# Question Bank-1

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Advanced Mathematics2 –ME212

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# Objectives

To teach students the following opinions through out the course

1. Sequences and Series
2. Functions of several variables
3. Vector in the plain and space
4. Limits and continuity in higher dimensions
5. Partial derivatives ,Higher order Partial derivatives, Implicit differential, Direction derivatives and gradients

Example: Describe the formula for the following sequences

$$2, 4, 6, 8, 10, 12, \ldots$$

Solution:

$a_n = 2n$

Example: Describe the formula for the following sequences

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \ldots\right\}$$

$$\{b_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \ldots\right\}$$

Solution

$$\{a_n\} = \sqrt{n},$$

$$\{b_n\} = \frac{n-1}{n},$$

**THEOREM** The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

**EXAMPLE** Find thr values of the following functions as n ➡ infinity

(a) $\frac{\ln(n^2)}{n} = \frac{2\ln n}{n} \to 2 \cdot 0 = 0$ — Formula 1

(b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$ — Formula 2

(c) $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$ — Formula 3 with $x = 3$ and Formula 2

(d) $\left(-\frac{1}{2}\right)^n \to 0$ — Formula 4 with $x = -\frac{1}{2}$

(e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$ — Formula 5 with $x = -2$

(f) $\frac{100^n}{n!} \to 0$ — Formula 6 with $x = 100$

Example: Find the values of the following functions as n $\longrightarrow \infty$.

(a) $$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=\lim_{n\to\infty}1-\lim_{n\to\infty}\frac{1}{n}=1-0=1$$

(b) $$\lim_{n\to\infty}\frac{4-7n^6}{n^6+3}=\lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)}=\frac{0-7}{1+0}=-7.$$

Example: show that the following series converge if p>1.

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$$

**Solution** If $p>1$, then $f(x)=1/x^p$ is a positive decreasing function of $x$. Since

$$\int_1^{\infty}\frac{1}{x^p}\,dx=\int_1^{\infty}x^{-p}\,dx=\lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$=\frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}}-1\right)$$

$$=\frac{1}{1-p}(0-1)=\frac{1}{p-1},$$

the series converges by the Integral Test.

**EXAMPLE** Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

**Solution** We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

We can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1.$$

**EXAMPLE** Prove that the series converge

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

**EXAMPLE** Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$
$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$
$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

**EXAMPLE** Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

**Solution**

**(a)** The standard position vector $\mathbf{v}$ representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

**(b)** The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

**EXAMPLE** Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

**Solution** We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$. ■

**DEFINITION** Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

**EXAMPLE** To determine if two vectors are orthogonal, calculate their dot product.

**(a)** $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

**(b)** $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

## Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and $c$ is a scalar, then

**1.** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

**2.** $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

**3.** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

**4.** $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

**5.** $\mathbf{0} \cdot \mathbf{u} = 0$.

**DEFINITION** The **dot product** $\mathbf{u} \cdot \mathbf{v}$ ("**u** dot **v**") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

**EXAMPLE**

**(a)** $$\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -6 - 4 + 3 = -7$$

**(b)** $$\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$$

**THEOREM 1—Angle Between Two Vectors** The angle $\theta$ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

**EXAMPLE** Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

**Solution** We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians}.$$

**EXAMPLE** Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

**Solution**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

**EXAMPLE** Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

**Solution** The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

■

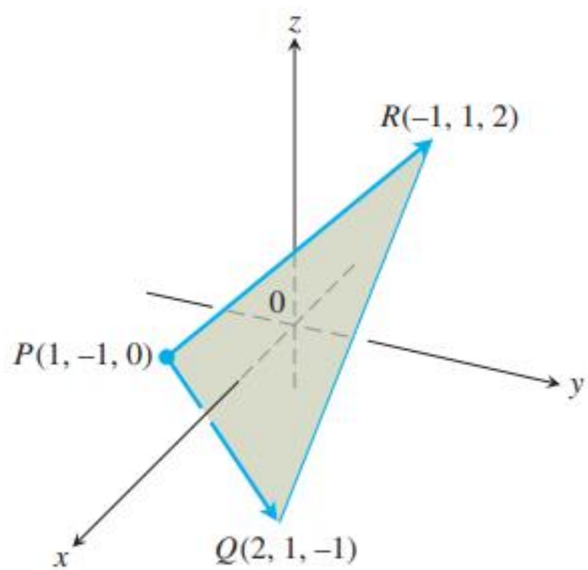


FIGURE .31

**EXAMPLE** Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

**Solution** Since $f$ is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 14.4. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

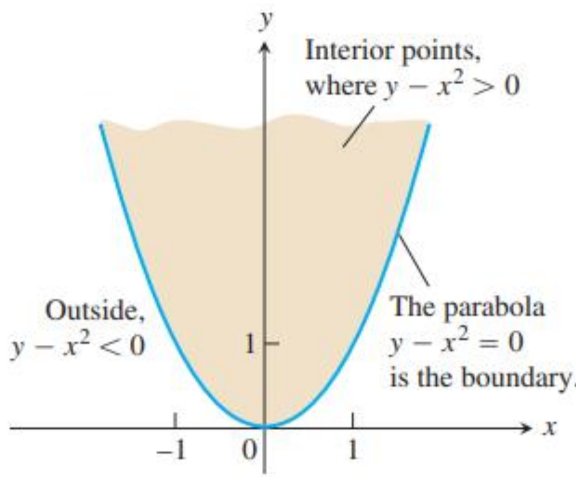


**FIGURE .4** The domain of $f(x, y)$ in Example consists of the shaded region and its bounding parabola.

**EXAMPLE** Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of $f$ in the plane.

**Solution** The domain of $f$ is the entire $xy$-plane, and the range of $f$ is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure .5.

The level curve $f(x, y) = 0$ is the set of points in the $xy$-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure 14.5) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \qquad \text{or} \qquad x^2 + y^2 = 49$$

$$f(x, y) = 100 - x^2 - y^2 = 75, \qquad \text{or} \qquad x^2 + y^2 = 25.$$

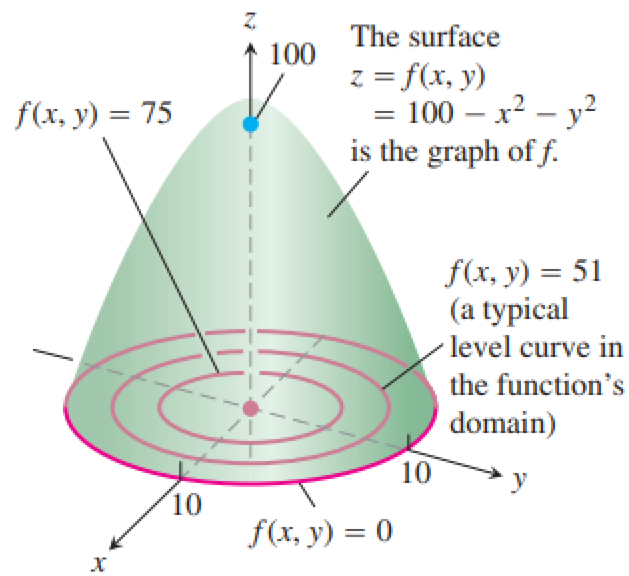


**FIGURE .5** The graph and selected level curves of the function $f(x, y)$ in Example 3.

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of $f$. ■

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of $f$. Figure .6 shows the contour curve $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour curve lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.

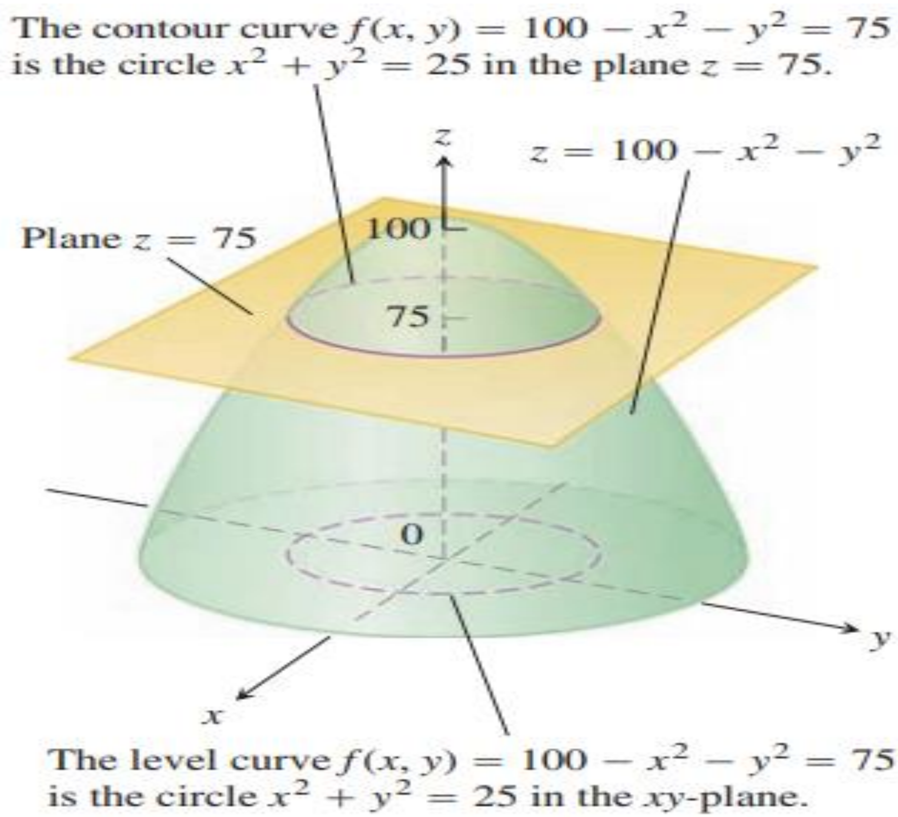


**FIGURE .6** A plane $z = c$ parallel to the $xy$-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

**EXAMPLE** Differentiate the following powers of $x$.

**(a)** $x^3$ **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\dfrac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

**Solution**

**(a)** $\dfrac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ **(b)** $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3}$

**(c)** $\dfrac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ **(d)** $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$

**(e)** $\dfrac{d}{dx}(x^{\ 4/3}) = -\dfrac{4}{3}x^{\ (4/3)\ \ 1} = \cdot\ 3$

**(f)** $\dfrac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \dfrac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

**EXAMPLE** Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

**Solution**

**(a)** From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

**(b)** This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for $y$ and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

**EXAMPLE** Find the derivative of **(a)** $y = \dfrac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

**Solution**

**(a)** We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.\end{aligned}$$

**(b)** $\dfrac{d}{dx}(e^{-x}) = \dfrac{d}{dx}\left(\dfrac{1}{e^x}\right) = \dfrac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \dfrac{-1}{e^x} = -e^{-x}$

$y = e^x \sin x$:
$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \sin x$$
$$= e^x \cos x + e^x \sin x$$
$$= e^x (\cos x + \sin x)$$

$y = \frac{\sin x}{x}$:
$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$$
$$= \frac{x \cos x - \sin x}{x^2}$$

## Derivative of the Cosine Function

$$\frac{d}{dx}(\cos x) = -\sin x.$$

## Derivatives of the Other Basic Trigonometric Functions

Because sin $x$ and cos $x$ are differentiable functions of $x$, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

**EXAMPLE** Find $y''$ if $y = \sec x$.

**Solution** Finding the second derivative involves a combination of trigonometric derivatives.

$$
\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}
$$

## EXAMPLE

$y = 5e^x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x)$$

$$= 5e^x - \sin x$$

$y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

$$y = \frac{\cos x}{1 - \sin x}:$$

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$

# References


1. Thomas' Calculus" 11th edition
2. Calculus Early Transcendental Functions" by Ron Larson and Bruce Edwards