Chapter Two

Second Order Differential Equations

1. Special Types of second order equations

Certain types of second order differential equations, of which the general form is:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0\,, \qquad \ldots (1)$$

Can be reduced to first order equations by a suitable change of variables.

Type1 Equations with dependent variable missing.

When eq. (1) has the special form

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0\,, \qquad \ldots (2)$$

We can reduce it to first order equation by substituting

$$P = \frac{dy}{dx}\,, \qquad \frac{d^2y}{dx^2} = \frac{dP}{dx}$$

Then eq. (2) takes the form

$$F\left(x, P, \frac{dP}{dx}\right) = 0\,,$$

which is of the first order in P.

Example.1

Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \qquad [xy'' + y' = 0]$

Solution: Let $P = \frac{dy}{dx}$, then $\frac{d^2y}{dx^2} = \frac{dP}{dx}$

by substituting $x\frac{dP}{dx} + P = 0 \Rightarrow x\frac{dP}{dx} = -P$

$$\Rightarrow \int \frac{dP}{P} = \int -\frac{dx}{x}$$

$$\Rightarrow \ln|P| = -\ln|x| + \ln|c_1|$$

$$= \ln\left|\frac{c_1}{x}\right|$$

$$\Rightarrow P = \frac{c_1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{c_1}{x}$$

$$\Rightarrow \int dy = \int c_1 \frac{dx}{x} \Rightarrow \boxed{y = c_1 \ln|x| + c_2}$$

Example.2

Solve $\frac{d^2y}{dx^2} = a\sqrt{1 + (\frac{dy}{dx})^2}$, where a is a constant. $[\, y'' = a\sqrt{1 + (y')^2}\,]$

Solution: Let $P = \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(P) = \frac{dP}{dx}$,

then by substituting we have:

$$\frac{dP}{dx} = a\sqrt{1+P^2}$$

$$\Rightarrow \int \frac{dP}{\sqrt{1+P^2}} = \int a\,dx \Rightarrow \sinh^{-1} P = ax + C_1 \Rightarrow P = \sinh(ax+C_1)$$

and $P = \frac{dy}{dx}$; so $\frac{dy}{dx} = \sinh(ax+C_1) \Rightarrow \int dy = \int \sinh(ax+C_1)dx$

$$\Rightarrow \boxed{y = \frac{1}{a}\cosh(ax+C_1)}$$

Type 2 Equations with dependent variable missing.

When eq. (1) doesn't contain x explicitly but has the form

$$F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0\,, \qquad \ldots (3)$$

the substitutions to use are

$$P = \frac{dy}{dx} \quad and \quad \frac{d^2y}{dx^2} = P\frac{dP}{dy}$$

Then eq. (3) takes the form

$$F\left(y, P, P\frac{dP}{dy}\right) = 0\,,$$

Which is of the first order in P.

Example.3

Solve $\frac{d^2y}{dx^2} + y = 0 \quad [\, y'' + y = 0\,]$

Solution: Let $\frac{dy}{dx} = P$, $\frac{d^2y}{dx^2} = \frac{dP}{dx} = \frac{dP}{dy} \cdot \frac{dy}{dx} = P\frac{dP}{dy}$

Then we proceed as follows:

$$\left(P \cdot \frac{dP}{dy} + y = 0\right) * dy$$

$$\Rightarrow PdP + ydy = 0 \Rightarrow \left[\frac{P^2}{2} + \frac{y^2}{2} = \frac{C_1^2}{2}\right] * 2$$

$$\Rightarrow P \overset{\pm}{=} \sqrt{C_1^2 - y^2} \Rightarrow \int \frac{dy}{\sqrt{C_1^2 - y^2}} = \int \pm dx, \text{ then } \sin^{-1}\frac{y}{C_1} = \pm(x + C_2) \Rightarrow \boxed{y = C_1 \sin^{\pm}(x + C_2)}$$

since C_1 is an arbitrary, there is no need for the $\pm$ sign, and we have $y = C_1 \sin(x + C_2)$ as the general solution.

Example.4

Solve the differential equation $y'' - y = 0$

Solution: Let $P = \frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2} = P\frac{dP}{dy}$ (type.2)

By substituting $\Rightarrow P\frac{dP}{dy} - y = 0$

$$\Rightarrow P\frac{dP}{dy} = y$$

$$\Rightarrow PdP = y\,dy$$

$$\Rightarrow \left[\frac{P^2}{2} = \frac{y^2}{2} + \frac{{C_1}^2}{2}\right] * 2$$

$$\Rightarrow P^2 = y^2 + {C_1}^2$$

$$\Rightarrow P = \pm\sqrt{y^2 + C_1^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\sqrt{y^2 + C_1^2}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y^2 + C_1^2}} = \int \pm x$$

$$\Rightarrow \sinh^{-1}\frac{y}{C_1} = \pm x + C_2$$

$$\Rightarrow \frac{y}{C_1} = \sinh[C_2 \pm x]$$

$$\boxed{\therefore\ y = C_1 . \sinh[C_2 \pm x]}$$

where $\boxed{\int \frac{dy}{\sqrt{y^2 + C_1^2}} = \sinh^{-1}\frac{y}{C_1}}$ and $\boxed{\int \frac{dy}{\sqrt{y^2 - C_1^2}} = \cosh^{-1}\frac{y}{C_1}}$

Note. The hyperbolic functions: $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ have derivatives:

$$\frac{d(\sinh u(x))}{du} = \cosh u(x).\frac{du}{dx}, \qquad \frac{d(\cosh u(x))}{du} = \sinh u(x).\frac{du}{dx}$$

Example.5

Solve the differential equation $y'' + y' = 0$

Solution: Let $P = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ (type 1)

By substituting $\Rightarrow \frac{dP}{dx} + P = 0$

$$\Rightarrow \frac{dP}{dx} = -P$$

$$\Rightarrow \frac{dP}{P} = -dx$$

$$\Rightarrow \int \frac{dP}{P} = -\int dx$$

$$\Rightarrow \ln|P| = -x + c_1$$

$$\Rightarrow \cancel{e^{\ln|P|}} = -x + c_1$$

$$\Rightarrow \frac{dy}{dx} = e^{-x+c_1}$$

$$= e^{-x} \cdot e^{c_1} \quad , \quad \text{let } c_2 = e^{c_1}$$

$$\Rightarrow dy = c_2 e^{-x} dx$$

$$\Rightarrow \int dy = c_2 \int e^{-x} dx$$

$$\Rightarrow \boxed{y = -c_2 e^{-x} + c_3}$$

H.W/ Solve 1. $\frac{d^2y}{dx^2} + \omega^2 y = 0$ $(\omega \; is \; a \; constant \; \neq 0)$

2. $y\, y'' + y'^2 = 0$