2. Homogeneous Equations

Any differential equation $[\frac{dy}{dx} = f(x, y)]$ that we can put it into the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \quad \dots (1)$$

Is called homogeneous.

Let $v = \frac{y}{x}$, where v is a new independent variable, then $\frac{dy}{dx} = v * 1 + x * \frac{dv}{dx}$ and (1) becomes $v + x * \frac{dv}{dx} = F(x), \dots (2)$

$\Longrightarrow$ Eq.(2) can be solved by separation of variables:

$$\Longrightarrow x * \frac{dv}{dx} = F(v) - v$$

$$\Longrightarrow \frac{dv}{F(v) - v} = \frac{dx}{x}, \quad \dots \ (3)$$

After (3) is solved, the solution of the original equation is obtained when we replace v by $\frac{y}{x}$.

Example. 1

Solve the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$

Solution. Let $y = vx \Longrightarrow \frac{dy}{dx} = \frac{x+vx}{x} = \frac{x(1+v)}{x} \Longrightarrow \mathrm{F(v)} = 1 + v$

$$then \ \frac{dv}{F(v) - v} = \frac{dx}{x} \Longrightarrow \int \frac{dv}{1 + v - v} = \int \frac{dx}{x}$$

$$\Longrightarrow \int dv = \int \frac{dx}{x} \Longrightarrow v = \ln x + \ln c$$

So $\frac{y}{x} = \ln|cx| \Longrightarrow y = x \ln|cx|$ is a particular solution.

Example 2. Solve the equation $\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$

Solution. Let $y = vx \Rightarrow \frac{dy}{dx} = \frac{x \cdot vx}{x^2 - (vx)^2} = \frac{\cancel{x^2} \cdot v}{\cancel{x^2}[1 - v^2]}$

$$\Rightarrow F(v) = \frac{v}{1 - v^2}$$

$$\text{then } \frac{dx}{x} = \frac{dv}{F(v) - v}$$

$$\frac{dx}{x} = \frac{dv}{\frac{v}{1-v^2} - v} = \frac{dv}{\frac{v - v(1-v^2)}{1-v^2}} = \frac{dv}{\frac{\cancel{v} - \cancel{v} + v^3}{1-v^2}} = \frac{dv}{\frac{v^3}{1-v^2}}$$

$$\int \frac{dx}{x} = \int \frac{1-v^2}{v^3} dv \Rightarrow \int \frac{dx}{x} = \int \left(v^{-3} - \frac{1}{v}\right) dv$$

$$\ln x + \ln c = \frac{v^{-2}}{-2} - \ln v$$

$$\ln cx + \ln v = \frac{-1}{2v^2}$$

$$\ln|cxv| = \frac{-1}{2v^2} \quad \Rightarrow e^{\ln(c\cancel{x}\cdot\frac{y}{\cancel{x}})} = e^{\frac{-1}{2\frac{y^2}{x^2}}}$$

$$\Rightarrow cy = \exp\left(\frac{-x^2}{2y^2}\right) \; ; \; y = \frac{1}{c}\exp\left(\frac{-x^2}{2y^2}\right)$$

$$\Rightarrow \boxed{y = k\, e^{-\frac{x^2}{2y^2}}} \text{, where } k = \frac{1}{c}$$

Ex.3 Solve $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$

Solution. First we need to show that this equation is homo.

$\Rightarrow$ let $y = vx$, $\frac{dy}{dx} = \frac{x^2+3(vx)^2}{2x(vx)}$

$$= \frac{x^2+3v^2x^2}{2x^2v} = \frac{\cancel{x^2}(1+3v^2)}{2\cancel{x^2}v} = F(v)$$

$$\Rightarrow F(v) = \frac{1+3v^2}{2v}$$

So, $\frac{dx}{x} = \frac{dv}{F(v)-v} \Rightarrow \frac{dx}{x} = \frac{dv}{\frac{1+3v^2}{2v} - v}$

$$\frac{dx}{x} = \frac{dv}{\frac{1+3v^2-2v^2}{2v}} \Rightarrow \int\frac{dx}{x} = \int\frac{2v\,dv}{1+v^2}$$

$$\Rightarrow \ln x + \ln c = \ln(1+v^2)$$

$$\Rightarrow \cancel{\ln}\, cx = \cancel{\ln}(1+v^2)$$

$$\Rightarrow cx = 1 + \frac{y^2}{x^2}$$

$$\Rightarrow \frac{y^2}{x^2} = cx - 1$$

$$\Rightarrow \boxed{y^2 = x^2(cx-1)}$$

3. Exact equations

An equation that can be written in the form

$$M(x,y)dx + N(x,y)dy = 0\,, \quad . . . \ (1)$$

And having the property that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad . . . \ (2)$$

Is said to be exact, because its left side is an exact differential.

To solve this equation, we must find a function $f(x,y)$ such that

$$df = M\,dx + N\,dy \qquad . . . \ (3)$$

Then (1) becomes $df = 0$, and the solution $f(x,y) = c$, where c is an arbitrary constant.

We know that

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}dy$$

So that $M = \frac{\partial f}{\partial x}$ *and* $N = \frac{\partial f}{\partial y}$, where $M = M(x,y)$ *and* $N = N(x,y)$.

$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ because the derivative of f with respect to x and y are continuous for all real values.

The method of finding $f(x,y)$ to satisfy (3) is as follows:

1. $f(x,y) = \int_x M(x,y)dx + g(y)\,, \qquad . . . \ (4)$
2. *then* $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int_x M\,dx + g(y)\right], \qquad . . . \ (5)$
3. *since eq.* (1) *is exact* differential,

   $\therefore N = \frac{df}{dy} \Rightarrow N = \frac{\partial}{\partial y}\left[\int_x M\,dx\right] + \frac{\partial g}{\partial y}\,, \quad . . . \ (6)$

   where g is a function of y only.

   $$\Rightarrow \frac{dg}{dy} = N - \frac{\partial}{\partial y}\int_x M\,dx\,, \qquad . . . \ (7)$$

4. $$g(y) = \int \frac{d(g(y))}{dy}\,dy\,, \qquad . . . \ (8)$$

5. at last, we put (8) in (4) to find $f(x,y)$.

Example.1

Find a function $f(x,y)$ whose differential is $df = (x^2 + y^2)dx + 2xy\,dy$

Sol. We set $M = x^2+y^2$ and $N = 2xy$

$$\frac{\partial M}{\partial y} = 2y \quad , \frac{\partial N}{\partial x} = 2y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

that's mean the expression $(x^2+y^2)dx + 2xy\,dy$ is exact such that $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$,

then $f(x,y) = \int_x (x^2+y^2)dx + g(y)$

$$\Rightarrow \boxed{f(x,y) = \frac{x^3}{3} + y^2x + g(y)} \quad ——(1)$$

$$\frac{\partial f}{\partial y} = 0 + 2yx + g'(y)$$

since $N = \frac{\partial f}{\partial y}$, $2yx = 2yx + g'(y)$

$$\Rightarrow g'(y) = 0 \Rightarrow \boxed{g(y) = C} \quad ——(2)$$

put (2) in (1):

$$\boxed{f(x,y) = \frac{x^3}{3} + y^2x + C}$$

Example.2

Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

Sol. Here $M = 5x^4 + 3x^2y^2 - 2xy^3$, $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 0 + 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2 - 0$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now $f(x,y) = \int (5x^4 + 3x^2y^2 - 2xy^3)dx$

$$\boxed{f(x,y) = x^5 + x^3y^2 - x^2y^3 + g(y)} \quad ——(1)$$

$$\frac{\partial f}{\partial y} = 0 + \cancel{2x^3y} - \cancel{3x^2y^2} + g'(y) = N = \cancel{2x^3y} - \cancel{3x^2y^2} - 5y^4$$

$$\Rightarrow g'(y) = -5y^4 \Rightarrow g(y) = \int -5y^4 dy = -y^5 + C$$

$$\therefore \boxed{g(y) = -y^5 + C} \quad ——(2)$$

put (2) in (1) $\Rightarrow \boxed{f(x,y) = x^5 + x^3y^2 - x^2y^3 - y^5 + C}$

4. Integrating factors (linear)

A differential equation of first order, which is also linear, can be put in the standard form:

$$\frac{dy}{dx} + P(x) = Q(x)\,, \qquad ...\ (1)$$

where P&Q are functions of x.

One method for solving Eq.1 is to find a function $IF = IF(x)$ such that if Eq.1 multiplied by IF , the left side becomes the derivative of the product $IF.y$

That is, we multiply Eq. (1) by IF,

$$IF\frac{dy}{dx} + IF.Py = IF.Q\,, \qquad ...\ (1')$$

This function is called an integrating factor for Eq.1 with its help, $(1')$ becomes

$$\frac{d}{dx}(IF.y) = IF.Q(x)\,,\ with\, IF = e^{\int p\,dx}$$

whose solution is $y = \frac{\int IF.Q\,dx + C}{IF}$.

Example.1

Solve the equation $x\,\frac{dy}{dx} - 3y = x^2$

Solution. We put the equation in standard form,

$$\frac{dy}{dx} - \frac{3}{x}y = x\,, \qquad P(x) = \frac{-3}{x}\,, \quad Q(x) = x$$

$$IF = e^{\int P dx} = e^{\int \frac{-3}{x}dx}$$

$$IF = e^{-3\ln x} = \cancel{e}^{\cancel{\ln} x^{-3}} = \frac{1}{x^3} \qquad \Rightarrow\ y = \frac{\int IF.\,Q\,dx + c}{IF}$$

$$\Rightarrow y = \frac{\int \frac{1}{x^3}\cdot x\,dx + c}{\frac{1}{x^3}}$$

$$\Rightarrow y = \left[\int x^{-2}dx + c\right] * x^3$$

$$\Rightarrow y = \left[\frac{-1}{x} + c\right] * x^3$$

$$\Rightarrow \boxed{y = cx^3 - x^2}$$ is the solution

Example. 2

Solve the equation $\frac{dy}{dx} + 2y = e^{-x}$

Sol. Consider that: $P(x) = 2$, $Q(x) = e^{-x}$

then $IF = e^{\int P dx}$

$IF = e^{\int 2 dx} = e^{2x}$

$\Rightarrow y = \frac{\int IF . Q dx + C}{IF} = \frac{\int e^{2x} . e^{-x} dx + c}{e^{2x}}$

$\Rightarrow y = \frac{\int e^{x} dx + c}{e^{2x}} = e^{-2x}(e^{x} + C)$

$\Rightarrow \boxed{y = e^{-x} + c e^{-2x}}$ is the general solution ■

Example. 3

Solve the equation $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

Sol. The standard form is

$\frac{dy}{dx} + \frac{3}{x} . y = \frac{\sin x}{x^3}$, $P(x) = \frac{3}{x}$ and $Q(x) = \frac{\sin x}{x^3}$

$\Rightarrow IF = e^{\int \frac{3}{x} dx}$ $\Rightarrow IF = e^{3 \ln x} = \cancel{e}^{\cancel{\ln} x^3} = x^3$

Then $y = \frac{\int \cancel{x^3} . \frac{\sin x}{\cancel{x^3}} dx}{x^3} \Rightarrow \boxed{y = \frac{-\cos x + c}{x^3}}$ is the general solution ■

Example. 3 Solve the eq. $(x \, dy + y \, dx = y \, dy) \div dy$

Sol. $(x + y . \frac{dx}{dy} = y) \div y$

$\Rightarrow \boxed{\frac{dx}{dy} + \frac{1}{y} x = 1} \Rightarrow P(y) = \frac{1}{y}$ and $Q(y) = 1$

$IF = e^{\int \frac{1}{y} dy} = \cancel{e}^{\cancel{\ln} y} = y \Rightarrow x = \frac{\int IF . Q \, dy + C}{IF}$

$\Rightarrow x = \frac{\int y . 1 \, dy + C}{y} \Rightarrow x = \frac{1}{y}(\frac{y^2}{2} + C)$

$\boxed{x = \frac{y}{2} + \frac{C}{y}}$ ■

5. Equations reducible to the linear form (Bernoulli's equation).

Some non-linear DE can be transformed to linear DE by change the dependent variable.

$$y' + P(x)y = Q(x)y^n$$

It is called Bernoulli's equation.

Multiplying by $(y^{-n}) \Rightarrow y^{-n}y' + P(x)y^{1-n} = Q(x)$

Let $u = y^{1-n} \Rightarrow u' = (1-n)y^{-n}y'$

$\therefore \left[\frac{u'}{1-n} + P(x)u = Q(x)\right] * (1-n)$

$\Rightarrow u' + (1-n)P(x)\,u = (1-n)Q(x)$

Transform to linear $\Rightarrow u' + P_1(x)u = Q_1(x)$.

Example.1

Solve $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Solution. Multiply the equation by $(\frac{1}{y^2})$

$\Rightarrow \frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{1}{x} y^{-1} = \frac{1}{x^2}$

Now let $u = y^{-1} \Rightarrow u' = -y^{-2}y' = -y^{-2}\frac{dy}{dx}$

$\Rightarrow \left[-u' + \frac{1}{x}u = \frac{1}{x^2}\right] * (-1)$

$\Rightarrow u' - \frac{1}{x}u = \frac{-1}{x^2}$ where $P = \frac{-1}{x}$ and $Q = \frac{-1}{x^2}$

then $IF = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$,

$u = \frac{\int IF\, Q\, dx + c}{IF}$

$u = \frac{\int \frac{1}{x} \cdot \frac{-1}{x^2} dx + c}{\frac{1}{x}} = \frac{-\int \frac{1}{x^3} dx + c}{\frac{1}{x}}$

$u = x\left[+\frac{1}{2x^2} + c\right]$, but $u = y^{-1} \Rightarrow \frac{1}{y} = \frac{1}{2x} + cx$

$= \frac{1 + 2cx^2}{2x}$

Then $\boxed{y = \frac{2x}{1+2cx^2}}$ ∎

Example. 2

Solve the equation $y' + y\tan x = (\sec x)\, y^3$

Sol. Multiply both sides by y^{-3}

$\Rightarrow y^{-3}y' + y^{-2}\tan x = \sec x$ ---- (1)

Let $\boxed{u = y^{-2}} \Rightarrow u' = -2y^{-3}y'$

multiply eq. (1) by (-2),

$\underbrace{-2y^{-3}y'}_{u'} - 2\underbrace{y^{-2}}_{u}\tan x = -2\sec x$

$u' - (2\tan x)\, u = -2\sec x, \qquad P(x) = -2\tan x, \; Q(x) = -2\sec x$

then $IF = e^{\int -2\tan x\, dx}$

$IF = e^{-2\int \frac{\sin x}{\cos x} dx} = e^{2\ln\cos x} = \cancel{e}^{\cancel{\ln}\cos^2 x} = \cos^2 x$

$\Rightarrow u(x) = \dfrac{\int IF \cdot Q\, dx + c}{IF}$

$\Rightarrow u(x) = \dfrac{\int \cos^2 x \cdot (-2\sec x)\, dx + c}{\cos^2 x} \qquad \left(\sec x = \frac{1}{\cos x}\right)$

$\Rightarrow u = \dfrac{-2\int \cos x\, dx + c}{\cos^2 x} \qquad \Rightarrow y^{-2} = \dfrac{-2\sin x + c}{\cos^2 x}$

$\Rightarrow \boxed{y^2 = \dfrac{\cos^2 x}{-2\sin x + c}}$ the general sol.