_A = \frac{180°[2q+1]}{n-m}$$

or $$\phi_A = \frac{180°[2q+1]}{3} \qquad \text{and } q = 0, 1, \ldots, n-m-1,$$

where $n - m - 1 = 4 - 1 - 1 = 2$

$$\phi_A = 60°, \quad 180°, \quad 300°$$

The pole at the origin approaches zero at $s = -3$, the pole at $s = -10$ goes to $-\infty$ on the real axis, and the complex poles go to zeroes along asymptotes making an angle of 60° and 300° with the real axis as k goes from 0 to ∞. Figure 7.4 is a sketch of the root locus plot.

7.3.1 Addition of Poles and Zeroes

The root locus method gives a graphic picture of the movement of the poles of the closed-loop system with the variation of one of the system parameters that needs to be selected by the designer. Later in this chapter we discuss how the relative

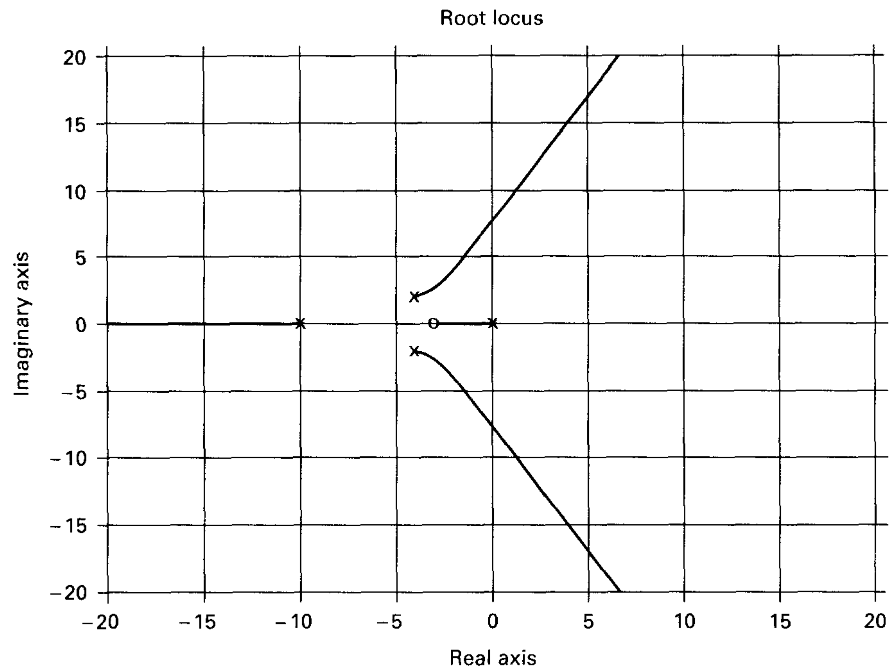

FIGURE 7.4
Root locus plot for Example Problem 7.3.

stability of the system and its performance can be obtained from the root locus diagram.

In many cases it is not possible to satisfy all the performance specifications using a single parameter such as the system gain. This requires the designer to add some form of compensation to the basic control system. The compensators may be electrical circuits, mechanical devices, or electromechanical devices that are added to the system to improve its performance. The compensators may be added to either the forward or feedback path. The compensator has a transfer function composed of poles and zeroes. Before discussing various methods of providing compensation to a control system it would be useful to examine the influence of the addition of poles and zeroes to the loop transfer function $G(s)H(s)$. We will do this by way of a simple example.

EXAMPLE PROBLEM 7.4. Construct a root locus plot from the transfer function $G(s)H(s)$ given by

$$G(s)H(s) = \frac{k}{s(s + p_1)}$$

then examine how the locus is affected by the addition of one of the following to the original transfer function.

i. simple pole ii. multiple pole iii. simple zero.

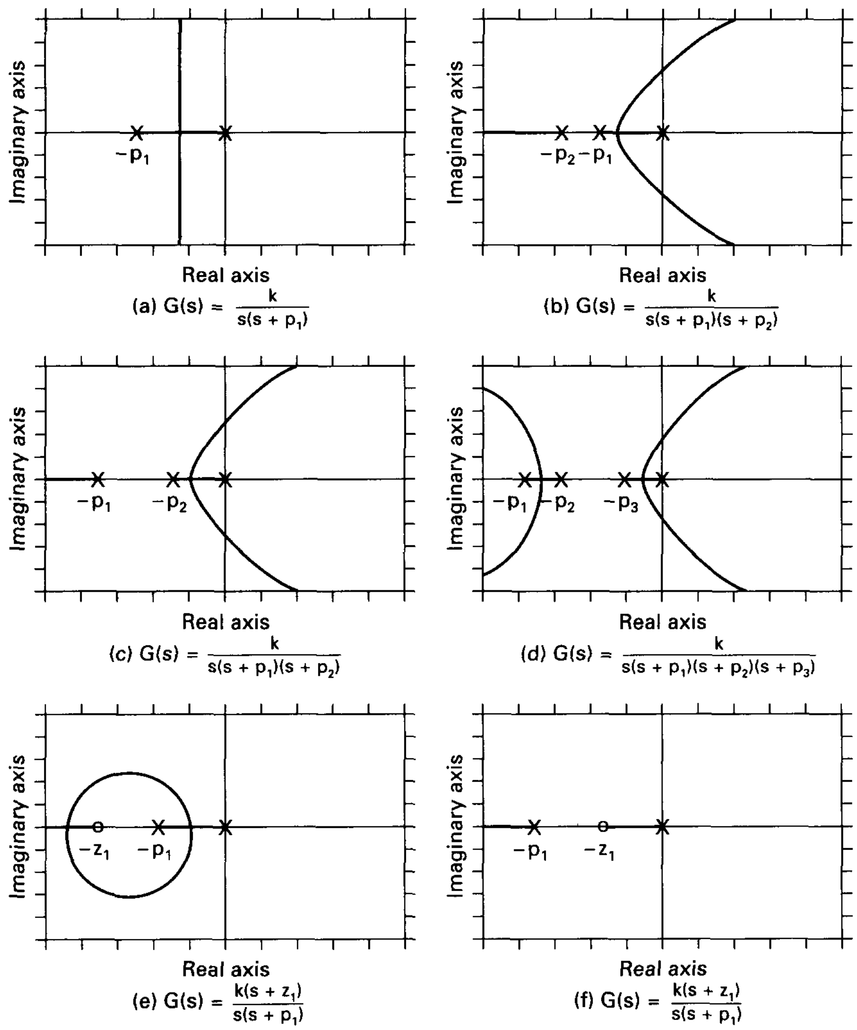

FIGURE 7.5
Sketch of root locus plot for Example Problem 7.4.

Solution. The root locus plot can be easily constructed by the rules outlined in this chapter. A sketch of the root locus is shown in Figure 7.5. For this particular transfer function the system is stable for $0 < k < \infty$. Now if we add a simple pole, $s + p_2$, to $G(s)H(s)$ the root locus will bend into the right half plane, which limits the range of k for which the system is stable. Notice that the plots for $p_1 > p_2$ or $p_2 > p_1$ have the same shape (see Figure 7.5(b) and (c)). The addition of yet another pole adds another branch of the locus that goes to zero at infinity, and the system can still become unstable if the system gain exceeds a certain value as shown in Figure 7.5(d). From this simple analysis we can conclude that the addition of a pole to a given transfer function causes the root locus plot to bend toward the right half portion of the complex plane. Thus, the additon of a simple pole tends to destabilize the system.

The addition of a simple zero, $s + z_1$, to the original transfer function, $G(s)H(s)$, will cause the root locus plot to bend further into the left half portion of the complex plane as illustrated in Figure 7.5(e) and (f). By adding a zero to $G(s)H(s)$, the system will be more stable than the original system.

The importance of this example is to show that the root locus plot of a control system can be altered by the addition of poles or zeroes. In practice a designer can use this idea to reshape the root locus contour so that the desired performance can be achieved. The compensator basically is a device that provides a transfer function consisting of poles or zeroes or both that can be chosen to move the root locus contour of the compensated system to the desired closed-loop pole configuration. Note that the addition of a compensator in general increases the order of the system.

7.4 FREQUENCY DOMAIN TECHNIQUES

The frequency response of a dynamic system was discussed in Chapter 6. The same techniques can be applied to the design of feedback control systems. The transfer function for a closed-loop feedback system can be written as

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{7.16}$$

If we excite the system with a sinusoidal input such as

$$r(t) = A_I \sin(\omega t) \tag{7.17}$$

the steady-state output of the system will have the form

$$c(t) = A_O \sin(\omega t + \phi) \tag{7.18}$$

The magnitude and phase relationship between the input and output signals is called the frequency response of the system. The ratio of output to input for a sinusoidal steady state can be obtained by replacing the Laplace transform variable s with $\mathrm{i}\omega$:

$$M(\mathrm{i}\omega) = \frac{G(\mathrm{i}\omega)}{1 + G(\mathrm{i}\omega)H(\mathrm{i}\omega)} \tag{7.19}$$

Expressing the previous equation in terms of its magnitude and phase angle yields

$$M(\mathrm{i}\omega) = M(\omega)\underline{/\phi(\omega)} \tag{7.20}$$

where

$$M(\omega) = \left|\frac{G(\mathrm{i}\omega)}{1 + G(\mathrm{i}\omega)H(\mathrm{i}\omega)}\right| \tag{7.21}$$

and

$$\phi(\omega) = \underline{/G(\mathrm{i}\omega)} - \underline{/[1 + G(\mathrm{i}\omega)H(\mathrm{i}\omega)]} \tag{7.22}$$

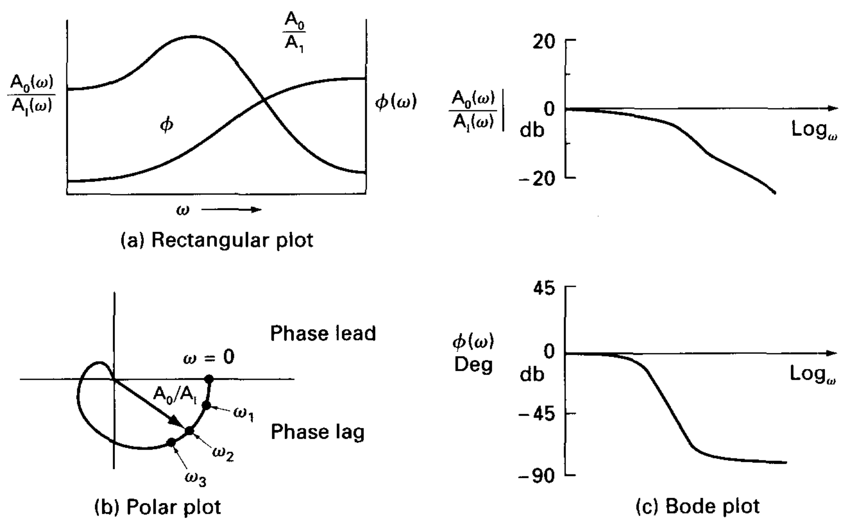

FIGURE 7.6
Various graphical ways of presenting frequency response data.

The frequency response information can be plotted in rectangular, polar, or logarithmic (Bode) plots. Figure 7.6 is a sketch of the various ways of presenting the frequency response data. The relationship between the frequency- and time-domain performance of a control system is discussed in the next section.

7.5 TIME-DOMAIN AND FREQUENCY-DOMAIN SPECIFICATIONS

The first step in the design of a feedback control system is to determine a set of specifications for the desired system performance. In the following section we shall present both time- and frequency-domain specifications and their relationship to one another for a second-order system. The transfer function of a second-order system can be expressed as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7.23}$$

where ζ is the damping ratio and ω_n is the undamped natural frequency of the system. Figure 7.7 shows the response to a step input of an underdamped second-order system. The performance of the second-order system is characterized by the overshoot, delay time, rise time, and settling time of the transient response to a unit step. The time response of a second-order system to a step input for an

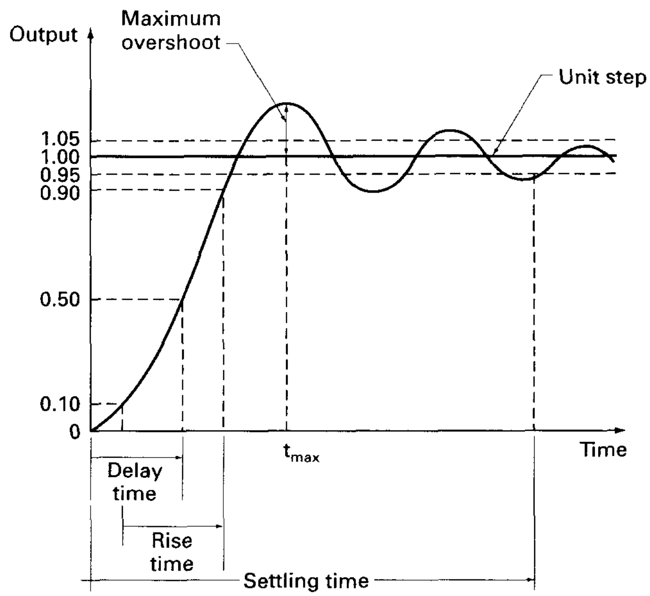

FIGURE 7.7
Time response of a second-order system.

underdamped system; that is, $\zeta < 1$, is given by Equations (7.24) and (7.25):

$$c(t) = 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2}\, t - \phi) \tag{7.24}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{-\zeta}\right) \tag{7.25}$$

The delay and rise time give a measure of how fast the system responds to a step input. Delay time t_d is the time it takes for the response to reach for the first time 50 percent of the final value of the response. The rise time t_r is the time required for the response to rise from 10 to 90 percent of the final value. The other two parameters of interest are the settling time and peak overshoot. Settling time t_s is the time it takes for the response to stay within a specified tolerance band of 5 percent of the final value. The peak overshoot is a measure of the oscillations about the final output. From the standpoint of control system design, we would like to have a system that responds rapidly with minimum overshoot. Equations (7.24) and (7.25) can be used to determine the relationships between the time-domain specifications t_d, t_r, and the like and the damping ratio ζ and undamped natural frequency ω_n. Table 7.3 is a summary of these relationships.

Figure 7.8 is a sketch of the typical magnitude and phase characteristics of a feedback control system. As in the time-domain analysis it is desirable to have a set of specifications to describe the control system performance in the frequency

TABLE 7.3
Time domain specifications

Delay time t_d	Rise time, t_r
$t_d \approx \dfrac{1 + 0.6\zeta + 0.15\zeta^2}{\omega_n}$	$t_r \approx \dfrac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n}$
Time to peak amplitude, t_p	**Settling time, t_s**
$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$	$t_s = \dfrac{3.0}{\omega_n\zeta}$

Peak overshoot, M_p

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

For a unit step

Percent maximum overshoot = $100 \exp(-\pi\zeta/\sqrt{1-\zeta^2})$

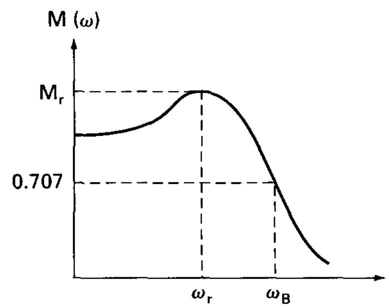

FIGURE 7.8
Frequency response of a closed-loop control system.

domain. In the frequency domain the design specifications are given in terms of the response peak M_r, the resonant frequency ω_r, the system bandwidth ω_B, and the gain and phase margins. The maximum value of $M(\omega)$, called the resonance peak, is an indication of the relative stability of the control system. If M_r is large the system will have a large peak overshoot to a step input. The resonant frequency, ω_r, is the frequency at which the resonance peak occurs. It is related to the frequency of the oscillations and speed of the transient response. The bandwidth ω_B is the band of frequencies from 0 to the frequency at which the magnitude $M(\omega)$ drops to 70 percent of the zero-frequency magnitude. The bandwidth gives an indication of the transient response of the system. If the bandwidth is large, the system will respond rapidly, whereas a small bandwidth will result in a sluggish control system.

The gain and phase margins are measures of the relative stability of the system and are related to the closeness of the poles of the closed-loop system to the iω axis.

For a second-order system the frequency domain characteristics M_r, ω_r, and ω_B can be related to the system damping ratio and the undamped natural frequency ω_n.

The relationships will be presented here without proof:

$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{7.26}$$

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \tag{7.27}$$

$$\omega_B = \omega_n\left[(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{1/2} \tag{7.28}$$

The peak response and the peak overshoot of the transient response in the time domain is given by the following approximation:

$$c(t)_{\max} \leq 1.17M_r \tag{7.29}$$

The phase margin of a second-order system can be related to the system damping ratio as follows:

$$\phi = \tan^{-1}\left[2\zeta\left(\frac{1}{(4\zeta^4 + 1)^{1/2} - 2\zeta^2}\right)^{1/2}\right] \tag{7.30}$$

This very formidable equation can be approximated by the simple relationship

$$\zeta \approx 0.01\phi \qquad \text{for} \qquad \zeta \leq 0.7 \tag{7.31}$$

The phase margin ϕ is in degrees.

From the preceding relationships developed for the second-order system the following observations can be made:

1. The maximum overshoot for a unit step in the time domain is a function of only ζ.
2. The resonance peak of the closed-loop system is a function of only ζ.
3. The maximum peak overshoot and resonance peak are related through the damping ratio.
4. The rise time increases while the bandwidth decreases for increases in system damping for a fixed ω_n. The bandwidth and rise time are inversely proportional to one another.
5. The bandwidth is directly proportional to ω_n.
6. The higher the bandwidth, the larger is the resonance peak.

7.5.1. Gain and Phase Margin from Root Locus

The gain and phase margin used to determine the relative stability of a control system using frequency response techniques also can be determined from the root locus plot. The gain margin can be estimated by taking the ratio of the gain when the locus crosses the imaginary axis to the gain selected for the system:

Gain margin

$$= \frac{\text{Value of system gain } k \text{ when locus crosses the imaginary axis}}{\text{Selected value of system gain } k} \tag{7.32}$$

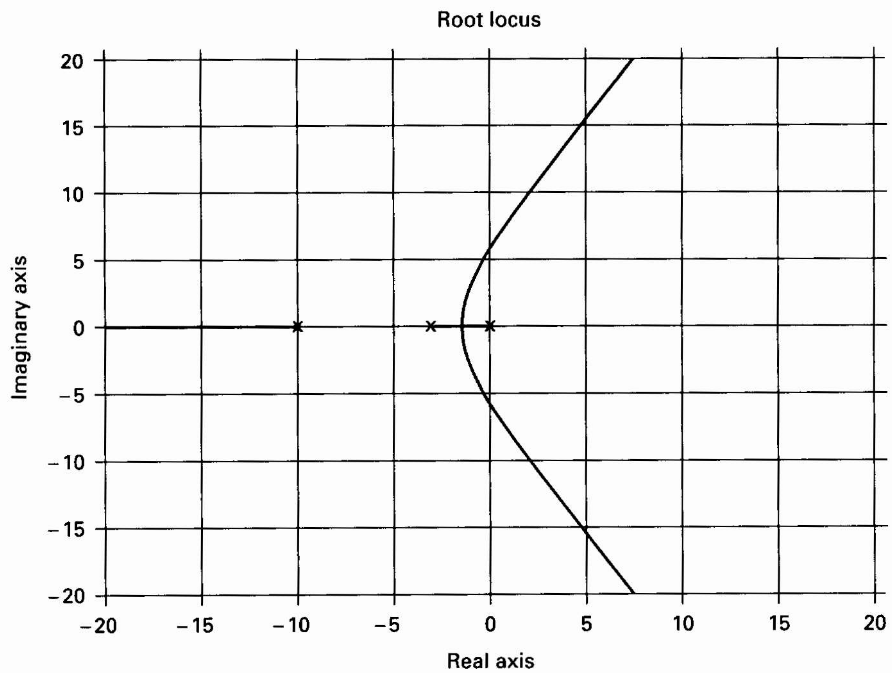

FIGURE 7.9

Root locus plot for the transfer function $G(s)H(s) = \dfrac{k}{s(s+3)(s+10)}$.

The value of ω at the intersection on the root locus is the phase crossover frequency. If the root locus plot has no branches that cross over the imaginary axis the gain margin is infinite.

The phase margin can be determined for the selected gain by estimating the frequency on the imaginary axis that satisfies the relationship

$$|G(i\omega_g)H(i\omega_g)| = 1 \tag{7.33}$$

The frequency can be determined by trial and error. The frequency that satisfies this relationship is called the gain crossover frequency. The phase margin can be calculated from the equation

$$\phi_{PM} = 180° + \arg G(i\omega_g)H(i\omega_g). \tag{7.34}$$

EXAMPLE PROBLEM 7.5. The root locus plot for a system having the following transfer function is given in Figure 7.9:

$$G(s)H(s) = \frac{k}{s(s+3)(s+10)}$$

Determine the following information:

(a) Select the system gain so that the dominant roots have a damping ratio, $\zeta = 0.6$.
(b) Estimate the settling time.
(c) Find the gain and phase margin for the gain selected in part (a).

Solution. To estimate the gain for a damping ratio, $\zeta = 0.6$, the value of s on the root locus that intersects the line of constant damping ratio of 0.6 needs to be determined. As was shown earlier the damping ratio is constant along radial lines drawn from the origin of the root locus diagram. The magnitude of the damping ratio is related to the angle θ as follows:

$$\zeta = \cos\theta$$

Solving for theta yields

$$\theta = \cos^{-1}[\zeta] = \cos^{-1}[0.6] = 53^\circ$$

The intersection of the line of constant damping ratio ($\theta = 53^\circ \Rightarrow \zeta = 0.6$) with the root locus occurs at $s = -1.2 + 1.65i$. The magnitude of the system gain at this point can be determined using the magnitude criteria:

$$|G(s)H(s)| = 1$$

or

$$\frac{|k|}{|s|\,|s+3|\,|s+10|} = 1$$

Substituting in the value of $s = -1.2 + 1.65i$ yields

$$\frac{|k|}{\left(\sqrt{(1.2)^2 + (1.65)^2}\right)\left(\sqrt{(1.8)^2 + (1.65)^2}\right)\left(\sqrt{(8.8)^2 + (1.65)^2}\right)} = 1$$

or

$$|k| = (2.04)\,(2.44)\,(8.95) = 44.55$$

The settling time t_s can be estimated from the approximate formula given in Table 7.3:

$$t_s = \frac{3.0}{\zeta\omega_n}$$

where $\zeta\omega_n$ is the magnitude of the real part of the complex root,

$$\zeta\omega_n = 1.2$$

Therefore

$$t_s = \frac{3.0}{\zeta\omega_n} = \frac{3.0}{1.2} = 2.5 \text{ s}$$

To determine the gain margin from the root locus plot we can use Equation (7.33). We need to determine the gain for the system when the root locus crosses the imaginary axis. From the root locus plot we can determine that $s = +5.5i$ at the crossover point. The gain is determined from the magnitude criteria

$$\frac{|k|}{|s|\,|s+3|\,|s+10|} = 1$$

where $s = +5.5i$ and

$$\frac{k}{(5.5)(6.26)(11.41)} = 1$$

or $k = 393$.

The gain margin can be calculated from Equation (7.33):

$$\text{Gain margin} = \frac{\text{Value of system gain } k \text{ when locus crosses imaginary axis}}{\text{Selected value of system gain } k}$$

$$= \frac{393}{44.55} = 8.82$$

The phase margin can be determined by finding the frequency ω_g, the gain crossover frequency, so that $|G(i\omega_g)H(i\omega_g)| = 1.0$.

$$\frac{44.55}{\omega_g\sqrt{\omega_g^2 + 3^2}\sqrt{\omega_g^2 + 10^2}} = 1$$

Solving this equation by trial and error yields $\omega_g = 1.3$.

The phase margin now can be estimated from Equation (7.34) where the arg $G(i\omega_g)\,H(i\omega_g)$ is found in the following way:

$$\arg G(i\omega_g)H(i\omega_g) = -\angle i\omega_g - \angle(i\omega_g + 3) - \angle(i\omega_g + 10)$$

$$= -90° - 23.4 - 7.4° = 120.8$$

$$\phi_{PM} = 180° - \arg G(i\omega_g)H(i\omega_g)$$

$$= 180° - 120.8° = 59.2°$$

7.5.2 Higher-Order Systems

Most feedback control systems are usually of a higher order than the second-order system discussed in the previous sections. However, many higher-order control systems can be analyzed by approximating the system by a second-order system. Obviously, when this can be accomplished, the design and analysis of the equivalent system is greatly simplified.

For a higher-order system to be replaced by an equivalent second-order system, the transient response of the higher-order system must be dominated by a pair of complex conjugate poles. These poles, called the dominant poles or roots, are located closest to the origin in a pole-zero plot. The other poles must be located far to the left of the dominant poles or near a zero of the system. The transient response caused by the poles located to the far left of the dominant poles will diminish rapidly in comparison with the dominant root response. On the other hand, if the pole is not located to the far left of the dominant poles, then the poles must be near a zero of the system transfer function. The transient response of a pole located near a zero is characterized by a very small amplitude motion, which can readily be neglected.

The transfer function of a second-order system can be expressed in terms of the system damping ratio, ζ, and the undamped natural frequency, ω_n, as follows:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7.35}$$

Consider the case where the system is underdamped; that is, $0 < \zeta < 1$. This implies that the second-order roots are complex. If the input is a unit step, that is, $R(s) = 1/s$, then the output is

$$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{7.36}$$

which can be inverted to the time domain as

$$C(t) = 1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t - \phi) \tag{7.37}$$

where

$$\phi = \tan^{-1}(\sqrt{1 - \zeta^2}/-\zeta). \tag{7.38}$$

The response is a damped sinusoidal motion.

Now, if we add a simple pole in the form $1/(1 + Ts)$ to Equation (7.35), the response to a step input would be given by

$$C(t) = 1 - \frac{T^2\omega_n^2}{1 - 2T\zeta\omega_n + T^2\omega_n^2} e^{-t/T} + \frac{e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t - \phi)}{\sqrt{1 - \zeta^2}(1 - 2\zeta T\omega_n + T^2\omega_n^2)} \tag{7.39}$$

and

$$\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} + \tan^{-1} \frac{T\omega_n \sqrt{1 - \zeta^2}}{1 - T\zeta\omega_n} \tag{7.40}$$

The pole is located at $s = -1/T$ and the smaller T is the farther the pole is from the imaginary axis. As the simple pole is moved farther to the left of the complex root the response of Equation (7.39) will approach that of Equation (7.37). This would occur when T is small and $1/T \gg \zeta\omega_n$. If we examine Equation (7.39) the second term vanishes much more quickly than the third term. The mathematical expression defining the third term approaches that of the second-order expression when T is small. A similar argument can be made for higher-order systems.

7.6 STEADY-STATE ERROR

The accuracy of a control system is measured by how well it tracks a given command input. Even if a system has good overall transient response it also must have good steady-state behavior. The accuracy of the control system is expressed in terms of the steady-state error to a given commanded input. The usual input signals used to evaluate the steady-state error are step, ramp, and parabolic input. Figure 7.10 shows a typical step, ramp, and parabolic input signal.

If we examine Figure 7.2 at the beginning of this chapter, an expression for the error signal can be developed. The error signal $E(s)$ can be shown to be

$$E(s) = \frac{R(s)}{1 + G(s)H(s)} \tag{7.41}$$

where $R(s)$ is the input signal and $G(s)H(s)$ is the loop transfer function. The steady-state error e_{ss} is the tracking error as time approaches a large value for a

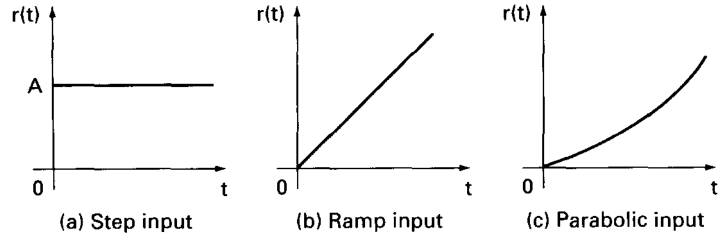

FIGURE 7.10
Typical input signals.

particular input command. Rather than inverting $E(s)$ back into the time domain and evaluating $e(t)$ as t goes to infinity we can use the final value theorem. This theorem states that if the Laplace transform of a function $f(t)$ is $F(s)$ and if the function $s\,F(s)$ is analytic on the imaginary axis and right half plane then

$$\underset{t \to 0}{\text{Limit}}\, f(t) = \underset{s \to 0}{\text{Limit}}\, sF(s) \tag{7.42}$$

The steady-state error can be found by applying the final value theorem:

$$e_{ss} = \underset{t \to \infty}{\text{Limit}}\, e(t) = \underset{s \to 0}{\text{Limit}}\, sE(s) \tag{7.43}$$

The steady-state error will depend on the input command $R(s)$ and the loop transfer function $G(s)H(s)$. The steady-state error for the three stipulated input signals is expressed in terms of error coefficients, which will be defined shortly. First we need to classify the open-loop transfer function. This is done by determining the order of the pole in $G(s)H(s)$ at the origin; that is $s = 0$. The loop transfer function $G(s)H(s)$ can be written in the pole-zero form as

$$G(s)H(s) = \frac{k}{s^l} \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \tag{7.44}$$

An alternate form of this expression is

$$G(s)H(s) = \frac{K(1 + T_{z_1}s)(1 + T_{z_2}s) \cdots (1 + T_{z_m}s)}{s^l(1 + T_{p_1}s)(1 + T_{p_2}s) \cdots (1 + T_{p_n}s)} \tag{7.45}$$

which is referred to as the time-constant form of the transfer function. The time constants are simply

$$T_{z_i} = \frac{1}{z_i} \qquad i = 1 \text{ to } m \tag{7.46}$$

$$T_{p_j} = \frac{1}{p_j} \qquad j = 1 \text{ to } n \tag{7.47}$$

and

$$K = k \frac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{n} p_j} \tag{7.48}$$

It is convenient to define the error constants in terms of the time constant form of the loop transfer function. The loop transfer function is classified in terms of the order of the pole at the origin. The system is referred to as a type 0, type 1, type 2, and so on depending on the value of the exponent of the pole at the origin, l; that is, $l = 0, 1, 2$, and so on.

Now let us return to defining the error constants. We first examine the tracking error to a step input. The step input can be expressed as

$$r = Au(t)$$

where A is the amplitude of the step and $u(t)$ is a unit step. The Laplace transform of the step input is given by $R(s) = A/s$. The steady-state error can be found using Equations (7.41) and (7.45) and the final value theorem:

$$e_{ss} = \operatorname*{Limit}_{t\to\infty} e(t) = \operatorname*{Limit}_{s\to 0} sE(s)$$

$$e_{ss} = \operatorname*{Limit}_{s\to 0} \frac{s(A/s)}{1 + G(s)H(s)}$$

$$e_{ss} = \operatorname*{Limit}_{s\to 0} \frac{A}{1 + G(s)H(s)} = \frac{A}{1 + \operatorname*{Limit}_{s\to 0} G(s)H(s)}$$

or finally

$$e_{ss} = \frac{A}{1 + K_p}$$

where K_p, called the positional error constant, is defined as

$$K_p = \operatorname*{Limit}_{s\to 0} G(s)H(s)$$

When the input signal is a ramp $r(t) = At$. The Laplace transform of a ramp input is $R(s) = A/s^2$. The steady-state error can be found as previously:

$$e_{ss} = \operatorname*{Limit}_{s\to 0} sE(s) = \operatorname*{Limit}_{s\to 0} \frac{s(A/s^2)}{1 + G(s)H(s)}$$

$$e_{ss} = \operatorname*{Limit}_{s\to 0} \frac{A}{s + sG(s)H(s)}$$

or

$$e_{ss} = \frac{A}{K_v}$$

where K_v is called the velocity error constant, is defined as

$$K_v = \operatorname*{Limit}_{s\to 0} sG(s)H(s)$$

The final input signal is that of a parabolic input or acceleration. The input signal is given as

$$r(t) = At^2/2$$

or in the Laplace domain

$$R(s) = A/s^3$$

where A is acceleration amplitude. The steady-state error for an acceleration input

TABLE 7.4
Steady-state errors

System type	Step input $r(t) = Au(t)$	Ramp input $r(t) = At$	Parabolic input $r(t) = At^2/2$
0	$\frac{A}{1 + K_p}$	∞	∞
1	0	$\frac{A}{K_v}$	∞
2	0	0	$\frac{A}{K_a}$

is as follows:

$$e_{ss} = \underset{s \to 0}{\text{Limit}}\ sE(s) = \underset{s \to 0}{\text{Limit}} \frac{sA/s^3}{1 + G(s)H(s)}$$

$$e_{ss} = \underset{s \to 0}{\text{Limit}} \frac{A}{s^2 + s^2 G(s)H(s)} = \frac{A}{K_a}$$

where K_a is the acceleration error constant, defined as

$$K_a = \underset{s \to 0}{\text{Limit}}\ s^2 G(s)H(s)$$

The steady-state error depends on the system type and input function. A summary of the steady-state error is given in Table 7.4.

EXAMPLE PROBLEM 7.6. Given the following transfer function, determine the steady-state error of the system to unit step, ramp, and parabolic inputs:

$$G(s)H(s) = \frac{k(s + 2)}{s(s + 1)(s + 4)(s + 5)}$$

Solution. The transfer function $G(s)H(s)$ is in the pole-zero form. Rewriting the transfer function in the time constant from yields

$$G(s)H(s) = \frac{2k(1 + 0.5s)}{20s(1 + s)(1 + 0.25s)(1 + 0.2s)}$$

$$= \frac{k}{10} \frac{(1 + 0.5s)}{s(1 + s)(1 + 0.25s)(1 + 0.2s)}$$

$$= \frac{K(1 + 0.5s)}{s(1 + s)(1 + 0.25s)(1 + 0.2s)}$$

where $K = k/10$.

This transfer function is a type 1 system because of the first-order pole at the origin. From Table 7.4 we see that the steady error is 0 for a step input, $1/K_v$ for the ramp input, and ∞ for the parabolic input. The velocity error constant K_v can be found as follows:

$$K_v = \underset{s \to 0}{\text{Limit}}\ sG(s)H(s)$$

$$K_v = \underset{s \to 0}{\text{Limit}} \frac{K(1 - 0.5s)}{(1 + s)(1 + 0.25s)(1 + 0.2s)}$$

$$K_v = K = \frac{k}{10}$$

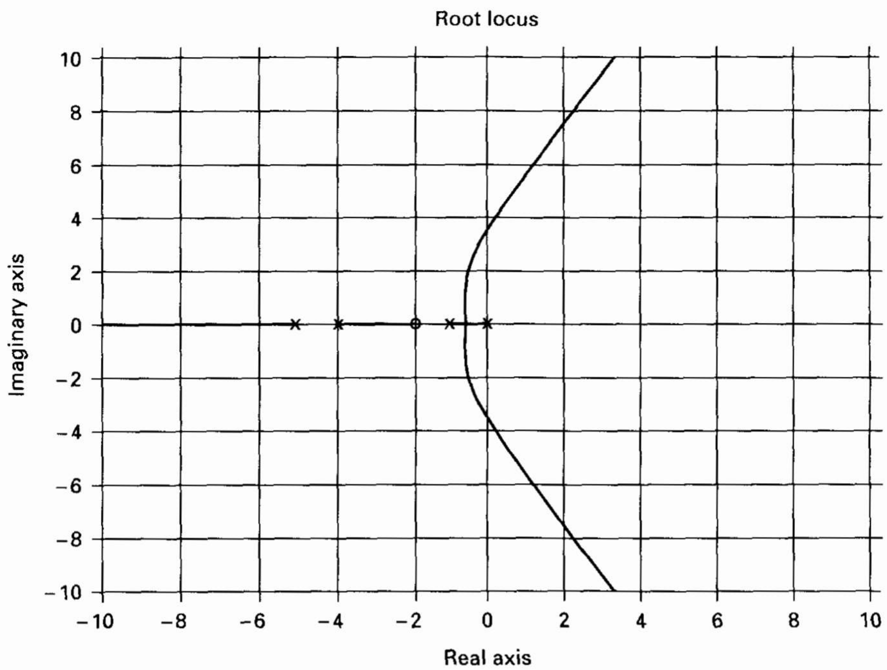

FIGURE 7.11
Root locus plot of $G(s)\,H(s) = \dfrac{k(s + 2)}{s(s + 1)(s + 4)(s + 5)}$.

The steady-state error for the ramp input is

$$e_{ss} = 10/k$$

As the system gain is increased, the steady-state error will decrease. However, for this particular example, the system gain is limited because too large a gain will cause the system to be unstable. Figure 7.11 shows the root locus plot for this system.

7.7 CONTROL SYSTEM DESIGN

In this section we will try to provide a simple overview of the design process in developing a new control system. Figure 7.12 is a simple flow chart indicating the basic elements in the design of a new product. Design often is divided into three phases: conceptual design, preliminary design, and detailed design. In conceptual design, the designer attempts to develop one or more concepts that can provide the overall system performance required by the customer. In the next phase, the preliminary design phase, additional analysis is performed to optimize the system. In

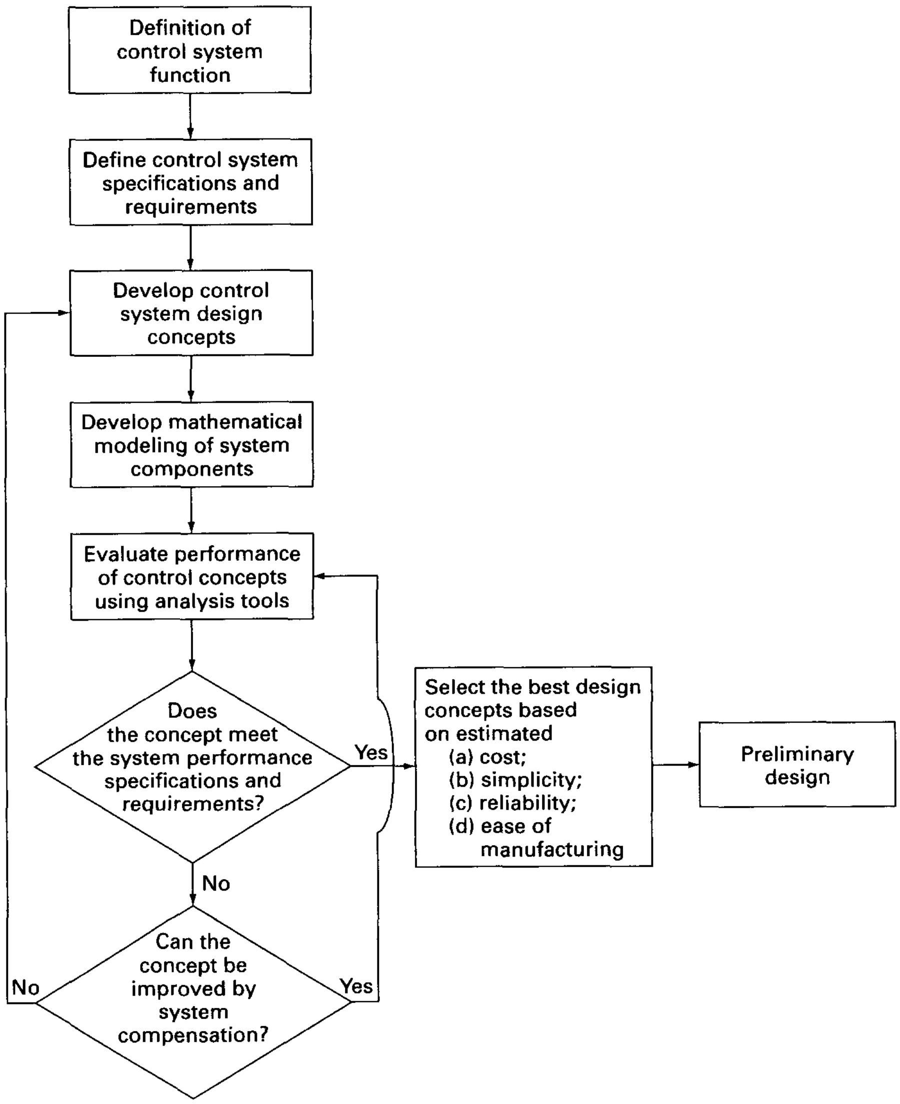

FIGURE 7.12
Flowchart of conceptual design process.

the final design phase the engineering team develops the detailed engineering drawings and stipulates the manufacturing details.

The design process begins with the recognition of a need for a new control system. This need may originate within the engineering department but is just as likely to come from the marketing or sales department through feedback from the company's customers. Regardless of how or where the idea originates, the recognition of the need for a new control system starts the engineering design process.

Once a product need is established this provides a definition of the purpose or function of the control system.

Having defined the purpose of the control system, the designer needs to identify its requirements and specifications. These consist of items such as control system performance, cost, reliability, maintainability, and other constraints. The performance of the system usually is given in terms of time or frequency domain characteristics or a combination of both. Time domain performance specifications include rise time, setting time, peak overshoot, steady-state error, and the like. On the other hand, the frequency domain specifications are given in terms of phase margin, gain margin, and so forth. Additional constraints may be weight and volume requirements, which might be critical in an aerospace application.

With the purpose and specifications defined the designer must develop one or more concepts to achieve the desired control function. The control system concepts in large part are based on the designer's creativity and experience. The concepts are simply ideas of how to implement the desired control function, which can be presented in the form of a simple block diagram. For example, if one were interested in designing a simple autopilot to maintain a wing's level attitude the control concept could be presented as shown in Figure 7.13.

The next phase of the design process is to evaluate the performance of each concept against the specifications. This requires the designer to develop the appropriate mathematical models for each of the design components, such as controller, actuators, plant, and sensor. The challenge at this point is to keep the mathematical model as simple as possible but accurate enough to retain the essential dynamic characteristics of each component.

Once the mathematical formulation is completed the control system can be analyzed using the techniques presented in this chapter or the state-space design methods presented in Chapter 9. These analysis methods allow the designer to evaluate the control system performance as a function of various control system design parameters. The performance of the control system concepts now can be compared with the desired performance. In practice, the designer often is faced with the problem that the concept does not meet all of the performance specifications. The designer basically has three options: One is to try to convince the potential customer that a particular performance specification is unrelated and not essential for the overall performance of the system if this indeed is the case. The second option is to select another control concept that can satisfy the specification. The third is to add some form of compensation to the concept to improve the system performance so that the specifications are satisfied.

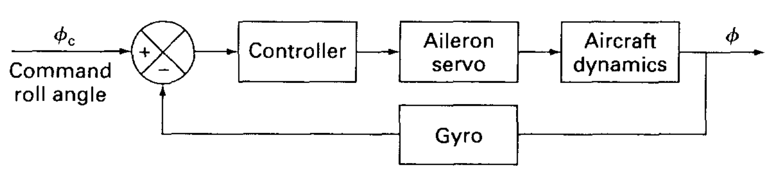

FIGURE 7.13
Wing-leveling autopilot.

7.7.1 Compensation

As stated in the previous section the ultimate test of a design concept is whether it meets the desired performance specifications. The control system performance is specified in terms of the transient behavior and the steady-state error. The transient performance in the time domain can be described in terms of the damping ratio, ζ, the peak overshoot, and the speed of the response as measured by the rise and settling time. The relative stability also can be specified in terms of frequency-domain performance indices such as the resonant peak, M_r, and gain and phase margins. The speed of response is measured by the resonant frequency, ω_r, and the system bandwidth, ω_B.

In general the designer on analyzing a control system concept finds that some but not all of the performance specifications are met by a particular control concept. Using the root locus analysis technique discussed earlier the designer can adjust the system gain to vary the control system performance; however, in most cases the designer cannot meet all the design performance objectives by gain adjustment alone. When the performance cannot be satisfied the designer can add an additional component to the control system, called a compensator. The purpose of the compensator is to improve the overall performance of the control system concept. Recall that when discussing the root locus techniques we examined the influence of the addition of either a simple pole, zero, or combination pole and zero to the root locus plot. We found that the addition of poles and zeroes allowed us to contour or change the shape of the root locus plot. The addition of some combination of poles and zeroes to a given control system transfer function represents a compensator. By selecting the parameter in the compensator the designer can change the shape of the root locus plot so that the overall performance specification can be met.

The compensators can be thought of as an additional transfer function $G_c(s)$ that can be added to either the forward or feedback path of the control system. As illustrated in Figure 7.14, when the compensator is added to the forward path it is called a cascade or series compensator and when it is placed in the feedback path it is called feedback or parallel compensator. In general, the compensators are electrical circuits or mechanical subsystems that provide the designer parameters that can be adjusted to improve the overall system performance.

7.7.2 Forward-Path Compensation

To examine how a compensator can be used to improve the performance of a control system we consider the simple control system shown in Figure 7.15. Suppose that the performance requirements are given in terms of the damping ratio and settling time as follows:

$$\zeta = 0.707$$

$$t_s < 3 \text{ s.}$$

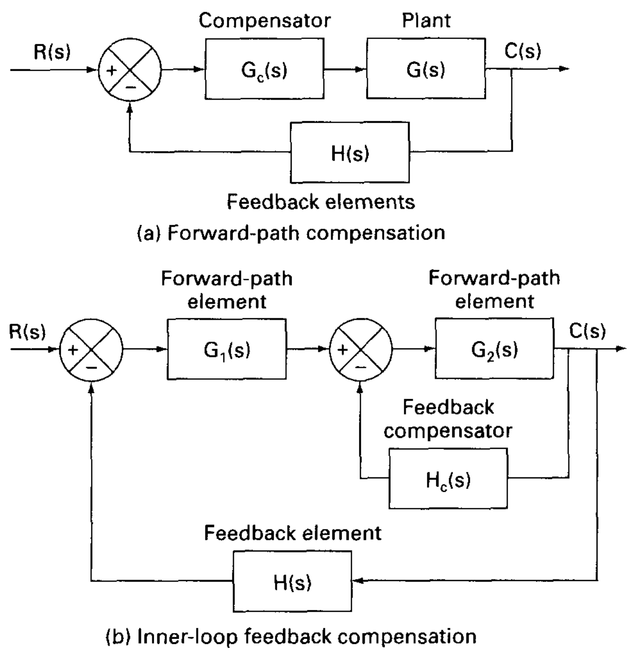

FIGURE 7.14
Series and parallel compensation.

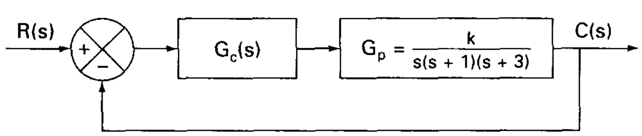

FIGURE 7.15
Control system with a forward-path compensator.

From the root locus plot shown in Figure 7.16 we can achieve the desired damping ratio by finding the gain for the point on the locus that interesects the radial line from the origin that makes an angle of 45° with respect to the negative real axis. The undamped natural frequency ω_n is the distance along the radial line of constant ζ from the origin to the root locus. For this case $\omega_n = 0.5$ rad/s.

The settling time which can be estimated by

$$t_s = \frac{3.0}{\zeta\omega_n} \tag{7.49}$$

for an $\omega_n = 0.5$ rad/s—the settling time is not less than 3 s. If the root locus plot could be made to intersect the $\zeta = 0.707$ line at a larger value of ω_n the settling time constraint could be met. As we noted earlier in this chapter a simple zero

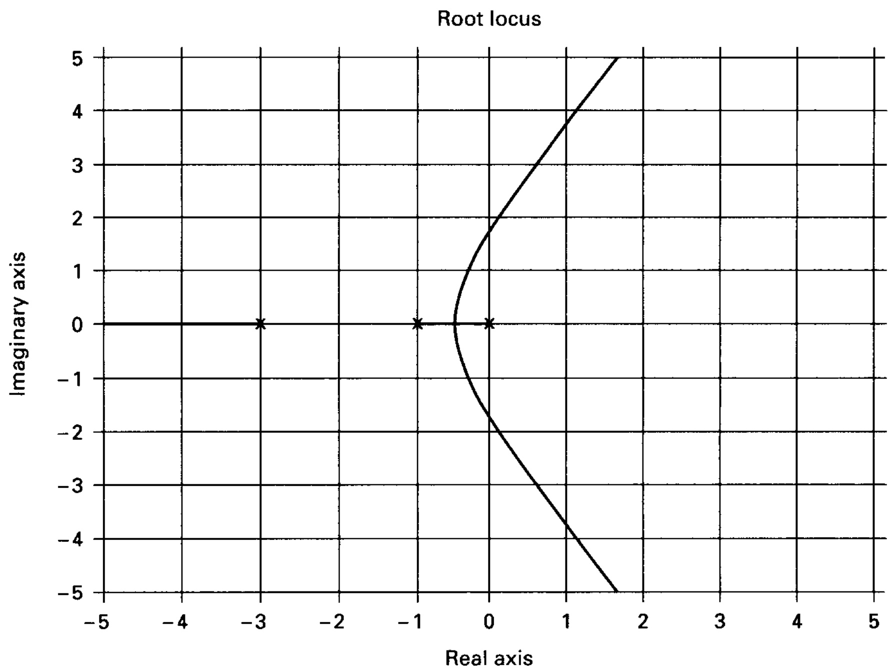

FIGURE 7.16

Root locus plot of $G(s)H(s) = \dfrac{k}{s(s+1)(s+3)}$.

added to an open-loop transfer function $G(s)H(s)$ causes the locus to bend more toward the left in the complex plane. Figure 7.17 is a root locus plot with the addition of a zero as $s = -1.1$. With the addition of the zero, the root locus plot bends toward the left. The value of ω_n for the damping ratio of 0.707 is now 1.98 rad/s, which yields a settling time less than 3 s.

Unfortunately a simple zero is not very practical. In practice we add a transfer function of the form

$$G_c(s) = \frac{s + z_c}{s + p_c} \tag{7.50}$$

where $z_c/p_c < 1$, or the compensator poles is located to the left of the compensator zero. Such a compensator is called a lead compensator. The designer can adjust the pole and zero location of the compensator to shape the root locus so that both the damping ratio and settling time specifications can be met. The movement of the compensator pole and zero is achieved by proper selection of the components in the electrical circuit. In summary the lead compensator can be used to improve the transient response characteristics of the control system.

It is possible to have a control system design with good transient characteristics but a large steady-state error. When the steady-state error is large a lag

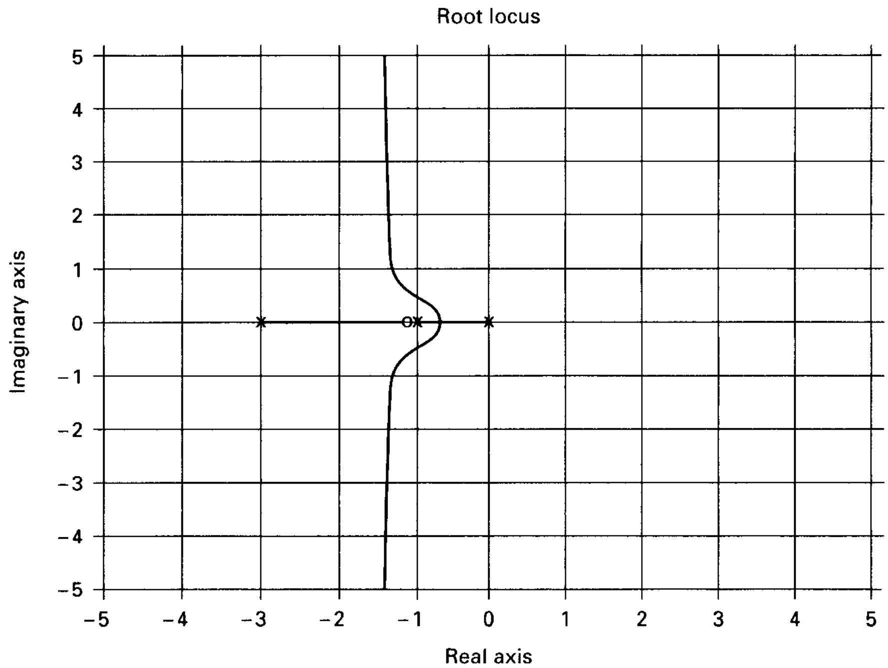

FIGURE 7.17
Root locus plot of $G(s)H(s) = \dfrac{k(s + 1.1)}{s(s + 1)(s + 3)}$.

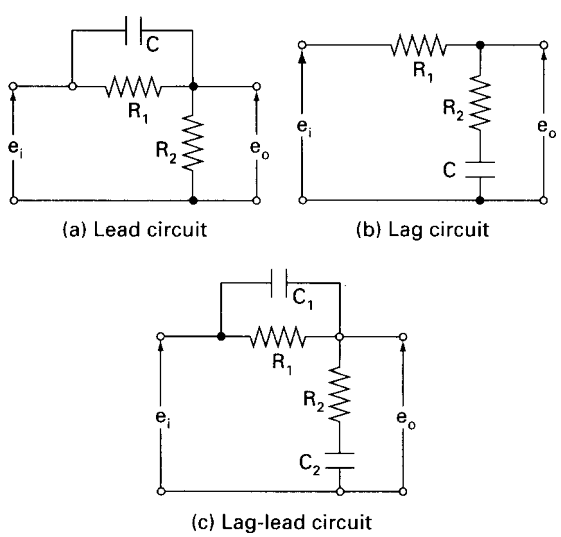

FIGURE 7.18
Electrical circuits used as a compensator.

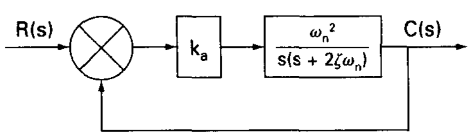

FIGURE 7.19
A second-order control system.

compensator can be used to improve the steady-state error. The lag compensator has the following form:

$$G_c(s) = \frac{(s + z_c)}{(s + p_c)} \tag{7.51}$$

where the compensator pole near the origin is located to the right of the compensator zero ($z_c/p_c > 1$).

For the case where both the transient and steady response are unsatisfactory a combination of a lag and lead compensator can be used. An example of a lag-lead compensator follows:

$$G_c(s) = \frac{(s + z_1)}{(s + p_1)} \frac{(s + z_2)}{(s + p_2)} \tag{7.52}$$

Figure 7.18 shows electrical circuits that could be used to create a lead, lag, or lag-lead compensator.

7.7.3 Feedback-Path Compensation

Feedback compensation can be used to improve the damping of the system by incorporating an inner rate feedback loop. The stabilizing effect of the inner loop rate feedback can be demonstrated by a simple example. Suppose we have the second-order system shown in Figure 7.19. The amplifier gain can be adjusted to vary the system response as shown in the accompanying root locus plot presented in Figure 7.20. The closed-loop transfer function for this system is given by

$$M(s) = \frac{k_a \omega_n}{s^2 + 2\zeta\omega_n s + k_a \omega_n^2}$$

Now if we add an inner rate feedback loop as shown in Figure 7.21, the closed-loop transfer function can be obtained as follows. The inner loop transfer functions are

$$G_1(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

$$H_1(s) = k_r s$$

which can be combined as

$$M(s)_{\text{I.L.}} = \frac{G_1(s)}{1 + G_1(s)H_1(s)}$$

$$= \frac{\omega_n^2}{s^2 + (2\zeta\omega_n + k_r\omega_n^2)s}$$

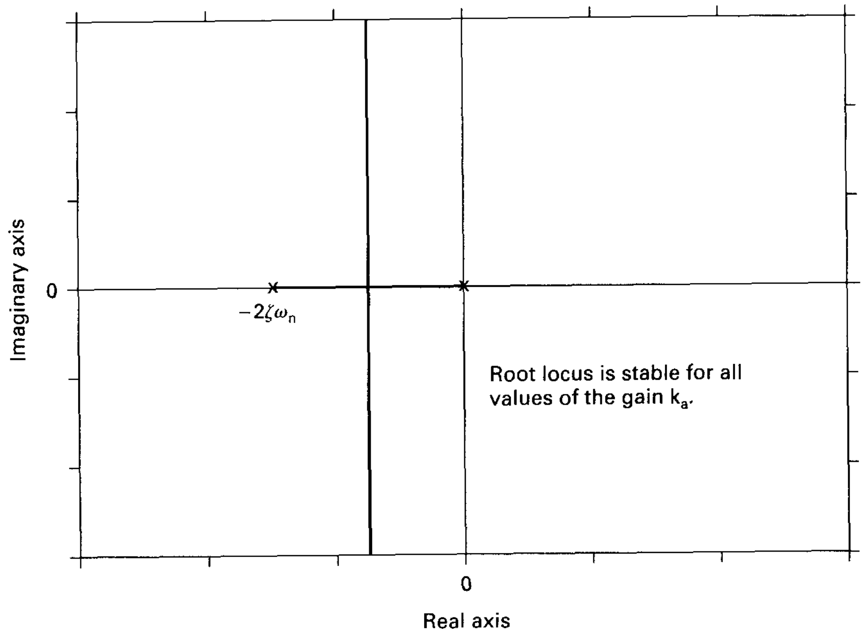

FIGURE 7.20
Root locus for second order system.

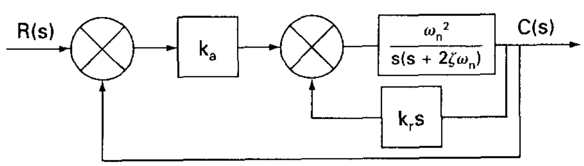

FIGURE 7.21
Control system with the addition of an inner rate feedback loop.

The closed-loop transfer function can be obtained by letting

$$G(s)_2 = \frac{k_a\omega_n^2}{s^2 + (2\zeta\omega_n + k_r\omega_n^2)s}$$

$$H_2(s) = 1$$

which can be combined as

$$M(s)_{\text{O.L.}} = \frac{G_2(s)}{1 + G_2(s)H_2(s)}$$

$$= \frac{k_a\omega_n^2}{s^2 + (2\zeta\omega_n + k_r\omega_n^2)s + k_a\omega_n^2}$$

If we compare the closed-loop transfer function for the cases with and without rate feedback we observe that in the closed-loop characteristic equation the damping term has been increased by $k_r\omega_n^2$. The gain k_r can be used to increase the system damping.

7.8 PID CONTROLLER

We have shown examples of various kinds of control concepts. The simplest feedback controller is one for which the controller output is proportional to the error signal. Such a controller is called a proportional control. Obviously the controller's main advantage is its simplicity. It has the disadvantage that there may be a steady-state error.

The steady-state error can be eliminated by using an integral controller

$$\eta(t) = k_i \int_0^t e(t)\,\mathrm{d}t \qquad \text{or} \qquad \eta(s) = \frac{k_i}{s} e(s) \tag{7.53}$$

where k_i is the integral gain. The advantage of the integral controller is that the output is proportional to the accumulated error. The disadvantage of the integral controller is that we make the system less stable by adding the pole at the origin. Recall that the addition of a pole to the forward-path transfer function was shown to bend the root locus toward the right half plane.

It is also possible to use a derivative controller defined as follows:

$$\eta(t) = k_d \frac{\mathrm{d}e}{\mathrm{d}t} \qquad \text{or} \qquad \eta(s) = k_d s e(s) \tag{7.54}$$

The advantage of the derivative controller is that the controller will provide large corrections before the error becomes large. The major disadvantage of the derivative controller is that it will not produce a control output if the error is constant. Another difficulty of the derivative controller is its susceptibility to noise. The derivative controller in its present form would have difficulty with noise problems. This can be avoided by using a derivative controller of the form

$$\eta(s) = k_d \frac{s}{\tau s + 1} e(s) \tag{7.55}$$

The term $1/(\tau s + 1)$ attentuates the high-frequency components in the error signal, that is, noise, thus avoiding the noise problems.

Each of the controllers—providing proportional, integral, and derivative control—has its advantages and disadvantages. The disadvanatages of each controller can be eliminated by combining all three controllers into a single PID controller, or proportional, integral, and derivative, controller.

The selection of the gains for the PID controller can be determined by a method developed by Ziegler and Nichols, who studied the performance of PID controllers by examining the integral of the absolute error (IAE):

$$\text{IAE} = \int_0^\infty |e(t)|\,\mathrm{d}t \tag{7.56}$$

From their analysis they observed that when the error index was a minimum the control system responded to a step input as shown in Figure 7.22. Note that the second overshoot is one quarter of the magnitude of the maximum overshoot. They called this the quarter decay criterion. Based on their analysis they derived a set of rules for selecting the PID gains. The gains k_p, k_i, and k_d are determined in terms

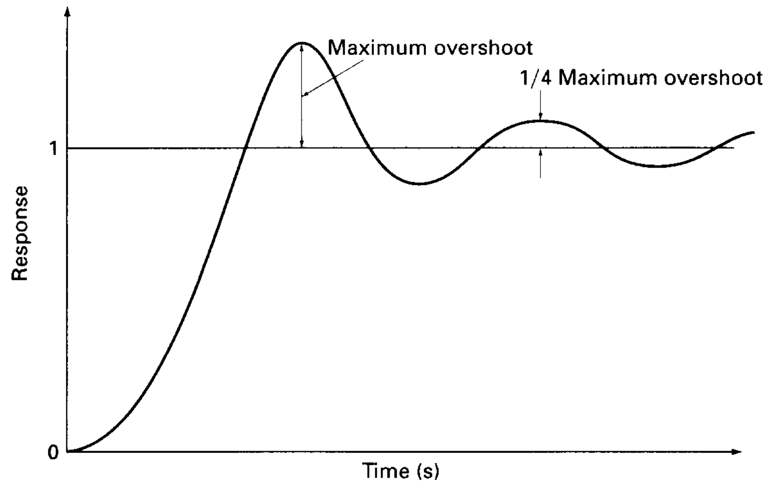

FIGURE 7.22
The quarter-decay response.

of two parameters, k_{p_u}, called the ultimate gain, and T_u, the period of the oscillation that occurs at the ultimate gain. Table 7.5 gives the values for the gains for proportional (P), proportional-integral (PI), and the proportional-integral-derivative (PID) controllers.

To apply this technique the root locus plot for the control system with the integral and derivative gains set to 0 must become marginally stable. That is, as the proportional gain is increased the locus must intersect the imaginary axis. The proportional gain, k_p, for which this occurs is called the ultimate gain, k_{p_u}. The purely imaginary roots, $\lambda = \pm i\omega$, determine the value of T_u:

$$T_u = \frac{2\pi}{\omega} \tag{7.57}$$

One additional restriction must be met: All other roots of the system must have negative real parts; that is, they must be in the left-hand portion of the complex s plane. If these restrictions are satisfied the P, PI, or PID gains easily can be determined.

EXAMPLE PROBLEM 7.7. Design a PID controller for the control system shown in Figure 7.23.

TABLE 7.5
Gains for P, PI, and PID controllers

Type of controller	k_p	k_i	k_d
P (proportional controller)	$k_p = 0.5k_{p_u}$		
PI (proportional-integral controller)	$k_p = 0.45k_{p_u}$	$k_i = 0.45k_{p_u}/(0.83T_u)$	
PID (proportional-integral-derivative controller)	$k_p = 0.6k_{p_u}$	$k_i = 0.6k_{p_u}/(0.5T_u)$	$k_d = 0.6k_{p_u}(0.125T_u)$

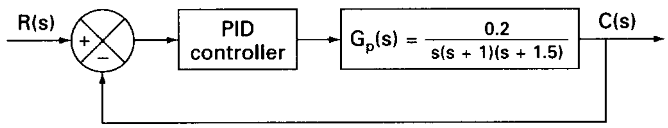

FIGURE 7.23
PID controller.

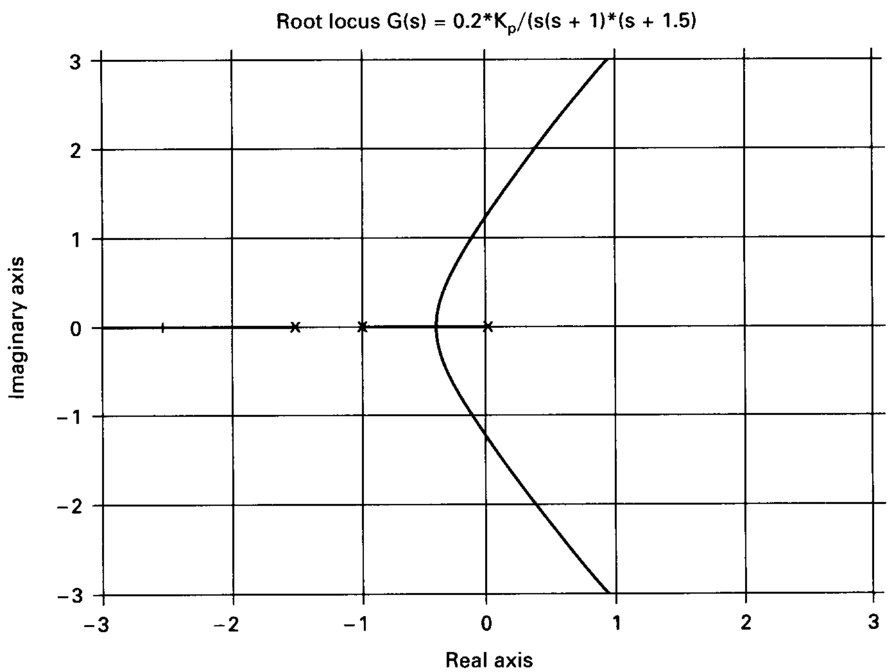

FIGURE 7.24
Root locus plot.

Solution. The gains of the PID controller can be estimated using the Ziegler-Nichols method provided the root locus for the plant becomes marginally stable for some value of the proportional gain k_p when the integral and derivative control gains have been set to 0. The root locus plot for

$$G(s) = \frac{0.2k_p}{s(s + 1)(s + 1.5)}$$

is shown in Figure 7.24. The root locus plot meets the requirements for the Ziegler-Nichols method. Two branches of the locus cross the imaginary axis and all other roots lie in the left half plane. The ultimate gain k_{p_u} is found by finding the gain when the root locus intersects the imaginary axis. The locus intersects the imaginary axis at $s = \pm 1.25i$. The gain at the crossover point can be estimated from the magnitude criteria:

$$\frac{|0.2|\, k_{p_u}}{|s|\,|s + 1|\,|s + 1.5|} = 1$$

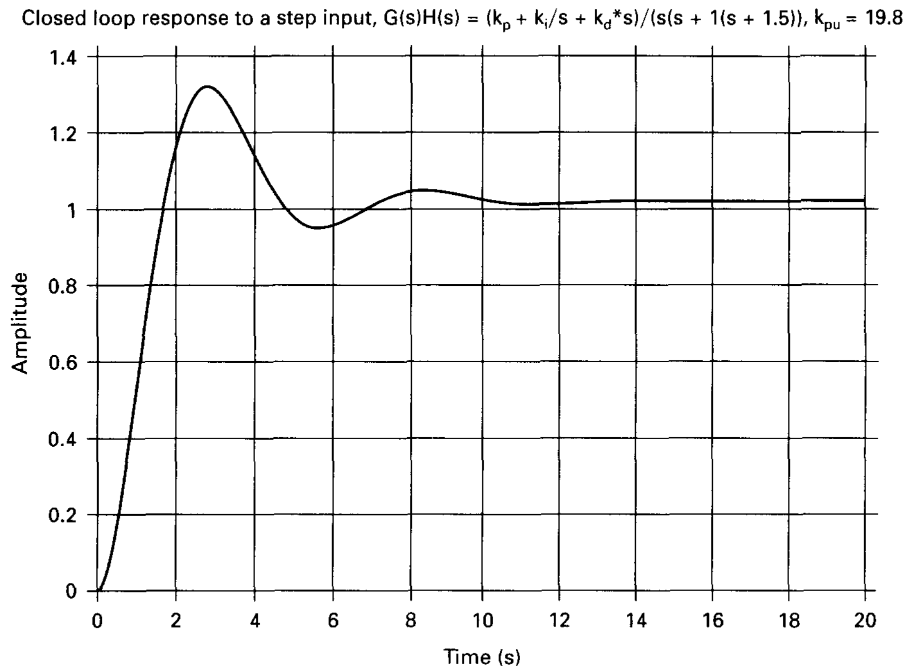

FIGURE 7.25
Transient response to a step input.

Substituting $s = 1.25i$ into the magnitude criteria yields

$$k_{p_u} = 19.8$$

The period of the undamped oscillation T_u is obtained as follows:

$$T_u = \frac{2\pi}{\omega} = \frac{2\pi}{1.25} = 5.03$$

Knowing k_{p_u} and T_u the proportional, integral, and derivative gains k_p, k_i, and k_d can be evaluated:

$$k_p = 0.6\ k_{p_u} = (0.6)(19.8) = 11.88$$

$$k_1 = 0.6\ k_{p_u}/(0.5T_u) = (0.6)(11.88)/[(0.5)(19.8)] = 0.72$$

$$k_d = 0.6\ k_{p_u}\ (0.125T_u) = (0.6)(19.8)(0.125)(5.03) = 7.47$$

The response of control system to a step input is given in Figure 7.25.

7.9 SUMMARY

In this chapter we examined some of the analytical tools available to the control system designer. The root locus technique allows the designer to examine the movement of the closed-loop poles of the control system as a function of one or more of the design variables. We also examined the relationship between the root location in the root locus diagram and the time and frequency domain performance of the system.

The conceptual design of a control system was presented. Once the control function has been identified, the designer must develop one or more concepts to meet the performance objectives of the control system. This phase of the design relies heavily on the designer's creativity and experience. Having developed some control system concepts the designer must evaluate the system performance. This requires mathematically modeling the various elements in the control system and selecting system parameters and analyzing the system performance using, for example, the root locus technique. In general, the designer usually will find that one or more of the concepts comes close to meeting the design objectives but that some of the requirements are not satisfied. In this case the designer must consider adding some form of compensating elements to the control system. We examined a number of compensators commonly used to improve control system performance. The type of compensation that needs to be added to a control system depends on what system performance specification needs to be improved.

PROBLEMS

Problems that require the use of a computer have a capital letter C after the problem number.

7.1. Given the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 + k = 0$$

find the range of values of k for which the system is stable.

7.2. Given the fourth-order characteristic equation

$$\lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda + k = 0$$

for what values of k will the system be stable?

7.3. Given the following characteristic equation determine the stability of the system using the Routh criterion. If the system is unstable determine the number of roots lying in the left portion of the complex plane.

$$\lambda^6 + 3\lambda^5 + 5\lambda^4 + 9\lambda^3 + 8\lambda^2 + 6\lambda + 4 = 0$$

7.4. The characteristic equations for several feedback control systems follow. Determine the range of values of k for which the following systems are stable:
(*a*) $s^3 + 3ks^2 + (k + 2)s + 4 = 0$
(*b*) $s^4 + 4s^3 + 13s^2 + 36s + k = 0$

7.5(C). The loop transfer function $G(s)H(s)$ is
(*a*) $\dfrac{k}{s(s^2 + 6s + 18)}$ (*b*) $\dfrac{k}{s(s + 2)(s + 5)}$ (*c*) $\dfrac{k(s + 4)}{s(s + 3)(s + 5)}$

(*d*) $\dfrac{k(s + 3)}{s^2 + 4s + 20}$ (*e*) $\dfrac{k(s + 4)}{(s^2 + 2s + 6)(s^2 + 4s + 8)}$

Sketch the root locus plot for variations of k, $0 \leq k \leq \infty$, for each transfer function. Check your results by using an appropriate root locus program.

7.6. Given the loop transfer function

$$G(s)H(s) = \frac{k}{s(s + 3)(s + 10)}$$

(*a*) Sketch the root locus plot for $G(s)H(s)$.
(*b*) Add a simple pole, $(s + 2)$, to $G(s)H(s)$ and examine the resulting root locus.
(*c*) Add a simple zero, $(s + 2)$, to $G(s)H(s)$ and examine the resulting root locus.

7.7. The root locus plot for the transfer function

$$G(s)H(s) = \frac{k}{s(s + 2)(s + 8)}$$

is shown in Figure P7.7.
(*a*) Estimate the system gain, k, when the system is critically damped.
(*b*) What is the value of the system gain, k, for which the system neutrally stable?

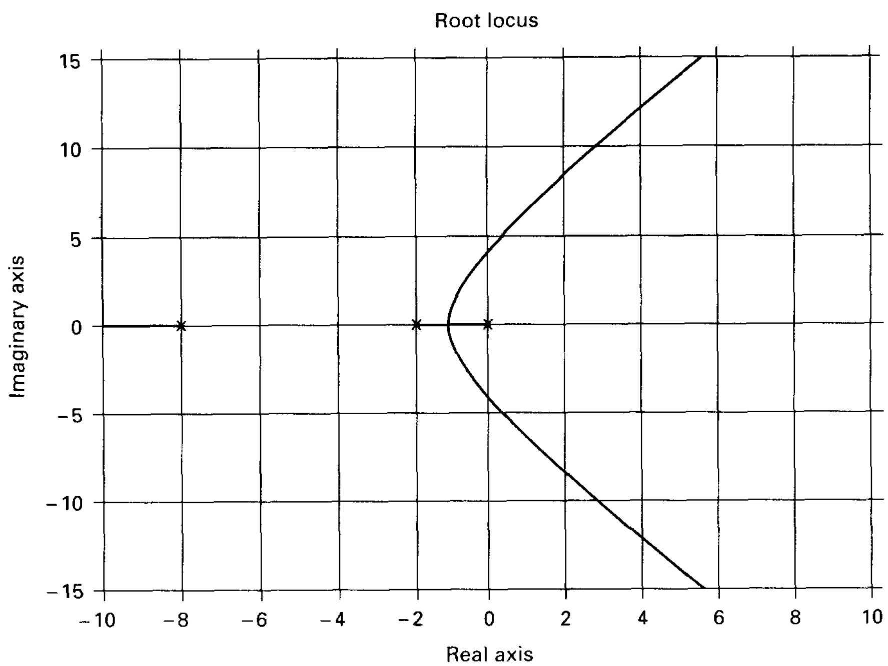

FIGURE P7.7

7.8. The single degree of freedom pitching motion of an airplane was shown to be represented by a second-order differential equation. If the equation is given as

$$\ddot{\theta} + 0.5\dot{\theta} + 2\theta = \delta_e$$

where the θ and δ_e are in radians, estimate the rise time, peak overshoot, and settling time for step input of the elevator angle of 0.10 rad.

7.9. Determine the frequency domain characteristic for Problem 7.8. In particular estimate the resonance peak, M_r, resonant frequency, ω_r, bandwidth, ω_B, and the phase margin.

7.10(C). The root locus plot for the loop transfer function

$$G(s)H(s) = \frac{k}{(s + 8)(s^2 + 6s + 13)}$$

is shown in Figure P7.10.

(*a*) Find the system gain when the damping ratio is $\zeta = 0.707$.

(*b*) Estimate the time-domain characteristic for the dominant roots for the gain determined in part (*a*).

(*c*) Estimate the frequency response characteristics, that is, gain and phase margin, from the root locus plot for the gain selected in part (*a*).

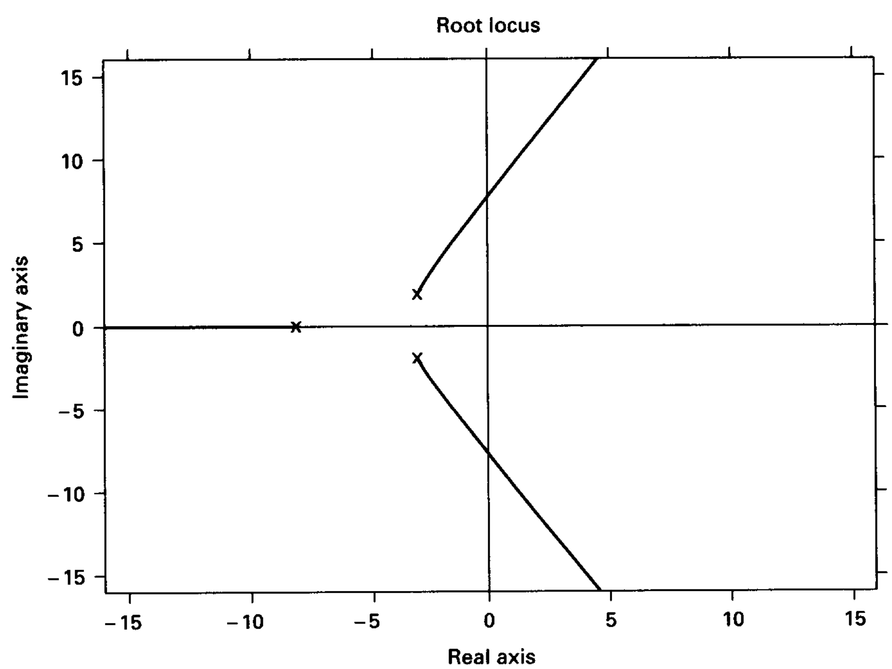

FIGURE P7.10

7.11. Calculate the position, velocity, and acceleration error constants K_p, K_v, and K_a for the loop transfer function $G(s)H(s)$ that follows:

(a) $\dfrac{10}{s(s+1)(s+10)}$

(b) $\dfrac{k}{s(1+0.1s)(1+s)}$

(c) $\dfrac{k}{s(s^2+4s+100)}$

(d) $\dfrac{s+2}{s(s^2+4s+6)}$

(e) $\dfrac{15\,(s+2)}{s^2(s+5)(s+3)}$

7.12. The lead compensator can be constructed from a simple electrical circuit shown in Figure P7.12. Show that the transfer function for this circuit can be written as

$$G(s) = \frac{e_o}{e_i} = a\frac{(T_1 s + 1)}{(aT_1 s + 1)}$$

where $a = R_2/R_1 + R_2$ and $T_1 = R_1 C$.

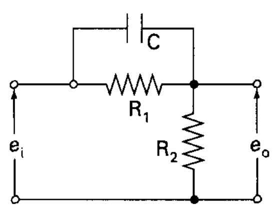

FIGURE P7.12
Lead circuit.

7.13. The lag compensator also can be constructed from a simple electrical circuit as shown in Figure P7.13. Show that the transfer function for this circuit can be written as

$$G(s) = \frac{e_o}{e_i} = \frac{T_2 s + 1}{(T_2/b)s + 1}$$

where $b = R_2/(R_1 + R_2)$
$T_2 = R_2 C$

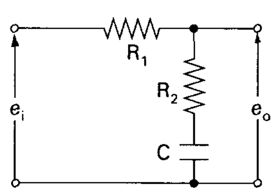

FIGURE P7.13
Lag circuit.

7.14(C). The control system shown in Figure P7.14 must meet the following performance specifications:

Damping ratio, $\zeta = 0.6$
Settling time, $t_s \leq 2.0$ s
Positional error constant, $K_p \geq 10$

(*a*) Assume that no compensation is used and estimate the system performance.
(*b*) Design a lead compensator to achieve this system performance.

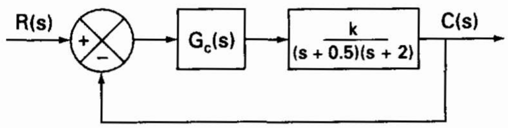

FIGURE P7.14

7.15. In the control system shown in Figure P7.15 rate feedback is to be used to increase the system damping. Estimate the gains k_a and k_r so that the system meets the following performance specifications:

Damping ratio, $\zeta = 0.7$
Settling time, ≤ 3.0 s

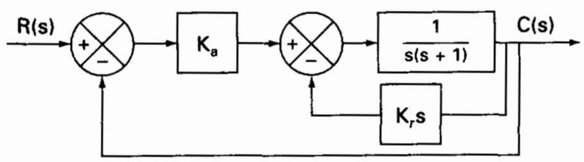

FIGURE P7.15

7.16(C). Given the control system shown in Figure P7.16 where the plant transfer function $G(s)$ is given by

$$G(s) = \frac{2.0}{s(s+1)(s+3)}$$

design a PID controller for this system.

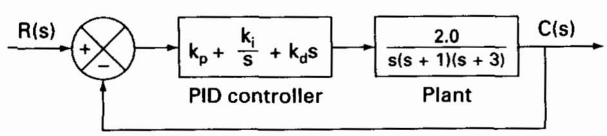

FIGURE P7.16

7.17(C). If the plant transfer function for Problem 7.16 is changed to

$$G(s) = \frac{7.0}{(s+5)(s^2+2s+5)}$$

design a PID controller for this system.

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