Tishk International University
Engineering Faculty
Computer Department

Discrete Mathematics

Topic: Chapter 2/ Set Theory

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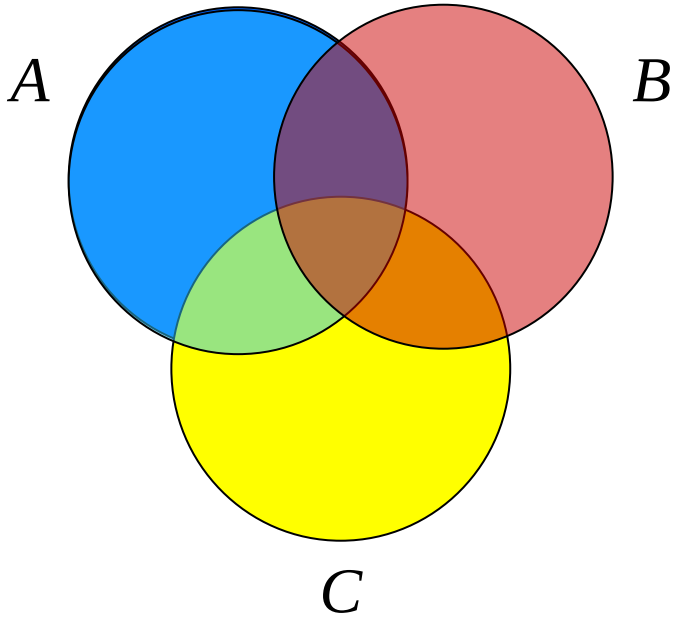
CHAPTER II

Set Theory

set theory is a branch of mathematical logic that studies sets



Objectives of this power point

- Definition of Basic Set Theory
- Types of some familiar sets
- Explanation of Universal Sets, Subsets, Power Set, and Proper Subsets
- Operations on sets (union, intersection, difference, complement, symmetric difference)
- Definition & explanation of Duality, Ordered Pairs, and Class of sets
- Examples

Basic Set Theory

- A set is just a collection of things having some common characteristics, and will be denoted by *A, B, C, X, Y, ….The things that make up a given set are called its elements or members, and will be denoted by a, b, c, x, y, …*
- Here is some of the *notation we use to talk about sets*. The simplest example of a set is Ø, the *empty set*, which has no *members or elements belonging to* it.
- The statement ‘’p belonge to A’’ is written p$\in A$

- The negation of $p \in A$ *is* $p \notin A$
- *{x | P} means "the set of all values of x that have* the property *P". For example,*

X= $\{ x \mid x \text{ is an even integer} \}$ *is the set of all even integers and*

Y= $\{ y\text{: there is an integer } n;\ y = 2n\}$

⇒ *X* = *Y means "X and Y have the same members*".

We abbreviate "if and only if" by "iff".

Notes. 1. order is not important. 2. repeated is ignored.

EXAMPLES A= {1, 2, 3, 4}

B= { x: x is positive integer, and x >3}

C= { y: y is integer and $1 \leq y \leq 4$} ⇒ A= C

S= { x: x is an odd integer}

Q= { x: there is an integer n; x= 2n+1} ⇒ S= Q

Some Familiar Sets

- Ø is the smallest set, with no members. We say that the *cardinality of Ø is zero, written | Ø | = 0.*

Much bigger are the number sets, which have infinitely many members (too many to write down).

- ℕ is Natural numbers, the set of all positive integers 1*, 2, 3,…* to infinity They are also called counting numbers as they are used to count objects.

(Warning: Other textbooks often use ℕ as the name for the set of non-negative integers 0*, 1, 2, 3,…*).

- ℤ is the set of all integers *…-2, -1, 0, 1, 2,…*

- ℝ is the set of all real numbers, consisting of all decimal numbers Including the integers, rational numbers like *-3/4* and 22/7 , and irrational numbers like $\sqrt{2}$ *and* π*.*

We can build new sets from these familiar sets by forming subsets, power sets unions, intersections, complements, and ordered pairs.

Universal Set and Subsets

- Universal Set. The numbers of all sete under investigation usually belong to some fixed large set called the universal set or univers.
- We write $X \subseteq Y$ *to say "X is a subset of Y",*

  by which we mean that every member of X *also* belongs to Y.
- $\emptyset \subseteq Y$ *since every member of* $\emptyset$ *is a member*

  of Y. *How do we know this? Since* $\emptyset$ *has no* members, we can say anything we like about its members, for example that they are all pink, and no-one can refute our claim by giving a counterexample.

Universal Set and Subsets

- If $X \subseteq Y$ *and* $Y \subseteq X$ *then* X *and* Y *must have* exactly the same members, so that $X = Y$.
- Sometimes all the sets of interest to us are subsets of some big set U, *which we call our universal set for that situation.*
- *To describe* subsets of U *we give some property the elements* have to satisfy, for example if $U = \mathbb{R}$ *we could use* the property $x > 0$ *to form the subset of positive* real numbers $\{x \mid x > 0\}$.

Power sets

Examples. 1. $N \subset Z \subset Q \subset R$

2. $U = \{1, 2, 3, \dots, 10\}, A = \{1, 3, 5, 7, 9\}, B = \{2, 4, 6, 8, 10\},$
$C = \{1, 3, 9\}\,,\; D = \emptyset.$
Then $D \subset B, C \subset A,\; C \subset U,\; A \subset U,\; A \not\subset B,\; B \subset U,$
$D \subset U, D \subset A, D \subset C.$

Power set. If A is a set, then the set of all subsets of A is called the power set of A and is denoted by P(A).

3. Let *X = {2, 3}. The subsets of X are Ø, {2}, {3} ,*
and *{ 2, 3}. If we collect these subsets of X as*
the elements of one big set we form the power set
P(X) = {Ø, {2}, {3}, { 2, 3} }.

4. Let A = $\{1, 2, 3\} \Rightarrow P(A) = \{\emptyset, A, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2,3\}\}$

Notes

- But there is a difference between the relations $\subseteq$ *and* $\in$, *for example* $\{2\} \subseteq X$ *but* $\{2\} \notin X$ *while* $\{2\} \in P(X)$ *but* $\{2\} \nsubseteq P(X)$.
  (We will use the symbol $\subseteq$ *between two sets while the symbol* $\in$ *between element and set)*

- The powerset $P(X)$ *is bigger* X *than the original* set X, *leading to the questions:*

Is $P(X)$ *always* bigger than X?
Can we predict the size of $P(X)$ if we know the size of X?

Theorem

1. $A \subseteq B$ *and* $B \subseteq A$ *if and only if* $A = B$
2. For any set A, we have $\emptyset \subset A \subset U$.
3. For any set A, we have $A \subseteq A$.
4. If $A \subset B$, *and* $B \subset C$, *then* $A \subset C$.

Proper Subset

Definition. If $A \subset B \; but \; A \neq B$, then we say that A is a proper subset of B.

Example 1 If $A = \{1,3\}$, $B = \{1,2,3\}$, $C = \{1,3,2\}$, then $A \subset C, B \subset C$, A is a proper subset of C but B is not a proper subset of C since B=C.

A set is finite if it has **n** distinct elements where **n** $\in \boldsymbol{N}$.

In this case **n** is called the cardinality of A and is denoted by card(A) or n(A).

Example 2 Let $A = \{x, y, z\}$, then card(A)=3 and the power set of A,

P(A)=$\{A, \emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}\} \Longrightarrow$

$$card(P(A)) = 2^{card(A)} = 2^3 = 8$$

Operations on Sets
1. Union $\Leftrightarrow \cup$

- $A \cup B = \{x: x \in A \text{ or } x \in B\}$, $A_1 \cup A_2 \cup \cdots \cup A_n = \cup_{k=1}^{k=n} A_k$
- Example: $\{2, 3\} \cup \{3, 4, 5\} = \{2, 3, 4, 5\}$.
- *Note that* the union does not get two copies of the element 3, even though 3 belongs to $\{2, 3\}$ *and also to* $\{3, 4, 5\}$. *We put a single copy of 3 into the union* of $X = \{2, 3\}$ *and* $Y = \{3, 4, 5\}$ *because 3 belongs* to at least one of X *and* Y. Notice that the union in our example is bigger than either of the two original sets.
- This invites two questions: Is $X \cup Y$ *always bigger than* (have more elements than) X *and* bigger than Y ?
- *Can we predict the size of* $X \cup Y$ if we know the sizes of X *and of* Y ?
- Notice that $X \subseteq X \cup Y$ *since every member of* X belongs to $X \cup Y$. *Is* Y *also a subset of* $X \cup Y$?

2.Intersections $\Leftrightarrow \cap$

- $A \cap B = \{x : x \in A \; and \; x \in B\}$, $A_1 \cap A_2 \cap \cdots \cap A_n = \cap_{k=1}^{k=n} A_k$
- If $A \cap B = \emptyset$, *then* A & B are disjoint sets.
- Example 1.

{2, 3} ∩ { 3, 4, 5 } = { 3 }.

Note that although 3 belongs to both *X = { 2, 3 } and Y = { 3, 4, 5 }, only one copy of 3 is put into the* intersection.

- Example 2.

{ 2, 3} ∩ { 6, 7, 8 } = Ø, since there are no elements common to *X ={ 2, 3 } and Y = { 6, 7, 8}*.

we say that the sets X and Y are disjoint.

Intersections

- Two questions to ponder are: Must $X \cap Y$ always be smaller (have fewer elements than) than X *and smaller than* Y ?
- Can we predict the size of $X \cap Y$ *if we know the* sizes of X *and of* Y ?
- Notice that $X \cap Y \subseteq X$ *since every member of* $X \cap Y$ *belongs to* X *as well as to* Y .
- *Is* $X \cap Y$ *also* a subset of Y ? *Is* $X \cap Y$ *a subset of* $X \cup Y$?

3.Difference $\Leftrightarrow A - B$

$A - B = \{x : x \in A \text{ and } x \notin B\} = A/B$

Example.1 Let $A = \{a, b, e\}$, $B = \{b, c, d, e\}$, then

$A - B = \{a\}$, $B - A = \{c, d\}$

Example.2 {2, 3} - { 3, 4, 5 } = { 2 }.

4.Complement of a set $\Leftrightarrow \bar{A}$ or A^c

- If we have a universal set U *containing* A, *then* $U - A$ *is* abbreviated to $\bar{A}$ or A^c *and we call* A^c *simply "the* complement of A *", thus* $A^c = \{x : x \notin A\}$

Example.3 Let $A = \{x : x \text{ is an integer and } x \geq 4\}$, then

$A^c = \{x : x \text{ is an integer and } x < 4\}$ Where U=R all real numbers, $A^c = U - A$

Complements

- Example: Suppose our universal set is $U = \mathbb{R}$.

  Let $Y = \{x \mid x \geq 0\}$. *Then* $Y^c = \{ x \mid x < 0\}$.

- Questions: Suppose we take any subset X *of* some universal set U.
- *What is* $X \cap X^c$ *equal to?*
- What is $X \cup X^c$ *equal to?*

5. Symmetric Difference $\Longleftrightarrow$ $A \oplus B$ *or* $A \triangle B$

- $\boldsymbol{A \triangle B = (A \cup B) - (A \cap B)}$

Example.4 $\{a, b\} \oplus \{a, c\} = (\{a, b\} \cup \{a, c\}) - (\{a, b\} \cap \{a, c\})$

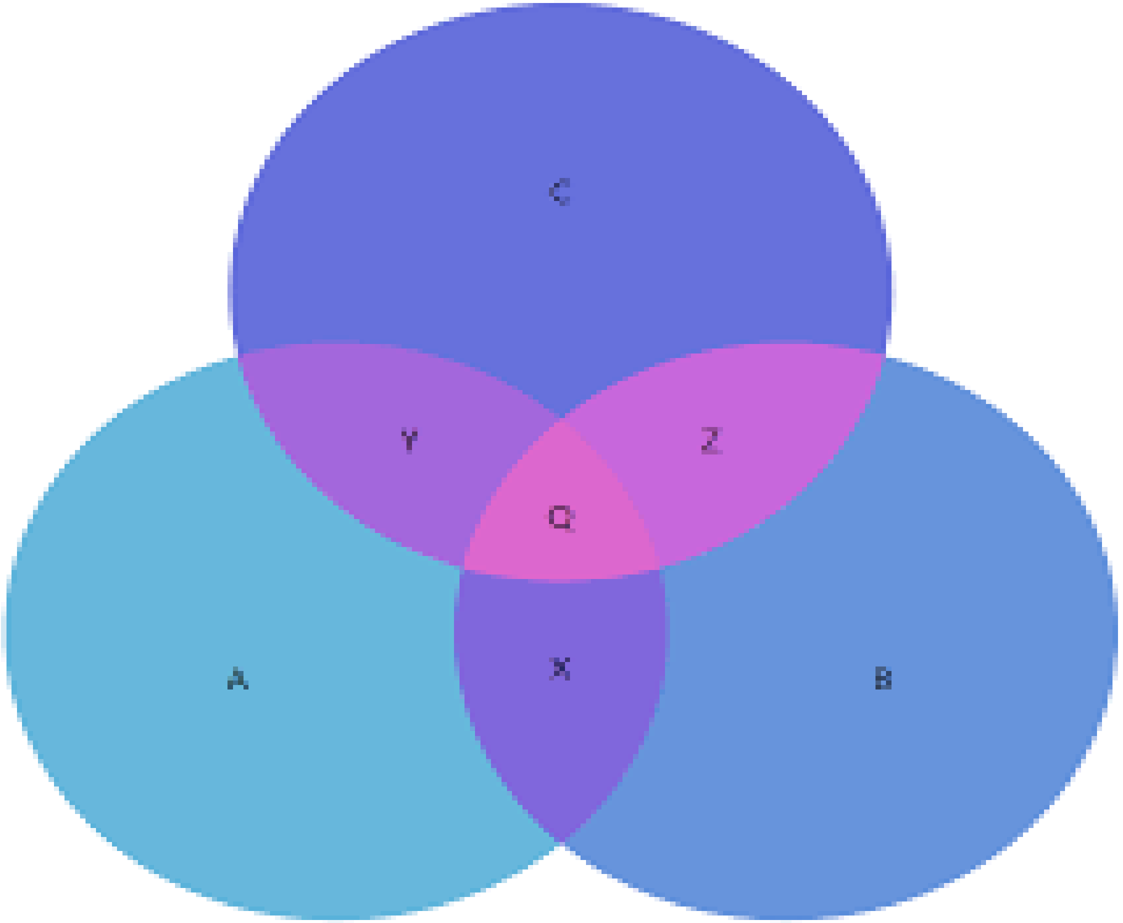
$= \{a, b, c\} - \{a\} = \{b, c\}$

$\{a, b\} \triangle \emptyset = \{a, b\}$

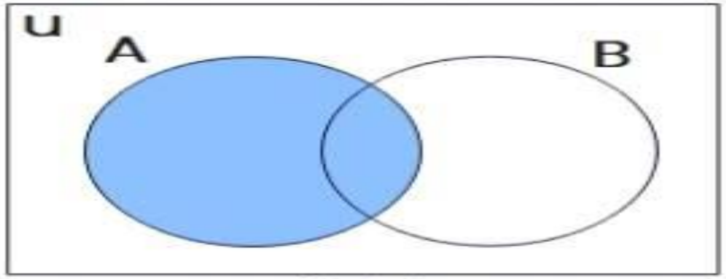
$\{a, b\} \triangle \{a, b\} = \emptyset$

Venn Diagrams

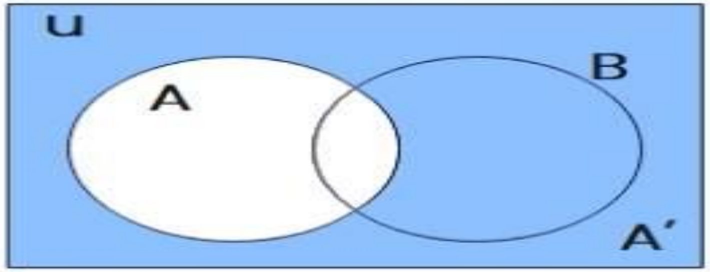
There is a very nice visual tool we can use to represent operations on sets. A Venn diagram displays sets as intersecting circles. We can shade the region we are talking about when we carry out an operation. We can also represent cardinality of a particular set by putting the number in the corresponding region.



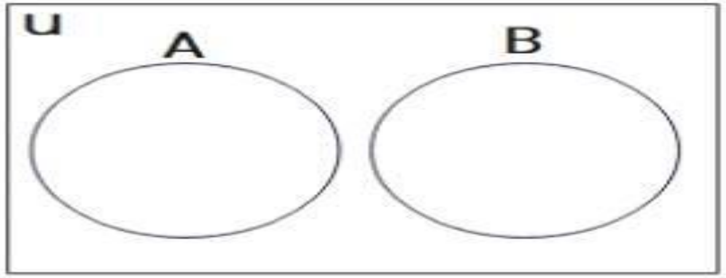
Operations of Sets and Venn Diagrams



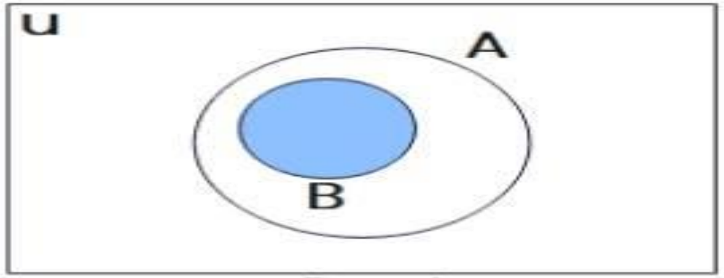
Set A



A′is the Complememt of A

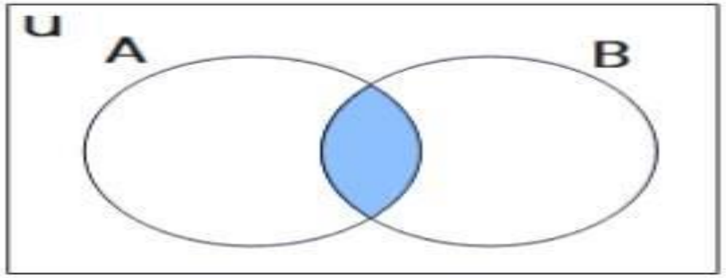


A & B are disoint sets



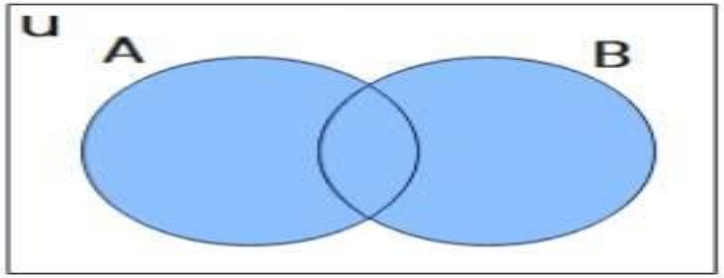
$B \subset A$

B is the proper subset of A



$A \cap B$

A & B are overlapping sets



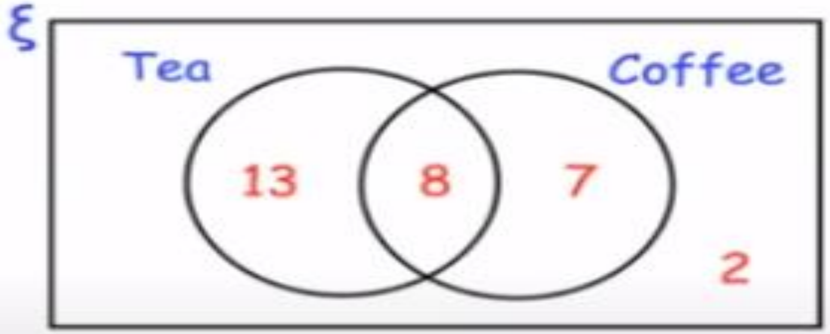
$A \cup B$

A & B are overlapping sets

Solving Problem

The Venn diagram shows information about the drinks some students like

How many students do not like tea or coffee?



How many students like tea and like coffee?

Homework

1. Let U= { 1, 2, 3, 4, …, 15, 16} and A = multiples of 3, B = multiples of 4. Complete the Venn diagram.

2. In a case of 24 students, 12 students play the piano, 13 students play the guitar, 4 students play neither instrument. How many students only play the guitar?

Duality

Suppose E is an equation of set algebra. The dual E^* *of* E is the equation obtained by replacing each occurrence of ∪,∩ , U *and* $\emptyset$ *in* E *by* ∩,∪, $\emptyset$, U respectively.

For example, the dual of

$$(U \cap A) \cup (B \cap A) = A \ \ is \ \ (\emptyset \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Algebra of sets are duals of each other and it is called the principle of duality. If any equation E is an identity then its' dual E^* is also an identity.

According to principle of duality "Dual of one expression is obtained by replacing AND (∧) with OR(∨) and OR with AND together with replacement of 1 with 0 and 0 with 1.

Ordered Pairs

- In the set {2, 3}, *the* order *in which we list the* members 2 and 3 is not important so {2, 3} = {3,2} *since the two descriptions single out the same* elements.
- But if the order of elements is important, we may change the braces *{ and } to parentheses and* write *(2, 3).*
- We read *(2, 3) as "the ordered pair with 2 as first* co-ordinate and 3 as second co-ordinate". That's mean $(2,3) \neq (3,2)$.
- We write $X \times Y$ *to say "the Cartesian product of X and Y".*
- $X \times Y$ *is the set of all ordered pairs (x, y) such* that $x \in X$ *and* $y \in Y$.
- Example: $\{2, 3\} \times \{5, 6\} = \{(2, 5), (2, 6), (3, 5), (3, 6)\}$.

Note that X and Y are sets and $X \times Y$ also is a set, but the elements of $X \times Y$ are ordered pairs.

Class of set

Let A be a set, then the set of sets that subset of A is called the class of subsets of A. That is the class of all subsets of A is a power set of A, P(A) and denoted by $\mathcal{A}$.

Note that class of B is $\mathcal{B}$, class of C is $\mathcal{C}$, class of D is $\mathcal{D}$, …etc.

Example.

Let $A = \{1, 2, 3, 4\}, and\ \mathcal{A}$ be the class of subsets of A which contain exactly three elements of A then

$$\mathcal{A} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

Let $\mathcal{B}$ be the class of subsets of A which contain 2 and two other elements of A, then

$$\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\} \Longrightarrow \mathcal{B}\ is\ a\ subclass\ of\ \mathcal{A}.$$