Final BOQ ASC 2024

SUMMERY

Definitions

Airplane performance deals with the determination of performance characteristics such as range, endurance, rate of climb, and takeoff and landing distance as well as flight path optimization

Flight dynamics is concerned with the motion of an airplane due to internally or externally generated disturbances. We particularly are interested in the vehicle's stability and control capabilities.

Aerodynamic Center: The point at which the pitching moment does not vary with angle of attack. ***Typically located at the 1/4-chord line.***

Aeroelasticity deals with both static and dynamic aeroelastic phenomena.

Static stability: is the initial tendency of the vehicle to return to its equilibrium state after a disturbance

dynamic stability: concerned with the time history of the motion of the vehicle after it is disturbed from its equilibrium point.

Pressure is the normal force per unit area acting on the fluid.: $P = F/A$

The relationship between pressure P, density ρ, and temperature T is given by the equation of state $P = \rho RT$

where R is a constant, the magnitude depending on the gas being considered. For air, R has a value 287 J/(kg .K) or 1718 ft2/(s^2 R)

Dynamic Pressure, Q, refers the pressure of the air moving over the aircraft and is given by $\mathbf{Q = (1/2)\,\rho V^2}$

Mach Number The ratio of an airplane's speed V to the local speed of sound(**a)**

M= V/a

atmospheric pressure therefore can be found to be;

$$\mathbf{P_{atm} = \rho g h}$$

Wing area: The reference area is usually the gross plan area of the wing, including that part within the fuselage, and is denoted S:

$$S = b\bar{c}$$

where b is the wing span and $\bar{c}$ is the standard mean chord of the wing.

Standard mean chord (smc)

For a straight tapered wing equation simplifies to $\bar{c} = \frac{S}{b}$

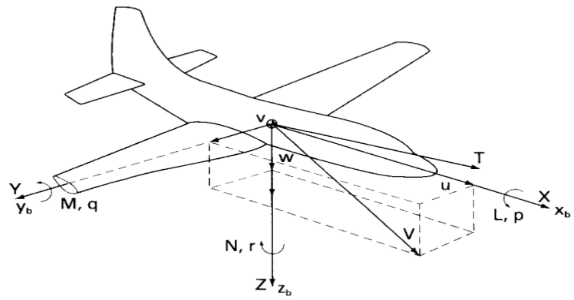
Aspect ratio: The aspect ratio of the aircraft wing is a measure of its spanwise slenderness and is denoted A and is defined as follows:

$$A = \frac{b^2}{S} = \frac{b}{\bar{c}}$$

Centre of gravity location: The centre of gravity, cg, of an aircraft is usually located on the reference chord as indicated in Fig. abov. Its position is quoted as a fraction of (c), denoted (h), and is measured from the leading edge of the reference chord as :

The cg position varies as a function of aircraft loading, the typical variation being in the range 10–40% of c. Or, equivalently, $0.1 \leq h \leq 0.4$.

AERODYNAMIC NOMENCLATURE



	Roll Axis x_b	Pitch Axis y_b	Yaw Axis z_b
Angular rates	p	q	r
Velocity components	u	v	w
Aerodynamic force components	X	Y	Z
Aerodynamic moment components	L	M	N
Moment of inertia about each axis	I_x	I_y	I_z
Products of inertia	I_{yz}	I_{xz}	I_{xy}

The aerodynamic forces are defined in terms of dimensionless coefficients, the flight dynamic pressure Q, and a reference area S as follows:

$X = C_x QS$	Axial Force	Net Force in the positive x-direction
$Y = C_y QS$	Side Force	Net Force in the positive y-direction
$Z = C_z QS$	Normal Force	Net Force in the positive z-direction

<u>Thus the forces on the aircraft are defined by the quantities Cx, Cy, and Cz.</u>

Moments are similarly defined

$L = C_l QSl_w$	Rolling Moment	Net Moment in the positive p-direction
$M = C_m QSl_w$	Pitching Moment	Net Moment in the positive q-direction
$N = C_n QSl_c$	Yawing Moment	Net Moment in the positive r-direction

where • S again is surface area of the plane (or another reference area).

• lω is the wingspan

• lc is the mean chord

The aerodynamic coefficients C_x, C_y, C_z, C_l, C_m, and C_n, primarily are a function of the Mach number, Reynolds number, angle of attack, and sideslip angle.

- Although not indicated in the figure, the variables φ, θ, ψ represent the angular rotations, relative to the equilibrium state, about the x, y, and z axes, respectively. Thus, p = φ' , q = θ', and r = ψ' , where the dots represent time derivatives.

The equations for **α** and **β** follow:

$$\alpha = \tan^{-1}\frac{w}{u} \qquad (17)$$

$$\beta = \sin^{-1}\frac{v}{V} \qquad (18)$$

$$V = (u^2 + v^2 + w^2)^{1/2} \qquad (19)$$

If the angle of attack and sideslip are small, that is, < 15°, then;

$$\boldsymbol{\alpha = w/u}$$

$\boldsymbol{\beta = v/V}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in radians.

$u = \vec{\mathbf{v}} \cdot \vec{x}$	velocity in the $\vec{x}$ direction
$v = \vec{\mathbf{v}} \cdot \vec{y}$	velocity in the $\vec{y}$ direction
$w = \vec{\mathbf{v}} \cdot \vec{z}$	velocity in the $\vec{z}$ direction

Forces and Moments: The motion of air creates forces and moments.

- Lift and Drag are measured at the aerodynamic center.
- Moment is measured as the moment about the aerodynamic center.

- Usually take standard form:

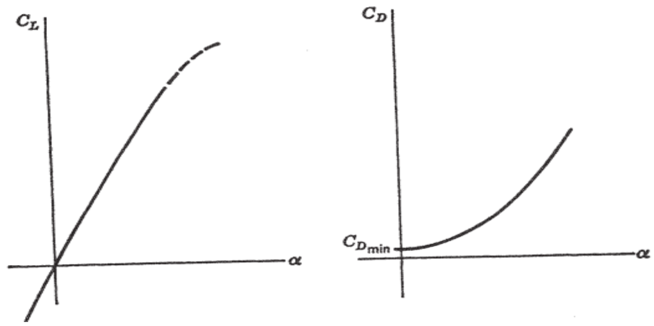
$$\mathbf{L = C_L QS, \qquad D = C_D QS, \quad and \qquad M = C_M\, QSl}$$

<u>Lift Coefficient C_L:</u>

Lift is given by $\quad \mathbf{L = C_L QS:} \qquad \mathbf{CL = C_{L0} + C_{L\alpha}\alpha}$

where • C_{L0} is the lift produced at steady-level flight. We define $C_{L0} = 0$ for an airfoil. However, for the aircraft overall, we want $C_{L0} > 0$ *(Don't want to fly nose-up all the time).*

• $C_{L\alpha} > 0$ is determined by the airfoil type and other factors *(Sweep, planform shape, winglets, Mach number, etc.).*



<u>Drag Coefficient C_D:</u> **$D = C_D QS$**

The drag coefficient, **C_D**, of an airfoil is related to the lift coefficient, C_L. It can be approximated as **$C_D = C_{D0} + KC_L^2$**

<u>Moment Coefficient</u>: Positive pitching moment is given by

$$\mathbf{M = C_M\, QSl}$$

General form: **$C_M = C_{M0} + C_{M\alpha}\,\alpha$**

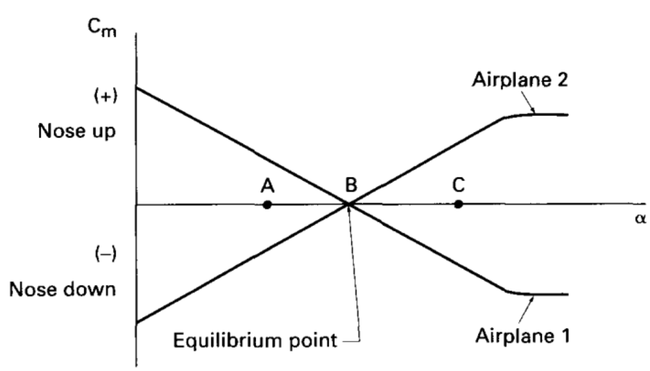
where • **C_{M0}** is the moment produced at steady-level flight.

• For the aircraft overall, we typically want **$C_{M0} > 0$**

<u>Longitudinal Static Stability</u>

t to have static longitudinal tability the aircraft pitching moment curve must have a negative slope. That is,

$$\frac{dc_{M,cg}}{d\alpha} = c_{M\alpha} < 0.$$



we also must have a positive intercept, that is, Cmo > 0 to trim at positive angles of attack

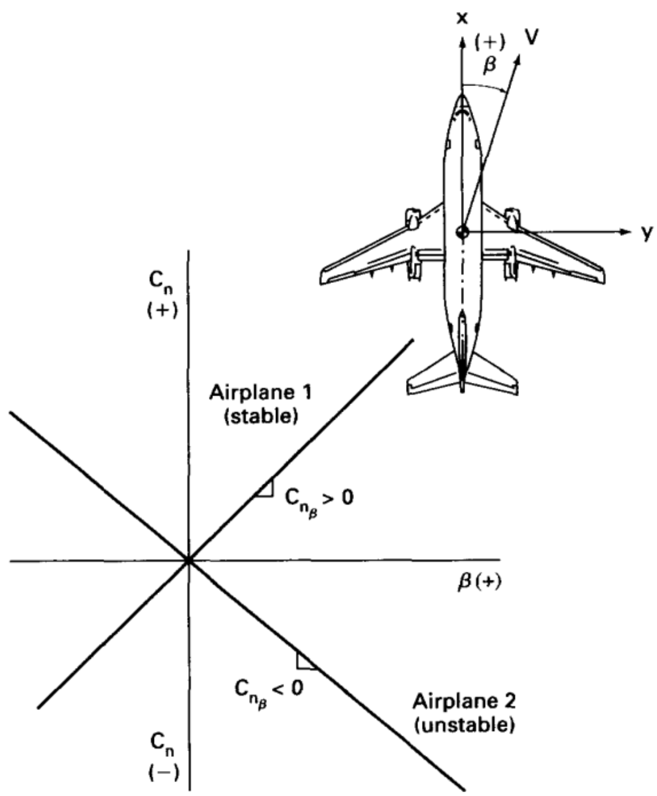
Longitudinal Balancing: **$\sum Fz = 0$, $\sum My = 0$.**

DIRECTIONAL STABILITY Directional, or weathercock, stability is concerned with the static stability of the airplane about the z axis.

It is desirable that the airplane should tend to return to an equilibrium condition when subjected to some form of yawing disturbance.

To have static directional stability the slope of the yawing moment curve must be positive, i.e:

$$C_{n\beta} > 0.$$



The side force acting on the vertical tail can be expressed as

$$Y_v = -C_{L_{\alpha_v}} \alpha_v Q_v S_v$$

where the subscript (v) refers to properties of the vertical tail. The angle of attack (α_v) that the vertical tail plane will experience can be written as

$$\alpha_v = \beta + \sigma$$ where σ is the sidewash angle

The moment produced by the vertical tail can be written as a function of the side force acting on it:

$$N_v = l_v Y_v = l_v C_{L_{\alpha_v}}(\beta + \sigma) Q_v S_v$$

or in coefficient form

$$\begin{aligned} C_n &= \frac{N_v}{Q_w Sb} = \frac{l_v S_v}{Sb}\frac{Q_v}{Q_w} C_{L_{\alpha_v}}(\beta + \sigma) \\ &= V_v \eta_v C_{L_{\alpha_v}}(\beta + \sigma) \end{aligned}$$

where $V_v = 1_v S_v/(Sb)$ is the vertical tail volume ratio and $\eta_v = Q_v/Q_w$, is the ratio of the dynamic pressure at the vertical tail to the dynamic pressure at the wing.

The contribution of the vertical tail to directional stability

$$C_{n_{\beta_v}} = V_v \eta_v C_{L_{\alpha_v}}\left(1 + \frac{d\sigma}{d\beta}\right)$$

Linear Transformation Relationship

Roll Angle ϕ : $\vec{v}' = R_1(\phi)\vec{v}$

Pitch Angle θ : $\vec{v}' = R_2(\theta)\vec{v}$

Yaw Angle ψ : $\vec{v}' = R_3(\psi)\vec{v}$

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Roll-Pitch-Yaw: $\vec{v}_{RPY} = R_1(\phi)R_2(\theta)R_3(\psi)\vec{v}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{V_B = D\ V_E}$$

$$\mathbf{D} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi \\ -\cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi \\ +\cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi \\ +\sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

By inverting the direction cosine matrix D the transformation from (V_B) to (V_E) is obtained as given by:

$$\mathbf{V_E} = \; = \mathbf{D^{-1}} \, \mathbf{V_B} = \mathbf{D^T V_B} = \boldsymbol{R_3^{-1}(\psi)} \; \boldsymbol{R_2^{-1}(\theta)} \; \boldsymbol{R_1^{-1}(\phi)} \; \mathbf{V_B}$$

$$\mathbf{D}^{-1} = \begin{bmatrix} \cos\psi\cos\theta & \begin{matrix}\cos\psi\sin\theta\sin\phi \\ -\sin\psi\cos\phi\end{matrix} & \begin{matrix}\cos\psi\sin\theta\cos\phi \\ +\sin\psi\sin\phi\end{matrix} \\ \sin\psi\cos\theta & \begin{matrix}\sin\psi\sin\theta\sin\phi \\ +\cos\psi\cos\phi\end{matrix} & \begin{matrix}\sin\psi\sin\theta\cos\phi \\ -\cos\psi\sin\phi\end{matrix} \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Rotational Transformation Matrix

The relationship between the angular velocities in the body frame (p, q, and r) and the Euler rates (**ψ’ , θ’, and ϕ’**) also can be determined from

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\Phi & C_\theta S_\Phi \\ 0 & -S_\Phi & C_\theta C_\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Above Equation can be solved for the Euler rates in terms of the body angular velocities:

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S_\Phi \tan\theta & C_\Phi \tan\theta \\ 0 & C_\Phi & -S_\Phi \\ 0 & S_\Phi \sec\theta & C_\Phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

EQUATIONS OF MOTIONS

The derivation of the equations, however, follows a very simple pattern starting from Newton's second law for translational and rotational motions. Newton's second law for translational motions is

$$\bar{F} = \frac{d}{dt}(m\,\bar{V})$$

where F is the sum of the externally applied forces and mV is linear momentum. Newton's second law for rotational motions is

$$\bar{G} = \frac{d}{dt}(\bar{H})$$

where G is the sum of the externally applied moments and H is angular momentum.

In order to derive the equations of motion, each side of Newton's equations are expanded to yield the following six nonlinear differential equations:

$$\text{Longitudinal}\begin{cases} F_x = m(\dot{U} + QW - RV) \\ F_z = m(\dot{W} + PV - QU) \\ G_y = \dot{Q}\,I_y - PR\,(I_z - I_x) + (P^2 - R^2)\,I_{xz} \end{cases}$$

$$\text{Lateral-Directional}\begin{cases} F_y = m(\dot{V} + RU - PW) \\ G_x = \dot{P}\,I_x + QR\,(I_z - I_y) - (\dot{R} + PQ)\,I_{xz} \\ G_z = \dot{R}\,I_z + PQ\,(I_y - I_x) + (QR - \dot{P})\,I_{xz} \end{cases}$$

Derivation of the right hand side (rhs) of the equations of motion

ASSUMPTION;

1. **The aircraft is a rigid body.**
2. **The earth and atmosphere are fixed in inertial space.**
3. **ASSUMPTION; Mass (m) is constant (dm/dt =0).**
4. **In addition, most motion of interest in stability and control takes place in a relatively short time.**

   A- Translational force Eq.

$$\bar{F} = \left.\frac{d(m\bar{V}_T)}{dt}\right|_{XYZ}$$

$$\left.\frac{d\bar{V}_T}{dt}\right|_{XYZ} = \left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T$$

$$\bar{F} = m \left[\left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T \right]$$

They are defined as follows: $\mathbf{V_T = Ui + Vj + Wk}$

and $\boldsymbol{\omega} \mathbf{= Pi + Qj + Rk}$

where P - roll rate , Q pitch rate, R - yaw rate

$$\bar{F} = m \left[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ U & V & W \end{vmatrix} \right]$$

$$\bar{F} = m \left[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + (QW - RV)\bar{i} - (PW - RU)\bar{j} + (PV - QU)\bar{k} \right]$$

$$\bar{F} = m \left[(\dot{U} + QW - RV)\bar{i} + (\dot{V} + RU - PW)\bar{j} + (\dot{W} + PV - QU)\bar{k} \right]$$

$$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$

$$F_x = m\,(\dot{U} + QW - RV)$$

$$F_y = m\,(\dot{V} + RU - PW)$$

$$F_z = m\,(\dot{W} + PV - QU)$$

B- Rotational Equations

A gain from Newton's second law

$$\bar{G} = \left.\frac{d(\bar{H})}{dt}\right|_{XYZ} \qquad (9)$$

G is the moment

This equation states the change in angular momentum, H, is equal to the applied

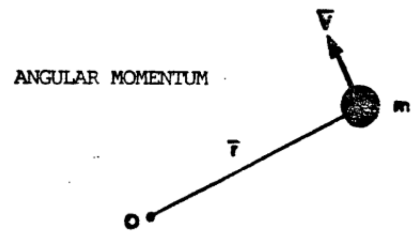
Linear Momentum = mV

Angular momentum is defined as H,

H = r X Linear Momentum and,

the angle between r and V is 90 degrees,

the magnitude of the angular momentum is **mrV**.



$$\overline{\text{Moment}} = \bar{r} \times \overline{\text{Force}}$$

$$\overline{\text{Angular Momentum}} = \bar{r} \times \overline{\text{Linear Momentum}}$$

$$\bar{H} = \int_V \rho_A \, [\bar{r} \times (\bar{\omega} \times \bar{r})] \, dV$$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then
$$\bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ x & y & z \end{vmatrix}$$

$$\bar{H} = \int_V \rho_A \, [\bar{r} \times (\bar{\omega} \times \bar{r})] \, dV$$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then
$$\bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ x & v & z \end{vmatrix} = (Qz - Ry)\bar{i} + (Rx - Pz)\bar{j} + (Py - Qx)\bar{k}$$

The moments of inertia are defined as

$$I_x = \int_V \rho_A (y^2 + z^2) dV$$
$$I_y = \int_V \rho_A (x^2 + z^2) dV$$
$$I_z = \int_V \rho_A (x^2 + y^2) dV$$

The products of inertia are defined as

$$I_{xy} = I_{yx} = \int_V \rho_A xy \, dV$$
$$I_{yz} = I_{zy} = \int_V \rho_A yz \, dV$$
$$I_{xz} = I_{zx} = \int_V \rho_A xz \, dV$$

$$\bar{H} = H_x\bar{i} + H_y\bar{j} + H_z\bar{k}$$

So that

$$H_x = PI_x - QI_{xy} - RI_{xz}$$
$$H_y = QI_y - RI_{yz} - PI_{xy}$$
$$H_z = RI_z - PI_{xz} - QI_{yz}$$

The xz-plane is a plane of symmetry. This causes two products of inertia, I_{xy} and I_{yz} to be zero.

$$H = (PI_x - RI_{xz})\, i + QI_y\, j + (RI_z - PI_{xz})\, k$$

The moment equation

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{xyz} + \bar{\omega} \times \bar{H}$$

which is $$\bar{G} = \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ H_x & H_y & H_z \end{vmatrix}$$

$$G_x = \dot{P}I_x + QR\,(I_z - I_y) - (\dot{R} + PQ)\, I_{xz}$$

$$G_y = \dot{Q}I_y - PR\,(I_z - I_x) + (P^2 - R^2)\, I_{xz}$$

$$G_z = \dot{R}I_z + PQ\,(I_y - I_x) + (QR - \dot{P})\, I_{xz}$$

LHS of the equations of motion:

LHS		RHS
Applied Forces and Moments	=	Observed Vehicle Motion
F_x	=	$m(\dot{U} + QW - PV)$
G_x	=	$\dot{P}I_x + QR(I_z - I_y) - (\dot{R} + PQ)I_{yz}$
etc.		

		SOURCE					
		Aero-dynamic	Direct Thrust	Gravity	Gyro-Scopic	Other	
LONGITUDINAL	F_x	X_A	X_T	X_g	0	X_{oth}	$= m\dot{U} + \cdots$ (19a)
	F_z	Z_A	Z_T	Z_g	0	Z_{oth}	$= m\dot{W} + \cdots$ (19b)
	G_y	M_A	M_T	0	M_{gyro}	M_{oth}	$= \dot{Q}I_y + \cdots$ (20a)
LATERAL-DIRECTIONAL	F_y	Y_A	Y_T	Y_g	0	Y_{oth}	$= m\dot{V} + \cdots$ (20b)
	G_x	L_A	L_T	0	L_{gyro}	L_{oth}	$= \dot{P}I_x + \cdots$ (21a)
	G_z	N_A	N_T	0	N_{gyro}	N_{oth}	$= \dot{R}I_z + \cdots$ (21b)

The vehicle motion can be thought of as two independent (decoupled) motions, each of which is a function only of the variables shown below.

1. Longitudinal Motion $(D, L, M_A) = f(U, \alpha, \dot{\alpha}, Q, \delta_e)$ (30B)

2. Lateral-Directional Motion $(Y_A, L_A, N_A) = f(\beta, \dot{\beta}, P, R, \delta_a, \delta_r)$

Steady Flight. Motion with zero rates of change of the linear and angular velocity components, i.e.,

$$\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = 0.$$

Straight Flight. Motion with zero angular velocity components, P, Q, and R = 0.

Symmetric Flight. Motion in which the vehicle plane of symmetry remains fixed in space throughout the maneuver. The unsymmetrical variables **P, R, V, , and ß are all zero** in symmetric flight.

S**teady straight symmetric flight**, the aircraft is assumed to be flying wings level with all components of velocity zero except Uo and Wo. Therefore, with reference to the body axis

$$V_T \simeq U_0 \simeq \text{constant}$$
$$W_0 \simeq \text{small constant} \therefore \alpha_0 \simeq \text{small constant}$$
$$V_0 \simeq 0 \therefore \beta_0 \simeq 0$$
$$P_0 \simeq Q_0 \simeq R_0 \simeq 0$$

Using this technique, the set of RHS equations become:

<u>Longitudinal</u>

"DRAG": $m(\dot{u} + qw - rv) = m\dot{u}$

"LIFT": $m(\dot{w} + pv - qU) = m\dot{w}$

"PITCH": $\dot{q}\, I_y - pr\,(I_z - I_x) + (p^2 - r^2)\, I_{xz} = \dot{q}\, I_y$

<u>Lateral-Directional</u>

"SIDE": $m(\dot{v} + rU - pw) = m\,\dot{v}$

"ROLL": $\dot{p}\, I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\, I_{xz} = \dot{P}\, I_x$

"YAW": $\dot{r}\, I_z + pq\,(I_y - I_x) + (qr - \dot{p})\, I_{xz} = \dot{r}\, I_z$

If we introduce the small-disturbance notation into the equations of motion, reduces the X-force equation to

$$\Delta X - mg\,\Delta\theta \cos\theta_0 = m\,\Delta\dot{u}$$

$$\Delta X = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T$$

or on rearranging

$$\left(m\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial X}{\partial u}\right)\Delta u - \left(\frac{\partial X}{\partial w}\right)\Delta w + (mg\cos\theta_0)\,\Delta\theta = \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T$$

The equation can be rewritten in a more convenient form by dividing through by the mass m:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\Delta\delta_e + X_{\delta_T}\Delta\delta_T$$

where Xu = dX/du/m, X, = dX/dw/m, and so on are aerodynamic derivatives divided by the airplane's mass.

Longitudinal equations

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w \Delta w + (g\cos\theta_0)\Delta\theta = X_{\delta_e}\Delta\delta_e + X_{\delta_T}\Delta\delta_T$$

$$-Z_u \Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\Delta\delta_e + Z_{\delta_T}\Delta\delta_T$$

$$-M_u \Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta_e}\Delta\delta_e + M_{\delta_T}\Delta\delta_T$$

Lateral equations

$$\left(\frac{d}{dt} - Y_v\right)\Delta v - Y_p \Delta p + (u_0 - Y_r)\Delta r - (g\cos\theta_0)\Delta\phi = Y_{\delta_r}\Delta\delta_r$$

$$-L_v \Delta v + \left(\frac{d}{dt} - L_p\right)\Delta p - \left(\frac{I_{xz}}{I_x}\frac{d}{dt} + L_r\right)\Delta r = L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

$$-N_v \Delta v - \left(\frac{I_{xz}}{I_z}\frac{d}{dt} + N_p\right)\Delta p + \left(\frac{d}{dt} - N_r\right)\Delta r = N_{\delta_a}\Delta\delta_a + N_{\delta_r}\Delta\delta_r$$

Longitudinal Dynamics:

Neglecting Z_q **and** $Z_{\dot{w}}$, the state space form yields

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta + M_{\dot{w}}Z_\delta & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix}$$

A- **Long Period Dynamics:**

$$\Delta\alpha = \frac{\Delta w}{u_0} \qquad \Delta\alpha = 0 \rightarrow \Delta w = 0 \qquad \textbf{(8)}$$

The homogeneous longitudinal state equations reduce to the following:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ \frac{-Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}$$

The eigenvalues of the long-period approximation are obtained by solving the equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0 \quad \text{or} \quad \begin{vmatrix} \lambda - X_u & g \\ \frac{Z_u}{u_0} & \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda X_u - Z g/u_o = 0 \quad \Rightarrow \quad \lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0$$

$$\omega_{n_p} = \sqrt{\frac{-Z_u g}{u_0}} \qquad \zeta_p = \frac{-X_u}{2\omega_{n_p}} \quad \Rightarrow \quad \omega_{n_p} = \sqrt{2}\,\frac{g}{u_0} \qquad \zeta_p = \frac{1}{\sqrt{2}}\frac{1}{L/D}$$

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ -\frac{Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ -\frac{Z_{\delta_e}}{u_0} & -\frac{Z_{\delta_T}}{u_0} \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_T \end{bmatrix} \qquad (17)$$

$$(s - X_u)\,\Delta u(s) + g\,\Delta\theta(s) = X_{\delta_e}\,\Delta\delta_e(s) + X_{\delta_T}\,\Delta\delta_T(s) \qquad (18)$$

$$\frac{Z_u}{u_0}\Delta u(s) + s\,\Delta\theta(s) = -\frac{Z_{\delta_e}}{u_0}\Delta\delta_e(s) - \frac{Z_{\delta_T}}{u_0}\Delta\delta_T(s) \qquad (19)$$

The transfer function $\Delta u(s)/\Delta\delta_e(s)$ and $\Delta\theta(s)/\Delta\delta_e(s)$ can be found by setting $\Delta\delta_T(s)$ to 0 and solving for the appropriate transfer function as follows:

$$(s - X_u)\frac{\Delta u(s)}{\Delta\delta_e(s)} + g\frac{\Delta\theta(s)}{\Delta\theta_e(s)} = X_{\delta_e} \qquad (20)$$

$$\frac{Z_u}{u_0}\frac{\Delta u(s)}{\Delta\delta_e(s)} + s\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = -\frac{Z_{\delta_e}}{u_0} \qquad (21)$$

$$\frac{\Delta u(s)}{\Delta\delta_e(s)} = \frac{\begin{vmatrix} X_{\delta_e} & g \\ \frac{-Z_{\delta_e}}{u_0} & s \end{vmatrix}}{\begin{vmatrix} s - X_u & g \\ \frac{Z_u}{u_0} & s \end{vmatrix}} \qquad \frac{\Delta u(s)}{\Delta\delta_e(s)} = \frac{X_{\delta_e}s + gZ_{\delta_e}/u_0}{s^2 + X_u s - \frac{Z_u g}{u_0}} \qquad (22)$$

In a similar manner $\Delta\theta(s)/\Delta\delta(s)$ can be shown to be

$$\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{-\frac{Z_{\delta e}}{u_0}s + \left(\frac{X_u Z_{\delta e}}{u_0} - \frac{Z_u X_{\delta e}}{u_0}\right)}{s^2 - X_u s - \frac{Z_u g}{u_0}} \quad (23)$$

The transfer functions can be written in a symbolic form in the following manner:

$$\frac{\Delta u(s)}{\Delta\delta_e(s)} = \frac{N^u_{\delta_e}(s)}{\Delta_p(s)} = \frac{A_u s + B_u}{As^2 + Bs + C} \quad (24)$$

$$\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{N^\theta_{\delta_e}}{\Delta_p(s)} = \frac{A_\theta s + B_\theta}{As^2 + Bs + C} \quad (25)$$

B- Short-Period Dynamics

An approximation to the short-period mode of motion can be obtained by assuming $\Delta u = 0$ and dropping the X-force equation. The longitudinal state-space equations reduce to the following:

$$\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$M_\alpha = \frac{1}{I_y}\frac{\partial M}{\partial \alpha} = \frac{1}{I_y}\frac{\partial M}{\partial(\Delta w/u_0)} = \frac{u_0}{I_y}\frac{\partial M}{\partial w} = u_0 M_w$$

We can write :

$Z_\alpha = u_0 Z_w$ and $M_{\dot{\alpha}} = u_0 M_{\dot{w}}$ Yields

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{u_0} & 1 \\ M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{u_0} & M_q + M_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix}$$

$|\lambda \mathbf{I} - \mathbf{A}| = 0$ which yields $\begin{vmatrix} \lambda - \frac{Z_\alpha}{u_0} & -1 \\ -M_\alpha - M_{\dot{\alpha}}\frac{Z_\alpha}{u_0} & \lambda - (M_q + M_{\dot{\alpha}}) \end{vmatrix} = 0$

$\lambda^2 - \left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)\lambda + M_q\frac{Z_\alpha}{u_0} - M_\alpha = 0$ from which we get:

$$\omega_{n_{sp}} = \left[\left(M_q \frac{Z_\alpha}{u_0} - M_\alpha\right)\right]^{1/2}$$ **and** $$\zeta_{sp} = -\left[M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right] \Big/ (2\omega_{n_{sp}})$$

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha/u_0 & 1 \\ M_\alpha + M_{\dot{\alpha}} Z_\alpha/u_0 & M_q + M_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e}/u_0 \\ M_{\delta_e} + M_{\dot{\alpha}} Z_{\delta_e}/u_0 \end{bmatrix} [\Delta\delta_e] \quad (8.1)$$

Taking the Laplace transform of this equation yields and rearrange , we can have the Transfer function:

$$(s - Z_\alpha/u_0) \frac{\Delta\alpha(s)}{\Delta\delta_e(s)} - \frac{\Delta q(s)}{\Delta\delta_e(s)} = Z_{\delta_e}/u_0 \qquad (11)$$

$$-(M_\alpha + M_{\dot{\alpha}} Z_\alpha/u_0) \frac{\Delta\alpha(s)}{\Delta\delta_e(s)} + [s - (M_q + M_{\dot{\alpha}})] \frac{\Delta q(s)}{\Delta\delta_e(s)} = M_{\delta_e} + M_{\dot{\alpha}} \frac{Z_{\delta_e}}{u_0} \quad (12)$$

$$\frac{\Delta\alpha(s)}{\Delta\delta_e(s)} = \frac{N^\alpha_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{\begin{vmatrix} Z_{\delta_e}/u_0 & -1 \\ M_{\delta_e} + M_{\dot{\alpha}} \frac{Z_{\delta_e}}{u_0} & s - (M_q + M_{\dot{\alpha}}) \end{vmatrix}}{\begin{vmatrix} s - Z_\alpha/u_0 & -1 \\ -(M_\alpha + M_{\dot{\alpha}} Z_\alpha/u_0) & s - (M_q + M_{\dot{\alpha}}) \end{vmatrix}} \quad (13)$$

$$\frac{\Delta\alpha(s)}{\Delta\delta_e(s)} = \frac{N^\alpha_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{A_\alpha s + B_\alpha}{As^2 + Bs + C}$$

$$\frac{\Delta q(s)}{\Delta\delta_e(s)} = \frac{N^q_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{\begin{vmatrix} s - Z_\alpha/u_0 & Z_{\delta_e}/u_0 \\ -(M_\alpha + M_{\dot{\alpha}} Z_\alpha/u_0) & M_{\delta_e} + M_{\dot{\alpha}} \frac{Z_{\delta_e}}{u_0} \end{vmatrix}}{\begin{vmatrix} s - Z_\alpha/u_0 & -1 \\ -(M_\alpha + M_{\dot{\alpha}} Z_\alpha/u_0) & s - (M_q + M_{\dot{\alpha}}) \end{vmatrix}} \quad (15)$$

$$\frac{\Delta q(s)}{\Delta\delta_e(s)} = \frac{N^q_{\delta_e}(s)}{\Delta_{sp}(s)} = \frac{A_q s + B_q}{As^2 + Bs + C} \qquad (16)$$

Roll Dynamics

The transfer function $\Delta p(s)/\delta_a(s)$ and $\Delta\phi(s)/\Delta\delta_a(s)$ can be obtained by taking the Laplace transform of the roll equation:

$$(s - L_p)\, \Delta p(s) = L_{\delta_a}\, \Delta\delta_a(s) \qquad \frac{\Delta p(s)}{\Delta\delta_a(s)} = \frac{L_{\delta a}}{s - L_p} \qquad (26)$$

But the roll rate Δp is defined as $\Delta\dot{\phi}$; therefore, $\Delta p(s) = s\Delta\phi(s)$ (**27**)

$$\frac{\Delta\phi(s)}{\Delta\delta_a(s)} = \frac{L_{\delta_a}}{s(s - L_p)} \qquad (\ \mathbf{28}\ )$$

Dutch Roll Approximation

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta/u_0 & -(1 - Y_r/u_0) \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_r}/u_0 & 0 \\ N_{\delta_r} & N_{\delta_a} \end{bmatrix} \begin{bmatrix} \Delta\delta_r \\ \Delta\delta_a \end{bmatrix}$$

$$(s - Y_\beta/u_0)\,\Delta\beta(s) + (1 - Y_r/u_0)\,\Delta r(s) = Y_{\delta_r}/u_0\,\Delta\delta_r(s)$$

$$-N_\beta\,\Delta\beta(s) + (s - N_r)\,\Delta r(s) = N_{\delta_a}\,\Delta\delta_a(s) + N_{\delta_r}\,\Delta\delta_r(s)$$

$$(s - Y_\beta/u_0)\frac{\Delta\beta(s)}{\Delta\delta_r(s)} + (1 - Y_r/u_0)\frac{\Delta r(s)}{\Delta\delta_r(s)} = Y_{\delta_r}/u_0 \qquad (\ \mathbf{28}\ )$$

$$-N_\beta\frac{\Delta\beta(s)}{\Delta\delta_r(s)} + (s - N_r)\frac{\Delta r(s)}{\Delta\delta_r(s)} = N_{\delta_r}$$

$$\frac{\Delta\beta(s)}{\Delta\delta_r(s)} = \frac{\begin{vmatrix} Y_{\delta_r}/u_0 & 1 - Y_r/u_0 \\ N_{\delta_r} & s - N_r \end{vmatrix}}{\begin{vmatrix} s - Y_\beta/u_0 & 1 - Y_r/u_0 \\ -N_\beta & s - N_r \end{vmatrix}} \qquad \frac{\Delta r(s)}{\Delta\delta_r(s)} = \frac{\begin{vmatrix} s - Y_\beta/u_0 & Y_{\delta_r}/u_0 \\ -N_\beta & N_{\delta_r} \end{vmatrix}}{\begin{vmatrix} s - Y_\beta/u_0 & 1 - Y_r/u_0 \\ -N_\beta & s - N_r \end{vmatrix}}$$

$$\frac{\Delta\beta(s)}{\Delta\delta_r(s)} = \frac{N_{\delta_r}^{\beta}(s)}{\Delta_{\mathrm{DR}}(s)} = \frac{A_\beta s + B_\beta}{As^2 + Bs + C} \qquad \frac{\Delta r(s)}{\Delta\delta_r(s)} = \frac{N_{\delta_r}^{\beta}(s)}{\Delta_{\mathrm{DR}}(s)} = \frac{A_r s + B_r}{As^2 + Bs + C}$$

Control actuator:

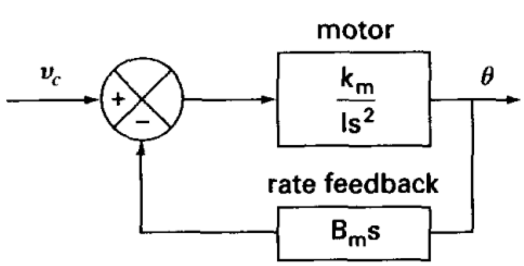


FIGURE 1
Motor with rate feedback.

$$T_m = k_m v_c \qquad I\ddot{\theta} = T_m \qquad \frac{\theta}{v_c} = \frac{k_m}{Is^2}$$

$$\frac{\theta}{v_c} = \frac{k}{s(\tau_m s + 1)} \quad \text{where} \quad \tau_m = \frac{I}{k_m B_m} \quad \text{and} \quad k = \frac{1}{B_m}$$

Pitch Displacement Autopilot: • The elevator servo transfer function can be represented as a first-order system:

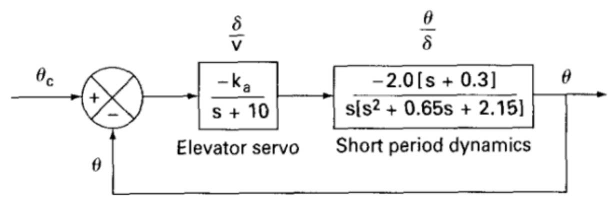
$$\frac{\delta_e}{v} = \frac{k_a}{\tau s + 1}$$

where δe, v, ka, - are the elevator deflection angle, input voltage, elevator servo gain, and of servomotor .

---- Time constants for typical servomotors fall in a range 0.05-0.25 s

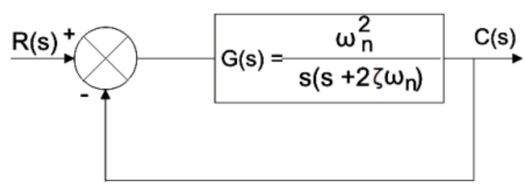
----- The short-period transfer function for the business jet can be shown to be

$$\frac{\Delta\theta}{\Delta\delta_e} = \frac{-2.0(s + 0.3)}{s(s^2 + 0.65s + 2.15)}$$



Time Response of Second-Order Control System:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Characteristic Equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \Rightarrow s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \;\; or \;\; -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

$Putting,\ \omega_d = \omega_n\sqrt{1-\zeta^2}$ And , α = ζωn Then , s = α ± jωd

<u>(1) Underdamped case ($0 < \zeta < 1$)</u>: In this case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

For a unit step input the output C(s) can be written:

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\mathcal{L}^{-1}[C(s)] = c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1 - \zeta^2}}{\zeta}\right), \quad \text{for } t \geq 0$$

The error of the signal of the response is given by:

$$e(t) = r(t) - c(t) = e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin\omega_d t\right) \quad \text{for } t \geq 0$$

---- frequency of the oscillation is ωd

---- time constant $\tau = 1/\zeta\omega_n = 1/\sigma$

(2) Critically damped case ($\zeta = 1$):

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written $C(s) = \dfrac{\omega_n^2}{(s + \omega_n)^2 s}$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \quad \text{for } t \geq 0$$

(3) Overdamped case ($\zeta > 1$): In this case, the two poles of $C(s)/R(s)$ are negative real and unequal. For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \quad \text{for } t \geq 0$$

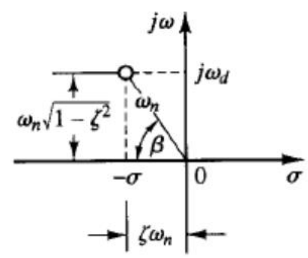
Definitions of transient-response specifications.:

. Delay time (td) is the time required to reach at 50% of its final value

Rise time (tr) is the time required to reach at final value by a under damped time response signal during its first cycle of oscillation. If the signal is over damped, then rise time is counted as the time required by the response to rise from 10% to 90% of its final value.

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

where β is defined in Figure



Peak time (tp) is simply the time required by response to reach its first peak

$$t_p = \frac{\pi}{\omega\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Maximum overshoot (Mp) is straight way difference between the magnitude of the highest peak of time response and magnitude of its steady

$$Maximum\ \%\ Overshoot = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$\%\ MP = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Settling time (ts) is the time required for a response to become steady.

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)}$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)}$$

. **Steady-state error (ess)** is the difference between actual output and desired output at the infinite range of time.

$$e_{ss} = \lim_{t\to\infty}[r(t) - c(t)] = \lim_{s\to 0} s[\,R/\,(1 + G(s)\,H(s))\,]$$

Damped Natural Frequency	$\omega_d = \omega_n \sqrt{1 - \zeta^2}$
	$\sigma = \zeta \omega_n$
	$\beta = \tan^{-1} \frac{\omega_d}{\sigma}$
Rise time	$t_r = \frac{\pi - \beta}{\omega_d}$
Peak time	$t_p = \frac{\pi}{\omega_d}$

Maximum overshoot	$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$ $= \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$
Settling time	$t_s = \frac{4}{\zeta \omega_n}$ (2% criterion)
	$t_s = \frac{3}{\zeta \omega_n}$ (5% criterion)

<u>Examples</u>

Example: Suppose an airplane is flying at 20 km at a speed of 200 m/s. The surface area is 30 m2 . The aircraft has a wingspan of 10 m. Suppose we have the following data :

Cx = 1.1 and Cy = 0.1 and Cz = 2.3 , the density of air is .08891 kg/m3 . The wind attach the airplane from the direction:

$$\vec{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 180 \\ 10 \\ 86.6 \end{bmatrix} m/s$$

Calculate Q, Z, X, lift to drag ratio (L/D), α, and β Solution

Solution

Then the dynamic pressure is

$$Q = \frac{1}{2}\rho V^2 = \frac{1}{2}.00891 * 200^2 = 178\frac{kg}{ms^2} = 178\frac{N}{m^2}$$

The Lift force is about **Z = Cz ∗ Q ∗ S = 2.3 ∗ 178 ∗ 30 = 12, 282N**

Likewise, the drag is about

X = Cx ∗ Q ∗ S = 1.1 ∗ 178 ∗ 30 = 5, 874N

This gives a Lift-to-Drag ration of about L/D = Cz/Cx = 2.1

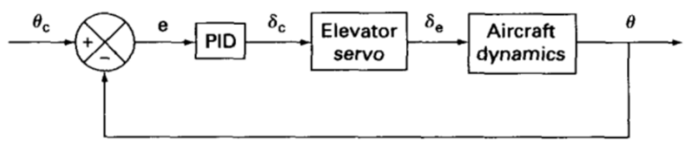
Then we can find angle of attack and sideslip as approximately

$$\alpha \cong \frac{w}{u} = .48rad = 27.5\,\text{deg} \qquad \text{and} \qquad \beta \cong \frac{v}{u} = .055rad = 3.18\,\text{deg}$$

Which contrast with the exact values of

$$\alpha = \tan^{-1}\frac{w}{u} = 25.7\,\text{deg} \qquad \text{and} \qquad \beta = \sin^{-1}\frac{v}{V} = 2.866\,\text{deg}$$

EXAMPLE. Use the PID controller for a pitch attitude autopilot as illustrated in Figure. The transfer functions for each component are given in Table



Control element	Parameters	Transfer function
PID	$k_p = ?$ $k_i = ?$ $k_d = ?$	$\frac{\delta_c}{e} = k_p + \frac{k_i}{s} + k_d s$
Elevator servo	$A = -0.1$ $\tau = 0.1$	$\frac{\delta_e}{\delta_c} = \frac{A}{\tau s + 1}$
Aircraft dynamics	$M_{\delta e} = -3\ s^{-2}$ $M_q = -2\ s^{-1}$ $M_\alpha = -5\ s^{-2}$	$\frac{\theta}{\delta_e} = \frac{M_{\delta e}}{s^2 - M_q s - M_\alpha}$

$$G(s)H(s) = \frac{3.0k_p}{(s+10)(s^2+2s+5)}$$

The root locus crosses the imaginary axis at s = jω = 5.13j.

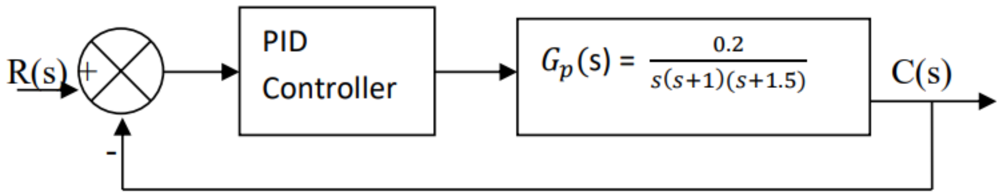
The gain $\mathbf{K_{pu}}$ of the system can be found from the magnitude criteria to be:

substitute s = jw in $\left|\frac{3.0k_p}{(s+10)(s^2+2s+5)}\right| = 1$, we get: $K_{pu} = 88.7$

And Tu = 2pi/w = 1.22. Hence gain for P, PI, PID controller:

P control	$k_p = 0.5k_{pu} = 44.35$
PI control	$k_p = 0.45k_{pu} = 39.92$ $k_i = 0.45k_{pu}/(0.83T_u) = 39.42$
PID control	$k_p = 0.6k_{pu} = 53.22$ $k_i = 0.6k_{pu}/(0.5T_u) = 87.24$ $k_d = 0.6k_{pu}(0.125T_u) = 8.12$

Example Problem: Design a PID controller for the controller for the control system shown in Figure. Given that the locus intersects the imaginary axis at s =



Solution: The ultimate gain ku is found by finding the gain when the root locus intersects the imaginary axis which is given at s = j ω = ±1.25j.

The gain crossover point can be determined from the magnitude criteria by :

$$|\,G(s)H(s)| = \frac{|0.2k_p|}{|s||s+1|s+1.5||} = 1$$

Substituting s = 1.25j into the magnitude criteria yields kpu = 19.8.

The period of un-damped oscillation Tu is obtained as follows: Tu = $2\pi/\omega = 2\pi$ /1.25 = 5.03

Knowing kpu and Tu the proportional, integral, and derivative gains kp, ki , and kd can be evaluated:

kp = 0.6 kpu = (0.6)(19.8) = 11.88,

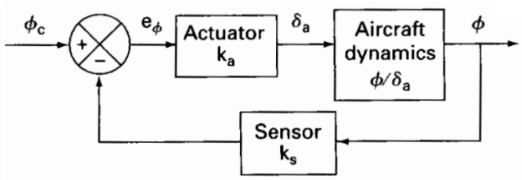
k i = 0.6 kpu /(0.5 Tu) = (0.6)(19.8)/[(0.5)(5.03)] = 4.73

kd = 0.6 kpu (0.125Tu) = (0.6)(19.8)(0.125)(5.03) = 7.47

E X A M P L E 2. Design a roll attitude control system to maintain a wings • level attitude for a vehicle having the following characteristics:

$$L_{\delta_a} = 2.0/s^2 \qquad L_p = -0.5/s$$

The system performance is to have a damping ratio, ζ = 0.707, and an undamped natural frequency, ω_n = 10 rad/s. Roll attitude control system is shown in the block diagram in Figure And the line of constant ζ = 0.707 intersect the root locus at s = -0.25 + 0.25j



The roll angle to aileron input transfer function for an airplane

$$\frac{\Delta\phi(s)}{\Delta\delta_a(s)} = \frac{L_{\delta_a}}{s(s - L_p)}$$

Solution:

$$G(s) = \frac{\Delta\delta_a(s)}{e(s)}\frac{\Delta\phi(s)}{\Delta\delta_a(s)} = k_a\frac{L_{\delta_a}}{s(s - L_p)}$$

$$G(s)H(s) = \frac{k}{s(s - L_p)} \qquad k = k_a L_{\delta_a} \qquad G(s)H(s) = \frac{k}{s(s + 0.5)}$$

Given $\zeta = 0.707 = \cos\theta$ which means $\theta = 45^0$

The line of constant $\zeta = 0.707$ will intersect the root locus at s = -0.25 + 0.25j.

Sub this in:

$$|G(s)| = \frac{|k|}{|s|\ |s + 0.5|} = 1 \quad \text{where } s = -0.25 + 0.25i. \qquad k = 0.0139$$

The undamped natural frequency is much lower than specified: $\omega_n = 0.35$ rad/s

Example.1 : Consider the second order system for an aircraft where ζ=0.6 and wn=5 rad/sec. find the rise time tr , peak time tp, maximum overshoot Mp, and settling time ts when the airplane is subjected to a unit-step gust?

Solution: Given ζ=0.6 and ω_n=5 rad/sec

$$\Omega_d = \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-0.6^2} = 4$$

$$\sigma = \zeta\,\omega_n = 0.6 * 5 = 3$$

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p:

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

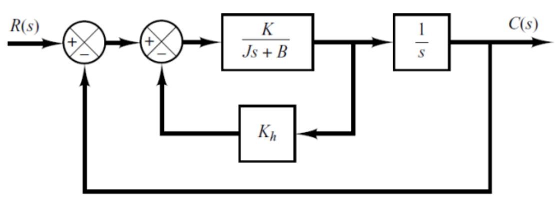
Settling time t_s: For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

Example.2: For the aircraft control system with velocity feedback shown in figure, determine the values of gain K and velocity feedback constant Kh so that the maximum overshoot in the unit-step perturbation response is 0.2 and the peak time is 1 sec. With these values of K and Kh, obtain the rise time and settling time. Assume that J=1kg-m^2 and B=1 N-m/rad/sec.



Solution:

Given $M_p = 0.2$, J=1, and B=1

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2 \qquad \zeta = 0.456$$

$$t_p = \frac{\pi}{\omega_d} = 1 \qquad \omega_d = 3.14$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ $\omega_n = \omega_d / \sqrt{1-\zeta^2} = 3.14/\sqrt{1-0.456^2} = 3.528$

$$G1(s) = \frac{k/(Js+B)}{1+ \ kh\ k/(Js+B)} = \frac{k/J}{s+\frac{B}{J}+ \ kh\ k/J)} = \frac{k/J}{s+(B+ \ kh\ k)/J}$$

$$\text{Transfer Function} = \frac{G1/_s}{1+ G1/_s} = \frac{\frac{k/J}{s(s+\frac{B+ \ kh\ k}{J})}}{1+ \frac{k/J}{s(s+\frac{B+ \ kh\ k}{J})}}$$, substitude J=1 , B=1 and K $=k_h$ k

$$T.F = T(s) = \frac{\frac{k}{s(s+B+K)}}{1+ \frac{k}{s(s+B+K)}} = \frac{k}{s^2+s(1+K)+k}$$

From characteristic equation $s^2 + s(1+K) + k = 0$, $k = \omega_n^2 = (3.528)^2 = 12.5$, And $2\zeta\omega_n = 1+K = 1+K_h k$

Hence $K_h = (2\zeta\omega_n - 1)/k = (2*0.456*3.528 - 1)/12.5 = 0.178$

Rise time t_r: $t_r = \frac{\pi - \beta}{\omega_d}$ $\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.10$

$t_r = 0.65$ sec

Settling time t_s: For the 2% criterion, $t_s = \frac{4}{\sigma} = 2.48$ sec

For the 5% criterion, $t_s = \frac{3}{\sigma} = 1.86$ sec

Examples:

1. If Lift produced by wing is 350N then, determine lift coefficient.

   Given q = 35Pa and S=8.5 m^2.

   Solution: Lift coefficient = lift / q*S = 350/35*8.5 = 1.174.

2. For a symmetrical airfoil drag coefficient at zero lift is 0.05 and induced drag coefficient is 0.0025. Find the total drag coefficient.
   Solution:
   Total drag coefficient = drag coefficient at zero lift + induced drag coefficient

   = 0.05+0.0025 = 0.0525.

3. Determine sideslip angle for a steady level unaccelerated flight with
   [u, v, w] = [80, 2, 4.5].
   Solution: Given, v = 2, V = $[u^2+v^2+w^2]^{0.5}$ = $[80*80+2*2+4.5*4.5]^{0.5}$ = 80.1521.
   Sideslip angle = arcsine (v/V) = arcsine (2/80.1521) = 1.43°.

4. Determine the value of climb angle if, excess thrust is 40 unit and weight of the aircraft is 60 unit. Consider steady climb.

   Solution: Given, Excess thrust = T- D, steady flight (T-D) = 40

   (T- D) – W sin(climb angle) = 0

   Hence, Climb angle = arcsine (excess Thrust/weight) = arcsine (40/60) = 41.8°.

5. A wing is designed to operate with free stream velocity of 20m/s and air density of 1.225 kg/m^3. Find aerodynamic efficiency of given wing. Consider S as 8 m^2, CL as 0.9 and CD as 1.25.
   a) 0.72 b) 2 c) 3 d) 5.23
   Solution : Given, CL = 0.9, CD = 1.25
   Aerodynamic efficiency is defined as the ratio of CL and CD of the aircraft.
   Hence, Aerodynamic efficiency = CL/CD = 0.9/1.25 = 0.72.
6. An aircraft experiences sideslip of 4° and side wash at vertical tail is 1.2°. What will be the AOA at vertical tail?
   Solution: AOA = sideslip + side wash = 4°+1.2° = 5.2°.
7.

Problems

8. With the aid of a diagram showing a generalized set of aircraft body axes, define the parameter notation used in the mathematical modeling of aircraft motion1.

9. In the context of aircraft motion, what are the Euler angles? If the standard right handed aircraft axis set is rotated through pitch θ and yaw ψ angles only, show that the initial vector quantity (x0, y0, z0) is related to the transformed vector quantity (x, y, z) as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

10. Define the span, gross area, aspect ratio and mean aerodynamic chord of an aircraft wing.

11. If an aircraft is operating with dynamic pressure of the free stream q=20Pa and has area of wing is $10m^2$ then evaluate drag experience by the aircraft. Given drag coefficient
12. If an aircraft is operating with dynamic pressure of the free stream q=20Pa and has area of wing is $10m^2$ then evaluate drag experience by the aircraft. Given drag coefficient
13. An aircraft is flying in the north direction at a velocity of 60.5m/s under cross wind from the east to west of 5m/s. If the value of $\mathbf{Cn_{\beta}=0.02/deg}$, . Find sideslip angle β.
14. Find sideslip angle if [u, v, w] = [100, 5, 2.5]. Consider steady level flight.
15. Find lift to weight ratio if climb angle is 45°.
16. Find the approximate value of climb angle if Thrust is 1500N, drag is 1000N and weight of the aircraft is 2500N
17. Given ($F = d\, mV_T/dt$) , F is a force vector, m is a constant mass, and V_T is the velocity vector of the mass center. Find Fx , Fy , and Fz (if V_T = Ui + Vj + Wk and w = Pi + Qj + Rk) with respect to the fixed earth axis system.
18. Given H = $\int_v \rho A$ (r X v) dv where pAdV is the mass of a particle, with r as its radius vector from the eg, and V as its velocity, with respect to the eg. Find Hx with respect to the fixed earth axis system.
19. Define: L , M , N, P, Q, R
20. Define ψ, ϕ, θ, What are they used for? in what sequence must they be used? Explain the difference between ψ and ß.
21. What are the expressions for P, Q, R, in terms of Euler angles?

    D, L, M = f(, , , ,)

    Y, L, N = f(, , , , ,)

22. An aircraft is designed to be in steady level flight with weight of 1500N and CL of 1.0. Determine at which speed we need to design this aircraft so that it can achieve this requirement. Consider density as 1.2 kg/m^3 and reference area as $2m^2$.

MCQ

Choose correct answer, Describe your selection :

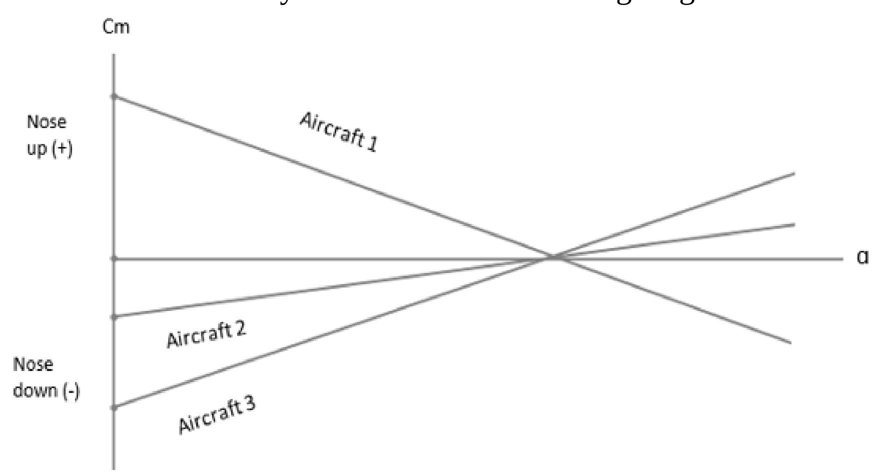
1. Which of the following is correct?
    a. a) Lift is equal to weight always b) Thrust is only proportional to nose of aircraft c) Aircraft which is statically stable may or may not be dynamically stable d) Drag is useful during takeoff
2. How do you define the lift coefficient?

   a) Ratio of aerodynamic lift to the dynamic lift b) Lift to drag

   c) Wing lift to weight of aircraft d) Thrust to weight

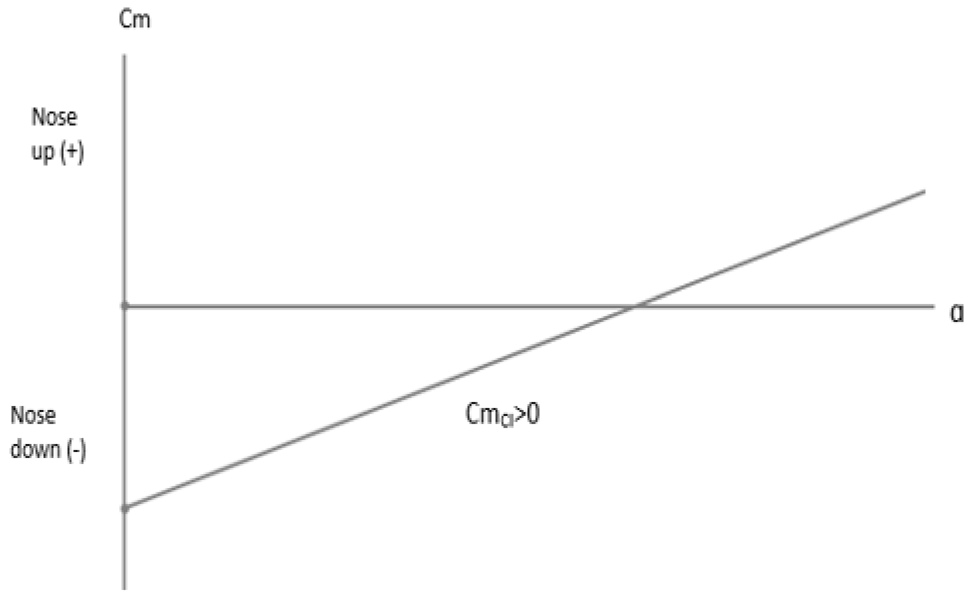
3. Which of the following is correct?
   a) $D = q*S*C_D*\rho$ b) $D = q*C_D$ c) $D = q$ d) $D = q*S*C_D$
4. For an incompressible flow, if local area velocity decreases then, the dynamic pressure will ________
   a) decrease b) increase c) constant d) independent of velocity
5. For ideal flow, total pressure along streamline will be _________.
   a) increases b) decreases c) constant d) always decreases by half
6. Aerodynamic forces are generated due to _______________
   a. a) shear effects only b) only pressure forces
   b. c) shear and pressure force acting on body. d) twisting of beam
7. The change in local air flow velocity will produce small change in skin friction drag.
   a. a) True b) False
8. How do you define the lift coefficient?
   a. a) Lift to drag b) Ratio of aerodynamic lift to the dynamic lift
   c) Wing lift to weight of aircraft d) Thrust to weight
9. Which of the following is correct?
   a. a) D = q b) D = q*S*CD*ρ c) D = q*CD d) D = q*S*CD
10. Aircraft is said to be statically stable if ___________
    a. it has more thrust than drag
    b. it has tendency to return to equilibrium state with the help of pilot's input
    c. it has more lift than weight always
11. How can we say that the aircraft has initial tendency to return to its original equilibrium position after being disturbed?
    a. If restoring force is not generated to oppose the disturbance
    b. If lift is same as weight always.
    c. If aircraft generates some restoring force or/and moment without any external help.
    d. If thrust loading is always unity.
12. Longitudinal stability means ___________
    a. stability about yawing axis
    b. stability about pitching axis
    c. stability about lateral axis
    d. stability about negative yawing axis
13. Which is the minimum requirement for pure directional stability?
    a) Positive zero lift pitching moment coefficient b) Negative lift curve slope
    c) Slope of yawing moment curve positive
    d) Negative pitching moment coefficient curve slope
14. Which of the following is correct to trim an aircraft at positive AOA?
    a) Every value of cm0 will trim at positive AOA b) Cm0 < 0
    c) Cm0 will not affect positive trim AOA d) Cm0 > 0

15. Drag which is **produced due to lift is called?**
    a) Induced drag   b) Parasite drag   c) Weight   d) Thrust drag
16. The change in local air flow velocity will produce small change in skin friction drag.
    a) True   b) False
17. Dynamic stability is more concerned about time.
    a) True   b) False
18. Stability about yawing axis is called as __________
    b) lateral stability   c) longitudinal stability   c) directional stability
    d) pitching moment stability
19. Yawing moment is positive if __________
    a) right wing comes forward   b) right wing goes back   c) if nose pitches up
    d) if nose pitches down
20. Stability about roll axis is called _____________
    a) lateral stability   b) directional stability   c) longitudinal stability   d) elevator control
21. Rolling moment will influence _______
    a) longitudinal stability   b) pitch axis stability   c) pitching stability only
    d) aircraft lateral stability
22. . If aircraft continues to go farther away from equilibrium position after being disturbed then the aircraft is called _______
    a) stable   b) unstable   c) statically stable   d) neutrally stable
23. Which of the aircraft will be statically stable based on following diagram?



    a) aircraft number 2   b) aircraft number 3   c) aircraft number 1
    d) same static stability for all 3 aircrafts

24. Following diagram represents ________



a) pitching moment diagram for stable aircraft b) lift curve slope
c) drag polar d) pitching moment coefficient diagram of unstable aircraft

25. Which of the following is correct?
a) Aircraft static longitudinal stability will be dependent upon the arrangement of different components
b) Lift is always same as weight
c) Static stability is similar to dynamic stability
d) All the aircrafts are statistically stable

Explanation: Static longitudinal stability of aircraft will be dependent upon the arrangement of different components such as wing placements, tail location etc. At cruise condition or at trim position lift will be same as weight. Static stability and dynamic stability are different. System can be statically stable but that doesn't mean that the system is dynamically stable as well.

26. Ratio of vertical distance travelled to the horizontal distance travelled is known as ___________
a) lift curve slope b) power required c) climb gradient d) thrust loss

27. Consider the vertical velocity of the aircraft is 10m/s and horizontal velocity is 12 m/s. Determine the value of climb gradient.
a) 1.89 b) 8 c) 2.483 d) 0.833

28. Which of the following is correct for steady level flight?
a) T>D b) D<T c) T=D d) L>W
View Answer

29. If an aircraft is in steady level flight and weighs 2500 kg then, find lift at the given flight condition.
a) 24.525KN b) 24525kg c) 2500 N d) 35000kg

30. Find climb angle of climb gradient is 0.6. Assume non accelerated flight.

a) 31°   b) 1.2   c) 4.5rad   d) 9.94°

Answer: a

Explanation: Given, climb gradient G = 0.6.

Now, climb angle = arctan (G) = arctan (0.6) = 30.9 ~ 31°.

31. Which of the following is correct for steady level flight?

a) T=D   b) T>D   c) D<T   d) L>W

Answer: a

Explanation: In steady level flight, flight path angle or climb angle is zero. Hence, conventional equation of motion reduces to the thrust T = Drag D. Steady level flight is unaccelerated and hence all the forces should give sum in respective direction as zero.

32. An aircraft has L/D as 12 and thrust required at this value is 100N. Now if L/D has doubled then what will be the new thrust required?

a) 50N   b) 500N   c) 100 N   d) 12.5N

33. In Steady level unaccelerated flight, lift is equal to weight.

a) True   b) False

Answer: a

Explanation: In steady level unaccelerated flight, lift is equal to weight of the aircraft. Drag is equal to Thrust of the aircraft. This is also called cruise flight. For steady level unaccelerated flight, Lift = weight = q*S*CL.

34.