# Lecture – 7

# Application of classical control theory

## 1. DYNAMIC STABILITY AND RESPONSE BEHAVIOR OF A SPRING–MASS–DAMPER SYSTEM AND ITS STABILITY CRITERIA

Figure 5.3 shows an example of a mechanical spring–mass–damper system. The position of the mass, x(t), is considered to be the 'output' of the system. The externally applied driving force, f(t), is considered to be the 'input' to the system.

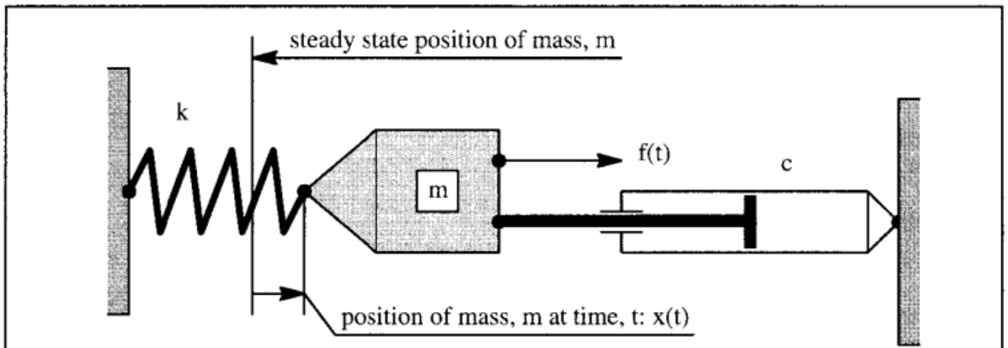



**Figure 5.3 Example of a Spring–Mass–Damper System**

The equation of motion for the system of Figure 5.3 can be written as follows:

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad (5.3)$$

It is useful to cast this equation in terms of accelerations rather than forces. This is done by dividing by the mass, m:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m}$$

The following two quantities will now be defined:

the undamped natural frequency : $\omega_n = \sqrt{\frac{k}{m}}$

the damping ratio : $\zeta = \frac{c}{2\sqrt{km}}$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{m} = f_1(t)$$

Applying the Laplace transform for non–zero initial conditions yields:

If the initial conditions are all equal to zero, it is possible to solve Eqn (5.9) for the ratio of the output Laplace transform, x(s), to the input Laplace transform, $f_1(s)$ :

$$\frac{x(s)}{f_1(s)} = G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (5.11)$$

This ratio is referred to as the **open loop transfer function** of the spring–mass–damper system.

Assume that the forcing function $f_1(t)$ is a unit step as illustrated in Figure 5.4. The Laplace transform of the unit step is:

$$f_1(s) = \frac{1}{s}$$

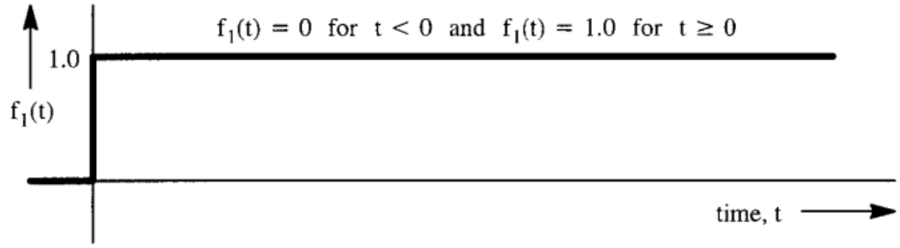



**Figure 5.4 Graphical Representation of a Unit Step Forcing Function**

The output of the system, in the s–domain, can now be written as:

$$x(s) = \frac{1}{s}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{5.13}$$

To find the corresponding time domain response, x(t), it is necessary to find the inverse Laplace transform of x(s). This is done with the method of partial fraction expansion. How this is done is explained in detail in Ref. (5.2). The usual procedure is to first find the roots of the characteristics equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{5.14}$$

The roots of this characteristic equation will be called $\lambda_1$ and $\lambda_2$. These roots are sometimes referred to as the 'eigen–values' of the system. It is now possible to write:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s - \lambda_1)(s - \lambda_2) \tag{5.15}$$

The system output in the s–domain can now be written as:

$$x(s) = \frac{1}{s}\left(\frac{1}{(s - \lambda_1)(s - \lambda_2)}\right) = \frac{A}{s} + \frac{B}{s - \lambda_1} + \frac{C}{s - \lambda_2} \tag{5.16}$$

The constants A, B and C may be determined with the theorem of residues (See Ref. 5.5). It is found that:

$$A = \frac{1}{\lambda_1\lambda_2} \qquad B = \frac{1}{\lambda_1(\lambda_1 - \lambda_2)} \qquad C = \frac{1}{\lambda_2(\lambda_2 - \lambda_1)} \tag{5.17}$$

It is now possible to rewrite Eqn (5.16) as:

$$x(s) = \frac{\frac{1}{\lambda_1\lambda_2}}{s} + \frac{\frac{1}{\lambda_1(\lambda_1 - \lambda_2)}}{s - \lambda_1} + \frac{\frac{1}{(\lambda_2(\lambda_2 - \lambda_1)}}{s - \lambda_2} \tag{5.18}$$

The roots of Eqn (5.14), $\lambda_1$ and $\lambda_2$, can be either both real or both complex. Both cases will be considered.

**Case 1: Both roots of Eqn (5.14) are real**

The inverse Laplace transform of the s–domain functions in Eqn (5.18) can be found directly from Table C1 in Appendix C. Doing so results in the following time–domain solution for x(t):

$$x(t) = \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1(\lambda_1 - \lambda_2)}e^{\lambda_1 t} + \frac{1}{\lambda_2(\lambda_2 - \lambda_1)}e^{\lambda_2 t} \tag{5.19}$$

Three possibilities present themselves:

1) If both roots, $\lambda_1$ and $\lambda_2$, are positive, $\lambda_1 > 0$ and $\lambda_2 > 0$, it is seen that x(t) will diverge to infinity. Such a system is said to be divergent. Note that if only one root is positive the system will still be divergent.

2) If both roots, $\lambda_1$ and $\lambda_2$, are negative, $\lambda_1 < 0$ and $\lambda_2 < 0$, it is seen that x(t) will converge toward the value $x(t) = 1/(\lambda_1\lambda_2)$. Such a system is said to be convergent.

3) If both roots, $\lambda_1$ and $\lambda_2$, are equal to zero, $\lambda_1 = 0$ and $\lambda_2 = 0$, it is seen that x(t) becomes undetermined. This can be shown by application of l'Hopital's Theorem. Such a system is said to be neutrally stable. Note that if only one root is zero, the system is still neutrally stable.

Figure 5.5 shows the various time–domain responses for the case of all–zero, all–negative and all–positive real characteristic equation roots.

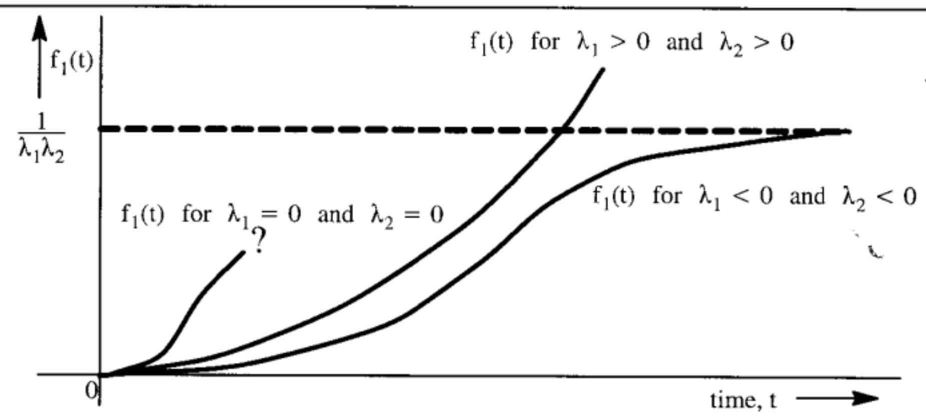



**Figure 5.5 Time Domain Responses of a Spring–Mass–Damper System with Real Characteristic Equation Roots**

**Case 2: Both roots of Eqn (5.14) are complex**

In this case, the complex roots must be each others conjugate:

$$\lambda_1 = n + j\omega \qquad \lambda_2 = n - j\omega \tag{5.20}$$

By substituting these forms of the roots into Eqn (5.19) it is found that:

$$x(t) = \frac{1}{n^2 + \omega^2} + \frac{e^{(n+j\omega)t}}{(n + j\omega)2j\omega} - \frac{e^{(n-j\omega)t}}{(n - j\omega)2j\omega} \tag{5.21}$$

With the help of De Moivre's Theorem (See Reference 5.5, page 467) this in turn can be written as follows:

$$x(t) = \frac{1}{n^2 + \omega^2}\left\{1 - e^{nt}\left(\cos\omega t - \frac{n}{\omega}\sin\omega t\right)\right\} \tag{5.22}$$

Clearly the system response in this case is oscillatory in nature.

Again, three possibilities present themselves:

1) It is seen from Eqn (5.22) that as long as the real part of the complex roots, n<0, the oscillatory terms will subside and the system will reach a final position given by:

$$x(t \rightarrow \infty) = \frac{1}{n^2 + \omega^2} = \frac{1}{\lambda_1\lambda_2} \tag{5.23}$$

Such a system is called oscillatory convergent (oscillatory stable).

2) Similarly it is seen from Eqn (5.22) that when the real part of the complex roots is positive, the amplitudes of the oscillations will tend to diverge. Such a system is called oscillatory divergent (oscillatory unstable).

3) If the real part of the complex roots is zero, the system will oscillate at a constant amplitude. Such a system is said to be neutrally stable.

Figure 5.6 shows examples of time–domain responses for the case of complex characteristic equation roots.

The following properties of the characteristic equation roots and their characteristics in the s–plane are to be noted. The reader is asked to refer to Figure 5.7.

First, the product of the characteristic equation roots is observed to be equal to the square of the undamped natural frequency as defined in Eqn (5.5):

$$\lambda_1\lambda_2 = n^2 + \omega^2 = \omega_n^2 \tag{5.24}$$

Notice that in the s–plane (see Figure 5.7) the undamped natural frequency is equal to the distance of each complex characteristic equation root (also called a pole) to the origin.

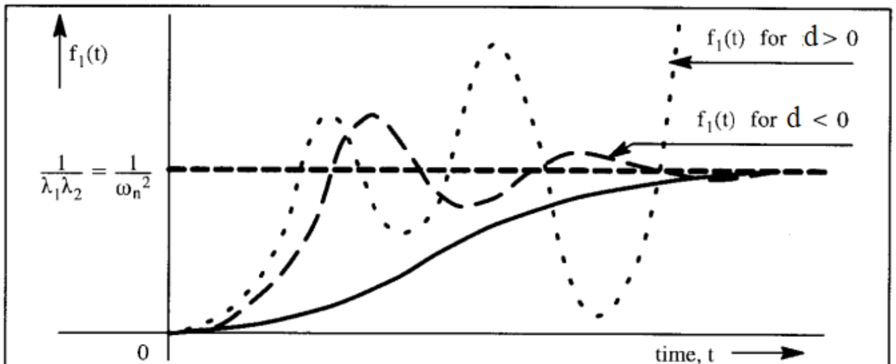



**Figure 5.6 Time Domain Responses of a Spring–Mass–Damper System with Complex Characteristic Equation Roots**

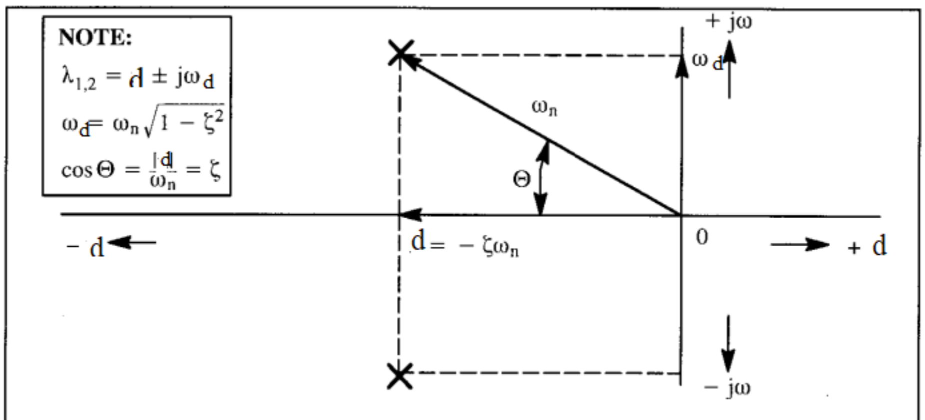



**Figure 5.7 Relationship Between Frequency, Undamped Natural Frequency and Damping Ratio**

$$\omega = \omega_n \sqrt{1 - \zeta^2} \tag{5.26}$$

From this result it can be observed that when the damping ratio becomes zero {this occurs when c=0 in Eqn (5.3)} the frequency of oscillation equals the undamped natural frequency:

$$\omega = \omega_n = \sqrt{\frac{k}{m}} \qquad \text{(for } c = 0\text{)} \tag{5.27}$$

The spring–mass–damper system output, x(t), as expressed by Eqn (5.22), can be cast in a format which reflects the system damping ratio and undamped natural frequency. The reader is asked to show that the result of doing this is as follows:

$$x(t) = \frac{1}{{\omega_n}^2}\left\{1 - e^{-\zeta\omega_n t}\left[\cos(\omega_n \sqrt{1 - \zeta^2}\,t) + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin(\omega_n \sqrt{1 - \zeta^2}\,t)\right]\right\} =$$

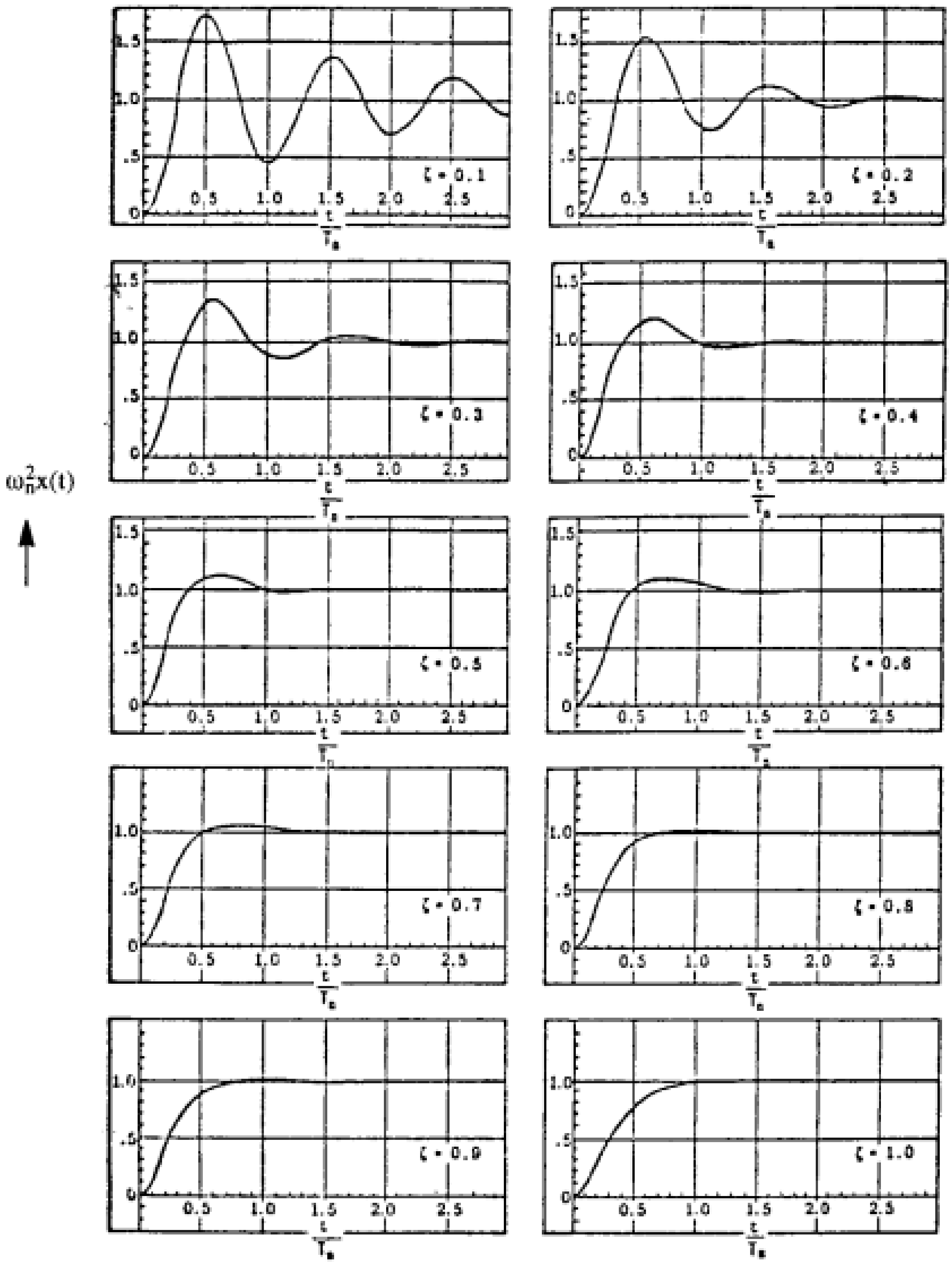
$$= \frac{1}{{\omega_n}^2}\left\{1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\sin\left((\omega_n \sqrt{1 - \zeta^2}\,t) + \psi\right)\right\} \text{ with :} \tag{5.28}$$

$$\psi = \arcsin\sqrt{1 - \zeta^2}$$

From the presentations so far, it is evident that the stability character of the response of a system such as shown in Figure (5.3) is determined entirely by:

a) the type of roots of the characteristic equation of that system

b) the sign of the real part of the roots of the characteristic equation




$$\omega_n^2 x(s) = \frac{1}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1\right)} \qquad T_n = \frac{2\pi}{\omega_n}$$

<u>**Figure 5.8 Normalize Response of a Second Order System to a Unit Step Input**</u>

## 2. LONGITUDINAL EQUATIONS AND TRANSFER FUNCTIONS

The small perturbation, longitudinal equations of motion of the airplane are represented by Eqns (5.1). Two observations are in order:

First, by using the substitution $q = \dot{\theta}$ and $w = U_1\alpha$ the equations can be cast in terms of the following variables: speed, u, angle − of − attack $\alpha$, and pitch − attitude angle, $\theta$

Second, the pitching moment of inertia, $I_{yy}$, is normally computed in a somewhat arbitrarily selected body–fixed axis system. Because the equations of motion are written in the stability axis system, $I_{yy}$, would have to be computed also in that system. However, because the stability axis system was obtained from any body–fixed axis system by rotation about the Y–axis, $I_{yy}$ remains the same. It will be seen in Section 5.3 that this will not be the case for the lateral–directional inertias.

Equations (5.1a–c) will now be rewritten in two steps:

**Step 1:** To obtain better insight into the physical characteristics of Equations (5.1) it is customary to divide both sides of the lift and drag force equations by the mass, m, and to divide both sides of the pitching moment equation by the pitching moment of inertia, $I_{yy}$. As a result all terms in the corresponding equations have the physical unit of linear or angular acceleration.

**Step 2:** To obtain better insight into the relative importance of the aerodynamic forces and moments, the so–called dimensional stability derivatives of Table 5.1 are introduced. How these derivatives come about is illustrated with one example. Consider the $C_{m_\alpha}$ term in Eqn (5.1c). This term will be re–written as follows:

$$\frac{\bar{q}_1 S\bar{c}C_{m_\alpha}\alpha}{I_{yy}} = M_\alpha\alpha \ , \ \text{where}: \ M_\alpha = \frac{\bar{q}_1 S\bar{c}C_{m_\alpha}}{I_{yy}} \qquad (5.29)$$

The newly defined dimensional stability derivative, $M_\alpha$, has the following very important physical meaning: it represents the pitch angular acceleration imparted to the airplane as a result of a unit change in angle of attack. This physical meaning can be generalized to apply to all dimensional stability derivatives of Table 5.1 by using the following definition:

**Definition**: Each dimensional derivative represents either the linear or angular acceleration imparted to the airplane as a result of a unit change in its associated motion or control variable.

The numerical magnitudes of these dimensional derivatives therefore give numerical clues about their relative importance. Their use in the equations of motion (5.1a–c) also results in a much 'cleaner' look for these equations. The resulting equations are presented as Eqns (5.30) in Table 5.2.

**Table 5.1 Definition of Longitudinal, Dimensional Stability Derivatives**

| | | | |
|---|---|---|---|
| $X_u = \dfrac{-\bar{q}_1 S(C_{D_u} + 2C_{D_1})}{mU_1}$ | $\dfrac{ft/sec^2}{ft/sec}$ | $M_u = \dfrac{\bar{q}_1 S\bar{c}(C_{m_u} + 2C_{m_1})}{I_{yy}U_1}$ | $\dfrac{rad/sec^2}{ft/sec}$ |
| $X_{T_u} = \dfrac{\bar{q}_1 S(C_{T_{x_u}} + 2C_{T_{x_1}})}{mU_1}$ | $\dfrac{ft/sec^2}{ft/sec}$ | $M_{T_u} = \dfrac{\bar{q}_1 S\bar{c}(C_{m_{T_u}} + 2C_{m_{T_1}})}{I_{yy}U_1}$ | $\dfrac{rad/sec^2}{ft/sec}$ |
| $X_\alpha = \dfrac{-\bar{q}_1 S(C_{D_\alpha} - C_{L_1})}{m}$ | $\dfrac{ft/sec^2}{rad}$ | $M_\alpha = \dfrac{\bar{q}_1 S\bar{c}C_{m_\alpha}}{I_{yy}}$ | $\dfrac{rad/sec^2}{rad}$ |
| $X_{\delta_e} = \dfrac{-\bar{q}_1 SC_{D_{\delta_e}}}{m}$ | $\dfrac{ft/sec^2}{rad}$ | $M_{T_\alpha} = \dfrac{\bar{q}_1 S\bar{c}C_{m_{T_\alpha}}}{I_{yy}}$ | $\dfrac{rad/sec^2}{rad}$ |
| $Z_u = \dfrac{-\bar{q}_1 S(C_{L_u} + 2C_{L_1})}{mU_1}$ | $\dfrac{ft/sec^2}{ft/sec}$ | $M_{\dot{\alpha}} = \dfrac{\bar{q}_1 S\bar{c}^2 C_{m_{\dot{\alpha}}}}{2I_{yy}U_1}$ | $\dfrac{rad/sec^2}{rad/sec}$ |
| $Z_\alpha = \dfrac{-\bar{q}_1 S(C_{L_\alpha} + C_{D_1})}{m}$ | $\dfrac{ft/sec^2}{rad}$ | $M_q = \dfrac{\bar{q}_1 S\bar{c}^2 C_{m_q}}{2I_{yy}U_1}$ | $\dfrac{rad/sec^2}{rad/sec}$ |
| $Z_{\dot{\alpha}} = \dfrac{-\bar{q}_1 S\bar{c}C_{L_{\dot{\alpha}}}}{2mU_1}$ | $\dfrac{ft/sec^2}{rad/sec}$ | $M_{\delta_e} = \dfrac{\bar{q}_1 S\bar{c}C_{m_{\delta_e}}}{I_{yy}}$ | $\dfrac{rad/sec^2}{rad}$ |
| $Z_q = \dfrac{-\bar{q}_1 S\bar{c}C_{L_q}}{2mU_1}$ | $\dfrac{ft/sec^2}{rad/sec}$ | | |
| $Z_{\delta_e} = \dfrac{-\bar{q}_1 SC_{L_{\delta_e}}}{m}$ | $\dfrac{ft/sec^2}{rad}$ | | |

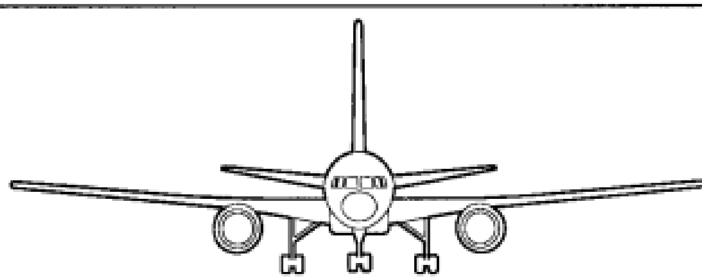


**Table 5.2 Development of the Perturbed Longitudinal Equations of Motion with Dimensional Stability Derivatives in Matrix Format**

$$\dot{u} = -g\theta\cos\theta_1 + X_u u + X_{T_u} u + X_\alpha \alpha + X_{\delta_e}\delta_e \tag{5.30a}$$

$$U_1\dot{\alpha} - U_1\dot{\theta} = -g\theta\sin\theta_1 + Z_u u + Z_\alpha \alpha + Z_{\dot{\alpha}}\dot{\alpha} + Z_q\dot{\theta} + Z_{\delta_e}\delta_e \tag{5.30b}$$

$$\ddot{\theta} = M_u u + M_{T_u} u + M_\alpha \alpha + M_{T_\alpha}\alpha + M_{\dot{\alpha}}\dot{\alpha} + M_q\dot{\theta} + M_{\delta_e}\delta_e \tag{5.30c}$$

Laplace transforming Eqns (5.30) for zero initial conditions:

$$(s - X_u - X_{T_u})u(s) \quad - X_\alpha \alpha(s) \quad g\cos\theta_1\theta(s) \quad = X_{\delta_e}\delta_e(s) \tag{5.31a}$$

$$-Z_u u(s) \quad \{s(U_1 - Z_{\dot{\alpha}}) - Z_\alpha\}\alpha(s) \quad \{-(Z_q + U_1)s + g\sin\theta_1\}\theta(s) \quad = Z_{\delta_e}\delta_e(s) \tag{5.31b}$$

$$-(M_u + M_{T_u})u(s) \quad -\{M_{\dot{\alpha}}s + M_\alpha + M_{T_\alpha}\}\alpha(s) \quad (s^2 - M_q s)\theta(s) \quad = M_{\delta_e}\delta_e(s) \tag{5.31c}$$

Writing Eqns (5.31) in matrix and transfer function format:

Transfer Function Matrix

$$\begin{bmatrix} (s - X_u - X_{T_u}) & -X_\alpha & g\cos\theta_1 \\ -Z_u & \{s(U_1 - Z_{\dot{\alpha}}) - Z_\alpha\} & \{-(Z_q + U_1)s + g\sin\theta_1\} \\ -(M_u + M_{T_u}) & -(M_{\dot{\alpha}}s + M_\alpha + M_{T_\alpha}) & (s^2 - M_q s) \end{bmatrix} \begin{Bmatrix} \dfrac{u(s)}{\delta_e(s)} \\ \dfrac{\alpha(s)}{\delta_e(s)} \\ \dfrac{\theta(s)}{\delta_e(s)} \end{Bmatrix} = \begin{Bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \end{Bmatrix} \tag{5.32}$$

System Matrix

Control Power Matrix

Equations (5.30) are Laplace transformed for zero initial conditions. The new variables are: u(s), $\alpha(s)$ and $\theta(s)$ respectively, while $\delta_e(s)$ is the Laplace transformed elevator input. The result is Eqns (5.31). Next, equations (5.31a–c) are divided by $\delta_e(s)$ . This gives rise to the so–called open–loop airplane transfer functions: $u(s)/\delta_e(s)$, $\alpha(s)/\delta_e(s)$ and $\theta(s)/\delta_e(s)$ . The open–loop transfer functions can now be thought of as the new 'variables'. By casting the equations in a matrix format the result is Eqns (5.32) which are also shown in Table 5.2. The airplane open–loop transfer functions can be determined with matrix algebra. Each transfer function is then expressed as a ratio of two determinants. The resulting determinant ratios are shown in Table 5.3 as Equations (5.33), (5.36) and (5.38) respectively.

Note, that the speed–to–elevator transfer function, $u(s)/\delta_e(s)$ , of Eqn (5.33) can be written as the following ratio of polynomials in the independent Laplace variable, s:

$$\frac{u(s)}{\delta_e(s)} = \frac{N_u}{\overline{D}_1} = \frac{A_u s^3 + B_u s^2 + C_u s + D_u}{A_1 s^4 + B_1 s^3 + C_1 s^2 + D_1 s + E_1} \qquad (5.40)$$

Similarly, the angle–of–attack–to–elevator transfer function, $\alpha(s)/\delta_e(s)$ , of Eqn (5.36) can be expressed as:

$$\frac{\alpha(s)}{\delta_e(s)} = \frac{N_\alpha}{\overline{D}_1} = \frac{A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha}{A_1 s^4 + B_1 s^3 + C_1 s^2 + D_1 s + E_1} \qquad (5.41)$$

Finally, the pitch–attitude–to–elevator transfer function, $\theta(s)/\delta_e(s)$ , of Eqn (5.38) can be written as:

$$\frac{\theta(s)}{\delta_e(s)} = \frac{N_\theta}{\overline{D}_1} = \frac{A_\theta s^2 + B_\theta s + C_\theta}{A_1 s^4 + B_1 s^3 + C_1 s^2 + D_1 s + E_1} \qquad (5.42)$$

It is seen that all transfer functions have the same denominator. When this denominator is set equal to zero the resulting equation is called the characteristics equation:

$$A_1 s^4 + B_1 s^3 + C_1 s^2 + D_1 s + E_1 = 0 \qquad (5.43)$$

The roots of this characteristic equation determine the dynamic stability character of the airplane. These roots and how they are affected by flight condition, by airplane mass, by airplane mass distribution (c.g. location and inertias), by airplane geometry and by the airplane aerodynamic characteristics will be discussed in Sub–section 5.2.2 – 5.2.6.

It is also seen from Eqns (5.40)–(5.42) that the numerators are all different. The numerator polynomials affect the **magnitude of the response** of an airplane to a control surface input. However, ONLY the denominators affect the **dynamic stability character of the response** (i.e. the frequency or time–constant behavior).

**Table 5.3 Longitudinal Airplane Transfer Functions**

$$\frac{u(s)}{\delta_e(s)} = \frac{\begin{vmatrix} X_{\delta_e} & -X_\alpha & g\cos\theta_1 \\ Z_{\delta_e} & \left\{s(U_1 - Z_{\dot\alpha}) - Z_\alpha\right\} & \left\{-(Z_q + U_1)s + g\sin\theta_1\right\} \\ M_{\delta_e} & -\left\{M_{\dot\alpha}s + M_\alpha + M_{T_\alpha}\right\} & (s^2 - M_q s) \end{vmatrix}}{\begin{vmatrix} (s - X_u - X_{T_u}) & -X_\alpha & g\cos\theta_1 \\ -Z_u & \left\{s(U_1 - Z_{\dot\alpha}) - Z_\alpha\right\} & \left\{-(Z_q + U_1)s + g\sin\theta_1\right\} \\ -(M_u + M_{T_u}) & -\left\{M_{\dot\alpha}s + M_\alpha + M_{T_\alpha}\right\} & (s^2 - M_q s) \end{vmatrix}} = \frac{N_u}{\overline{D}_1} \qquad (5.33)$$

$$\overline{D}_1 = A_1 s^4 + B_1 s^3 + C_1 s^2 + D_1 s + E_1 \quad , \text{ where :} \qquad (5.34)$$

$$A_1 = U_1 - Z_{\dot\alpha}$$

$$B_1 = -(U_1 - Z_{\dot\alpha})(X_u + X_{T_u} + M_q) - Z_\alpha - M_{\dot\alpha}(U_1 + Z_q)$$

$$C_1 = (X_u + X_{T_u})\left\{M_q(U_1 - Z_{\dot\alpha}) + Z_\alpha + M_{\dot\alpha}(U_1 + Z_q)\right\} + M_q Z_\alpha - Z_u X_\alpha + M_{\dot\alpha} g\sin\theta_1 +$$
$$- (M_\alpha + M_{T_\alpha})(U_1 + Z_q)$$

$$D_1 = g\sin\theta_1\left\{M_\alpha + M_{T_\alpha} - M_{\dot\alpha}(X_u + X_{T_u})\right\} + g\cos\theta_1\left\{Z_u M_{\dot\alpha} + (M_u + M_{T_u})(U_1 - Z_{\dot\alpha})\right\} +$$
$$+ (M_u + M_{T_u})\left\{-X_\alpha(U_1 + Z_q)\right\} + Z_u X_\alpha M_q +$$
$$+ (X_u + X_{T_u})\left\{(M_\alpha + M_{T_\alpha})(U_1 + Z_q) - M_q Z_\alpha\right\}$$

$$E_1 = g\cos\theta_1\left\{(M_\alpha + M_{T_\alpha})Z_u - Z_\alpha(M_u + M_{T_u})\right\} +$$
$$+ g\sin\theta_1\left\{(M_u + M_{T_u})X_\alpha - (X_u + X_{T_u})(M_\alpha + M_{T_\alpha})\right\}$$

$$N_u = A_u s^3 + B_u s^2 + C_u s + D_u \quad , \text{ where :} \qquad (5.35)$$

$$A_u = X_{\delta_e}(U_1 - Z_{\dot\alpha})$$

$$B_u = -X_{\delta_e}\left\{(U_1 - Z_{\dot\alpha})M_q + Z_\alpha + M_{\dot\alpha}(U_1 + Z_q) + Z_{\delta_e}X_\alpha\right\}$$

$$C_u = X_{\delta_e}\left\{M_q Z_\alpha + M_{\dot\alpha} g\sin\theta_1 - (M_\alpha + M_{T_\alpha})(U_1 + Z_q)\right\} +$$
$$+ Z_{\delta_e}\left\{-M_{\dot\alpha} g\cos\theta_q - X_\alpha M_q\right\} + M_{\delta_e}\left\{X_\alpha(U_1 + Z_q) - (U_1 - Z_{\dot\alpha})g\cos\theta_1\right\}$$

$$D_u = X_{\delta_e}(M_\alpha + M_{T_\alpha})g\sin\theta_1 - Z_{\delta_e}M_\alpha g\cos\theta_1 + M_{\delta_e}(Z_\alpha g\cos\theta_1 - X_\alpha g\sin\theta_1)$$

**Table 5.3 (Continued) Longitudinal Airplane Transfer Functions**

$$\frac{\alpha(s)}{\delta_e(s)} = \frac{\begin{vmatrix} (s - X_u - X_{T_u}) & X_{\delta_e} & g\cos\theta_1 \\ -Z_u & Z_{\delta_e} & \left\{-(Z_q + U_1)s + g\sin\theta_1\right\} \\ -(M_u + M_{T_u}) & M_{\delta_e} & (s^2 - M_q s) \end{vmatrix}}{\bar{D}_1} = \frac{N_\alpha}{\bar{D}_1} \qquad (5.36)$$

$$N_\alpha = A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha \quad , \text{ where :} \qquad (5.37)$$

$$A_\alpha = Z_{\delta_e}$$

$$B_\alpha = X_{\delta_e} Z_u + Z_{\delta_e}\left\{-M_q - (X_u + X_{T_u})\right\} + M_{\delta_e}(U_1 + Z_q)$$

$$C_\alpha = X_{\delta_e}\left\{(U_1 + Z_q)(M_u + M_{T_u}) - M_q Z_u\right\} + Z_{\delta_e} M_q (X_u + X_{T_u}) + \\ + M_{\delta_e}\left\{-g\sin\theta_1 - (U_1 + Z_q)(X_u + X_{T_u})\right\}$$

$$D_\alpha = -X_{\delta_e}(M_u + M_{T_u})g\sin\theta_1 + Z_{\delta_e}(M_u + M_{T_u})g\cos\theta_1 + \\ + M_{\delta_e}\left\{(X_u + X_{T_u})g\sin\theta_1 - Z_u g\cos\theta_1\right\}$$

$$\frac{\theta(s)}{\delta_e(s)} = \frac{\begin{vmatrix} (s - X_u - X_{T_u}) & -X_\alpha & X_{\delta_e} \\ -Z_u & \left\{s(U_1 - Z_{\dot{\alpha}}) - Z_\alpha\right\} & Z_{\delta_e} \\ -(M_u + M_{T_u}) & -\left\{M_{\dot{\alpha}} s + M_\alpha + M_{T_\alpha}\right\} & M_{\delta_e} \end{vmatrix}}{\bar{D}_1} = \frac{N_\theta}{\bar{D}_1} \qquad (5.38)$$

$$N_\theta = A_\theta s^2 + B_\theta s + C_\theta \quad , \text{ where :} \qquad (5.39)$$

$$A_\theta = Z_{\delta_e} M_{\dot{\alpha}} + M_{\delta_e}(U_1 - Z_{\dot{\alpha}})$$

$$B_\theta = X_{\delta_e}\left\{Z_u M_{\dot{\alpha}} + (U_1 - Z_{\dot{\alpha}})(M_u + M_{T_u})\right\} + Z_{\delta_e}\left\{(M_\alpha + M_{T_\alpha}) - M_{\dot{\alpha}}(X_u + X_{T_u})\right\} + \\ + M_{\delta_e}\left\{-Z_\alpha - (U_1 - Z_{\dot{\alpha}})(X_u + X_{T_u})\right\}$$

$$C_\theta = X_{\delta_e}\left\{(M_\alpha + M_{T_\alpha})Z_u - Z_\alpha(M_u + M_{T_u})\right\} + \\ + Z_{\delta_e}\left\{-(M_\alpha + M_{T_\alpha})(X_u + X_{T_u}) + X_\alpha(M_u + M_{T_u})\right\} + M_{\delta_e}\left\{Z_\alpha(X_u + X_{T_u}) - X_\alpha Z_u\right\}$$

**2.**
**3.**