2. Homogeneous Linear ODEs with Constant Coefficients

A second -order DE is called linear if it can be written

$$y'' + P(x)y' + q(x)y = r(x) \qquad . . . (A)$$

and nonlinear if it cannot be written in this form.

If $r(x) = 0$, then (A) reduces to

$$y'' + P(x)y' + q(x)y = 0 \qquad . . . (B)$$

And is called homogeneous. If $r(x) \neq 0$, then (A) is called nonhomogeneous.
For instance, $y'' + 25\, y = e^{-x}\cos x$ is nonhomogeneous while, $xy'' + y' + xy = 0$ is homogeneous.
We shall now consider second order homog. Linear ODEs whose coefficients a and b are constant.

$$y'' + a\, y' + b\, y = 0 \qquad . . . (1)$$

These equations have important applications, especially in connection with mechanical and electrical vibrations, as we shall see in the next section.

How to solve (1)? $\quad y'' + a\, y' + b\, y = 0$

Remember that the solution of the first-order linear ODE with a constant coefficient k

$y' + ky = 0$ *is* $y = ce^{-kx}$.

$y' = -ky \Rightarrow \frac{dy}{y} = -k\,dx \Rightarrow \ln y = -kx + \ln C \Rightarrow y = e^{-kx} \cdot e^{\ln C} \Rightarrow y = Ce^{-kx}$

This gives us the idea to try as a solution of (1) the function

$$y = e^{\lambda x} \qquad . . . (2)$$

Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \quad and \quad y'' = \lambda^2 e^{\lambda x}$$

Into our eq. (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

Hence if λ is a solution of the important characteristic equation

$$\lambda^2 + a\lambda + b = 0 \qquad . . . (3)$$

then $y = e^{\lambda x}$ is a solution of the ODE (1).

The roots of this quadratic eq. (3) are

$$\lambda_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

OR $\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right)$ and $\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}) \qquad . . . (4)$

(3) and (4) will be basic because our derivation shows that the functions

$$y_1 = e^{\lambda_1 x} \text{ and } y_2 = e^{\lambda_2 x} \quad . . . \quad (5)$$

are solutions of (1). Verify this by substituting (5) in (1).

From algebra the quadratic equation(3) may have three kinds of roots, depending on the sign of the discriminant a^2-4b,

(Case I) Two real roots if $a^2-4b>0$,

(Case II) A real double root if $a^2-4b=0$,

(Case III) Complex conjugate roots if $a^2-4b<0$.

(Case I) If $\lambda_1 \neq \lambda_2$, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

The corresponding general solution is

$$\boxed{y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}} \quad --- (6)$$

Example. 1 Solve $y'' - y = 0$

Solution. The characteristic eq. is $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\Rightarrow \lambda_1 = 1$, $\lambda_2 = -1$

Hence a basic of solutions is e^{x} and e^{-x} and gives the same general solution as before, $\boxed{y = C_1 e^{x} + C_2 e^{-x}}$

Example. 2 Solve the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5.$$

Solution. First to find the general solution,

the characteristic eq. is

$$\lambda^2 + \lambda - 2 = 0, \quad \text{where } a=1, b=-2$$

$\Rightarrow \lambda_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2-4b})$

Its roots are $\lambda_1 = \frac{1}{2}(-1+\sqrt{1+8}) = \frac{1}{2} * 2 = 1$

$\lambda_2 = \frac{1}{2}(-1-\sqrt{1+8}) = \frac{1}{2} * -4 = -2$

OR: $\lambda^2 + \lambda - 2 = 0$

$(\lambda+2)(\lambda-1) = 0$

$\lambda = -2, \lambda = 1$

So general solution is

$$\boxed{y = C_1 e^{x} + C_2 e^{-2x}}$$

Second To find the particular solution.

Since $y = c_1 e^{x} + c_2 e^{-2x} \Rightarrow y' = c_1 e^{x} - 2c_2 e^{-2x}$

$y(0) = 4 \Rightarrow 4 = c_1 e^{0} + c_2 e^{0}$ and $y'(0) = -5 \Rightarrow -5 = c_1 e^{0} - 2c_2 e^{0}$

Then $4 = c_1 + c_2$ and $-5 = c_1 - 2c_2 \Rightarrow$

$$\begin{array}{r} \cancel{c_1} + c_2 = 4 \\ -\cancel{c_1} \pm 2c_2 = \pm 5 \\ \hline 3c_2 = 9 \Rightarrow c_2 = 3 \\ \text{and } c_1 = 1 \end{array}$$

$\Rightarrow \boxed{y = e^{x} + 3e^{-2x}}$ is particular solution.

(Case II) If $\lambda_1 = \lambda_2 = \frac{-a}{2}$

If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = \frac{-a}{2}$, hence only one solution,

$$y = e^{-(a/2)x}$$

To obtain a second independent solution y_2 (needed for a basis), we use the method of reduction of order.

$\Rightarrow$ The corresponding general solution is

$$\boxed{y = (c_1 + c_2 x) e^{-\frac{a}{2}x}} \quad \text{--- (7)}$$

Warning. If λ is a simple root of (4), then $(c_1 + c_2 x) e^{\lambda x}$ with $c_2 \neq 0$ is not a solution of (1).

Example.3 Solve $y'' + 6y' + 9y = 0$

Solution. The characteristic eq. is $\lambda^2 + 6\lambda + 9 = 0$

$$\Rightarrow (\lambda + 3)^2 = 0$$

It has the double root $\lambda = -3$. Hence a basis is e^{-3x} and xe^{-3x}.

$\Rightarrow y = (c_1 + c_2 x) e^{-3x}$ is the general solution.

Example 4. Solve the initial value problem $y''+y'+0.25y=0$, $y(0)=3.0$, $y'(0)=-3.5$

Solution.

The chara. eq. is $\lambda^2+\lambda+0.25=0$

$(a=1,\ b=0.25)$

$\Rightarrow (\lambda+0.5)^2=0 \Rightarrow$ The general solution $y=(c_1+c_2x)e^{-0.5x}$

We need its derivative $y'=(c_1+c_2x)e^{-0.5x} * (-0.5)+e^{-0.5x}(0+c_2)$

$y'=-0.5(c_1+c_2x)e^{-0.5x}+c_2e^{-0.5x}$, $y'(0)=-3.5$

$\Rightarrow -3.5=-0.5(c_1+0)e^{0}+c_2e^{0}$

$\Rightarrow -3.5=-0.5c_1+c_2$ ---(1)

and $y=(c_1+c_2x)e^{-0.5x}$ with the condition $y(0)=3$

$\Rightarrow 3=(c_1+0)e^{0} \Rightarrow \boxed{c_1=3}$ --- (2)

put (2) in (1);

$\Rightarrow -3.5=-1.5+c_2 \Rightarrow \boxed{c_2=-2}$

$\Rightarrow y=(3-2x)e^{-0.5x}$ is the particular solution.

(Case III) Complex Roots $-\frac{1}{2}a+iw$ and $-\frac{1}{2}a-iw$

This case occurs if the discriminant a^2-4b of the characteristic equation (3) is negative.

$\Rightarrow \lambda_1$ and λ_2 are complex numbers.

$\Rightarrow$ We can obtain a basis of real solutions

$$y_1=e^{-\frac{a}{2}x}\cos wx,\quad y_2=e^{-\frac{a}{2}x}\sin wx,\quad w>0 \quad ---(8)$$

where $w^2=b-\frac{1}{4}a^2$ (We shall derive them systematically in the next course)

Hence a real general solution in Case III is

$$y=e^{-\frac{a}{2}x}(C_1\cos wx+C_2\sin wx), \quad ----(9)$$

C_1 and C_2 are arbitrary constant, w constant not equal zero.

Example.5

Solve $y'' + \omega^2 y = 0$

Solution. The characteristic equation is

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow \lambda^2 = -\omega^2 = i^2\omega^2$$

$$\Rightarrow \lambda = \pm\, i\,\omega \quad where\ i^2 = -1$$

$$\Rightarrow y = C_1 \cos\omega x + C_2 \sin\omega x$$

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Note. We can suppose that $\lambda_1 = \alpha + i\omega$, $\lambda_2 = \alpha - i\omega$ then the general solution is

$$y = e^{\alpha x}(\, C_1 \cos\omega x + C_2 \sin\omega x) , \quad . \ . \ . \quad (9')$$

Where $\alpha = \frac{-a}{2}$ $\quad and\ \omega = \frac{\sqrt{4b-a^2}}{2}$

Example.6

Solve the initial value problem

$$y'' + 0.4\, y' + 9.04\, y = 0\,, \quad y(0) = 0\,, \quad y'(0) = 3$$

Solution. The characteristic equation is

$$\text{P } \lambda^2 + 0.4\,\lambda + 9.04 = 0$$

$\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2}$, where $a = 0.4$, $b = 9.04$

$\lambda = \frac{-0.4}{2} \pm \frac{\sqrt{0.16 - 36.16}}{2}$

$\lambda = -0.2 \pm 3i \Rightarrow \alpha = -0.2$, $\omega = 3$

and a general solution is

$y = e^{-0.2x}(C_1 \cos 3x + C_2 \sin 3x)$

Now use the condition $y(0) = 0 \Rightarrow 0 = e^{0}(C_1 \cos 0 + C_2 \sin 0)$

$\Rightarrow C_1 = 0$

The remaining expression is

$y = C_2 e^{-0.2x} \cdot \sin 3x$

We need the derivative (chain rule)

$\Rightarrow y' = C_2(-0.2e^{-0.2x} \cdot \sin 3x + 3e^{-0.2x} \cdot \cos 3x)$

Since $y'(0) = 3$, then,

$3 = C_2(3e^{0}) \Rightarrow C_2 = 1$

$\Rightarrow$ The solution is

$\boxed{y = e^{-0.2x} \sin 3x}$

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Solving Problems

Q_1/ Solve the initial value problem. Check that your answer satisfies the ODEs as well as the initial condition.

① $y'' + 2ky' + (k^2+w^2)y = 0$, $y(0)=1$, $y'(0) = -k$.

Sol. The chara. eq. is:

$$\lambda^2 + 2k\lambda + (k+w^2) = 0 \Rightarrow \lambda = \frac{-2k \pm \sqrt{4k^2 - 4(k^2+w^2)}}{2}$$

$$\Rightarrow \lambda = -k \pm wi$$

$\therefore$ $y = e^{-kx}(c_1 \cos wx + c_2 \sin wx)$ is the general sol.

$$\Rightarrow y' = e^{-kx}(-c_1 w \sin wx + c_2 w \cos wx) - ke^{-kx}(c_1 \cos wx + c_2 \sin wx)$$

and $y(0)=1 \Rightarrow 1 = e^0(c_1 \cos 0 + \cancel{c_2 \sin 0})$

$$\Rightarrow \boxed{1 = c_1}$$

$$y'(0) = -k \Rightarrow -k = e^0(\cancel{-c_1 w \sin 0}^{=0} + c_2 w \overset{=1}{\cos 0}) - ke^0(c_1 \overset{=1}{\cos 0} + c_2 \cancel{\sin 0}^{=0})$$

$$-k = c_2 w - kc_1$$

$$c_2 w = k - k = 0 \Rightarrow \boxed{c_2 = 0}$$

$$\Rightarrow y = e^{-kx}\cos wx$$

Check the answer.

$$y = e^{-kx}\cos wx \Rightarrow y' = e^{-kx}\cdot(-w\sin wx) - ke^{-kx}\cos wx$$

$$y'' = e^{-kx}(-w^2\cos wx) + kwe^{-kx}\sin wx - ke^{-kx}(-w\sin wx) + k^2e^{-kx}\cos wx$$

substitute y, y' and y'' in the gerenal eq. and check if it's equal zero.

② $10y'' - 50y' + 65y = 0$, $y(0) = 1.5$, $y'(0) = 1.5$

Sol. The chara. eq. is

$$(10\lambda^2 - 50\lambda + 65 = 0) \div 10$$

$$\lambda^2 - 5\lambda + 6.5 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{25 - 4(6.5)}}{2} = \frac{5 \pm \sqrt{25-26}}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{5}{2} \pm \frac{1}{2}i = 2.5 \pm 0.5i$$

Then the general solution is

$$y = e^{2.5x}(c_1 \cos 0.5x + c_2 \sin 0.5x)$$

$$y' = 2.5e^{2.5x}(c_1 \cos 0.5x + c_2 \sin 0.5x) + e^{2.5x}(-0.5c_1 \sin 0.5x + 0.5c_2 \cos 0.5x)$$

using the cond. $y(0) = 1.5 \Rightarrow \boxed{1.5 = c_1}$

and $y'(0) = 1.5 \Rightarrow 1.5 = 2.5c_1 + 0.5c_2 \Rightarrow 1.5 = 2.5(1.5) + 0.5c_2$

$$\Rightarrow \boxed{c_2 = -4.5}$$

$\Rightarrow y = e^{2.5x}(1.5\cos 0.5x - 4.5\sin 0.5x)$ is a particular solution

Q2/ Find an ODE $y'' + ay' + by = 0$ for the given basis.

1. e^{2x}, e^{x} $\Rightarrow$ Solution. $\lambda_1 = 2,\ \lambda_2 = 1$

$\Rightarrow (\lambda - 2)(\lambda - 1) = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \boxed{y'' - 3y' + 2y = 0}$ is the ODE

2. $e^{0.5x}, e^{-3.5x}$

Solution. $\lambda_1 = 0.5,\ \lambda_2 = -3.5$

$\Rightarrow (\lambda - 0.5)(\lambda + 3.5) = 0$

$\Rightarrow \lambda^2 + 3\lambda - 1.75 = 0$

$\Rightarrow \boxed{y'' + 3y' - 1.75y = 0}$ is the ODE

3. $e^{x\sqrt{3}}, xe^{x\sqrt{3}}$

Solution. $\lambda_1 = \lambda_2 = \sqrt{3}$ $\Rightarrow$ characteristic equation is $(\lambda - \sqrt{3})^2 = 0$

$\Rightarrow \lambda^2 - 2\sqrt{3}\lambda + 3 = 0$

$\Rightarrow \boxed{y'' - 2\sqrt{3}y' + 3y = 0}$ is the ODE

4. $1, e^{-3x}$

Solution. $\lambda_1 = 0,\ \lambda_2 = -3 \Rightarrow \lambda(\lambda + 3) = 0$

$\Rightarrow \lambda^2 + 3\lambda = 0$

$\Rightarrow \boxed{y'' + 3y' = 0}$ is the ODE

5. e^{4x}, e^{-4x}

Solution. $\lambda_1 = 4,\ \lambda_2 = -4$

$\Rightarrow (\lambda - 4)(\lambda + 4) = 0$

$\Rightarrow \lambda^2 - 16 = 0 \Rightarrow \boxed{y'' - 16y} = 0$ is the ODE

6. $e^{(-1+i)x}, e^{-(1+i)x}$ $\qquad \lambda^2 + a\lambda + b = 0$

$\Rightarrow \lambda_{1,2} = -1 \pm i \quad (\lambda = \alpha \pm i\omega) \Rightarrow -1 \pm i = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

$\Rightarrow \alpha = -1,\ \omega = 1 \Rightarrow \dfrac{-a}{2} = -1$ and $\dfrac{\sqrt{a^2 - 4b}}{2} = 1$

$\Rightarrow a = 2$ and $\sqrt{4 - 4b} = 2 \Rightarrow \sqrt{1 - b} = 1 \Rightarrow 1 - b = 1$

$\Rightarrow b = 0 \Rightarrow \lambda^2 + 2\lambda + \cancel{b}^{\,0} = 0 \Rightarrow \boxed{y'' + 2y' = 0}$ is the ODE