TISHK UNIVERSITY

ENGINEERING FACULTY

COMPUTER ENGINEERING DEPARTMENT

COURSE NAME:

ENGINEERING MATHEMATICS

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Chapter one

Ordinary Differential Equations

1.1 Introduction

A **differential equation** is an algebraic equation that contains some **derivatives**. Differential equations are classified by:

a) Type (namely; ordinary or partial)
b) Order (is the order of the highest derivative that it contain)
c) Degree (is the highest power of the highest derivative)

For example, $(\frac{d^3y}{dx^3})^2 + (\frac{d^2y}{dx^2})^5 + \left(\frac{y}{x^2+1}\right) = e^x$ is an ordinary differential equation, of order 3 and degree 2

If the dependent variable y is a function of two or more independent variables, say $y = f\ (x, t)$ where x and t are independent variables, then partial derivatives of y may occur.

For example, $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ is a **partial differential equation**, of order 2 and degree one.

Examples

1. $\frac{dy}{dx} + 5y = 3t$ First Order Equation
2. $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 3y = 0$ Second Order Equation

In these two examples, ***y*** is the **dependent variable** and ***t*** and ***x*** are the **independent variables**, respectively.

1.2 Why study differential equations?

Some examples of field using differential equations in their analysis include:

- Solid machines and motion.
- Heat transfer and energy balance.
- Vibrational dynamics and seismology.
- Aerodynamics and fluid dynamics.
- Electronics and circuit design.
- Population dynamics and biological systems.
- Climatology and environmental analysis.
- Options trading and economics.

1.3 Some Applications of Differential Equations

To explain the application that leading to differential equations, let us illustrate the following examples:

Example.1 Newton's Law of Cooling

According to Newton's law of cooling, the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if T_m is the temperature of the medium and $T = T(t)$ is the temperature of the body at time t, then

$$\frac{dT}{dt} = -k\,(T - T_m)$$

$$\Rightarrow \frac{dT}{T-T_m} = -k\,dt$$

$$\Rightarrow \int \frac{dT}{T-T_m} = -k \int dt$$

$$\Rightarrow \ln|T - T_m| = -kt + C_1$$

$$\Rightarrow |T - T_m| = e^{-kt+C_1}$$

$$= Ce^{-kt}, \quad where\ C = e^{C_1} \text{ is a constant.}$$

Where k is a positive constant and the minus sign indicates that the temperature of the body increases with time if it's less than the temperature of the medium, or decreases if it's greater. We see that if T_m is constant then the solution is $T = T_m + (T_0 - T_m)\, e^{-kt}$.

Example.2 Motion

A drag racer accelerates from a stop so that its speed is **40*t* feet per second** *t* seconds after starting. How **far** will the car go in **8** seconds?

Given: $\frac{ds}{dt} = 40t$, where S(t) is time in seconds. **Find:** S(8) Ft

Solution: $\frac{dS}{dt} = 40t \quad \Longrightarrow \quad dS = 40t\,dt$

$$\Longrightarrow S(t) = \int 40t\,dt = 20t^2 + C$$

Apply the initial condition S(0)=0

S(0) = 0 = 20(0) + C $\Longrightarrow$ C = 0 $\Longrightarrow S(t) = 20\,t^2 \Longrightarrow S(8) = 20(8)^2 = 1280$ ft

$\Longrightarrow$ *the car travels* 1280 *feet in* 8 *secon*

1.4 First Order Differential Equations (ODEs)

1. Separable equations

If it is possible to write the equation in the form

$$\frac{dy}{dx} = f(x)g(y),$$

then the general solution is

$$\int \frac{dy}{g(y)} = f(x)dx$$

Example.1

Consider the differential equation $\frac{dy}{dx} = \frac{x}{y^2}$. Find the general solution, then find the particular solution that satisfies the initial condition $y(0) = 2$.

Solution. (a)

Step.1 separate the variables

$$y^2 dy = x\,dx$$

Step.2 Integrate both sides

$$\int y^2 dy = \int x\,dx$$

$$\Rightarrow \quad \frac{y^3}{3} = \frac{x^2}{2} + C$$

Step.3 Solve for the dependent variable:

$$y = (\tfrac{3}{2}x^2 + 3C)^{1/3}$$

$$= (\tfrac{3}{2}x^2 + D)^{1/3} \text{ , where D = 3C}$$

This is the general solution.

(b) Apply the initial condition, that is:

$$y(0) = 2 \Rightarrow (0, 2) = (x, y)$$

$$\Rightarrow \quad 2 = (\tfrac{3}{2}(0)^2 + D)^{1/3}$$

$$\Rightarrow \quad 2 = D^{1/3} \qquad \Longrightarrow D = 8$$

Thus, the particular solution we are looking for is

$$y = (\tfrac{3}{2}x^2 + 8)^{1/3}$$

■

Example.2

Solve the differential equation $(x+1)\frac{dy}{dx} = x(y^2+1)$

Solution. $[\,(x+1)dy = x(y^2+1)\,dx\,] \div (x+1)(y^2+1)$

$$\Rightarrow \int \frac{dy}{(y^2+1)} = \int \frac{x\,dx}{(x+1)}$$
$$\Rightarrow \quad \tan^{-1} y = \int \frac{x+1-1\,dx}{(x+1)} = \int \left[\frac{x+1}{x+1} - \frac{1}{x+1}\right] dx$$
$$\Rightarrow \quad \tan^{-1} y = [x - ln|x+1| + C] \quad general\ solution \qquad \blacksquare$$

Example.3

Solve the equation $\frac{dy}{dt} = ky$

Solution. $\int \frac{\mathrm{dy}}{\mathrm{y}} = \int \mathrm{k\,dt}$

$\Rightarrow \ln|y| = kt + C$

$\Rightarrow \mathrm{e}^{\ln|y|} = e^{kt+C}$

$\Rightarrow |\mathrm{y}| = e^{kt}.e^{C}$

$\Rightarrow \quad y = \pm e^{C}.\, e^{kt} = D.e^{kt} \qquad where\ \ D = \pm e^{C}$ $\blacksquare$

Example.4

Solve the equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

Solution. $\int (1-y^2)\mathrm{dy} = \int x^2\ \mathrm{dt}$

$$\Rightarrow \left[\ y - \frac{y^3}{3} = \frac{x^3}{3} + c\ \right] * 3$$
$$\Rightarrow [\ \ 3y - y^3 = x^3 + d\ \ ]\ \ where\ d = 3c \qquad \blacksquare$$

Example.5

Verify that y is a solution of the ODE. Determine from y the particular solution satisfying the given initial condition.

a. $y' = 0.5\,y\,, \quad y = ce^{0.5x}, \quad y(2) = 2$
b. $y' = y\tan x\ , \quad y = c\ sec\ x, \quad y(0) = \frac{\pi}{2}$

Solution. a. . $y = ce^{0.5x} \Longrightarrow y' = 0.5\ c\ e^{0.5x} \Longrightarrow y' = 0.5(y)$

$y(2) = 2 \quad \Longrightarrow$ $(2, 2)$ Satisfy the function $y = ce^{0.5x}$

$\Longrightarrow 2 = c\ e^{0.5(2)} = c\ e$

$\Longrightarrow c = \frac{2}{e}$

So $\quad y = \frac{2}{e}e^{0.5\,x} = 2e^{0.5x-1}$ is the particular solution $\blacksquare$

b. $y = c\, sec\, x \Longrightarrow y' = c \sec x \tan x \Longrightarrow y' = y \tan x$

$y(0) = \frac{\pi}{2} \Longrightarrow (0, \frac{\pi}{2})$ satisfy the function $y = c\, sec\, x$

$$\Longrightarrow \frac{\pi}{2} = c\, sec\, 0 = c.1$$

$$\Longrightarrow c = \frac{\pi}{2} \qquad \Longrightarrow y = \frac{\pi}{2} \sec x$$

is the particular solution ■

Example.6

State the order of the ODE. Verify that the given function is a solution (a, b and c are arbitrary constants)

a. $y'' + 2y' + 10y = 0\,, \quad y = 4e^{-x} \sin 3x$

b. $y''' = \cos x\,, \quad y = -sinx + a\,x^2 + bx + c$

Solution. $\boldsymbol{a}.$ $y'' + 2y' + 10y = 0$ *second order*

$y = 4e^{-x} \sin 3x \quad \Longrightarrow \quad y' = 4e^{-x}(3 \cos 3x) + (-4e^{-x}) \sin 3x$

$\Longrightarrow y' = 4e^{-x}(3 \cos 3x - \sin 3x)$ .....(1)

$\Longrightarrow y'' = 4e^{-x}(-9 \sin 3x - 3\, cos3x) - 4\, e^{-x}(3\, cos3x - \sin x)$.....(2)

Substitute eq.(1) &(2) in the ODE $\Longrightarrow$ $4e^{-x}(-9 \sin 3x - 3\, cos3x) - 4e^{-x}(3cos3x - \sin x) + 2 *$

$4e^{-x}(3 \cos 3x - \sin 3x) + 10\ (4e^{-x} \sin 3x)$

$= 4e^{-x}[-9 \sin 3x - 3\, cos\, 3x - 3 \cos 3x + \sin x + 6 \cos 3x - 2 \sin 3x + 10 \sin 3x] = 0$

■

b. $y''' = \cos x\,, \quad y = -sinx + a\,x^2 + bx + c$ (H.W)

H.W. Solve the initial value problems

1. $(4y - cosy)\frac{dy}{dx} - 3x^2 = 0\,,\ y(0) = 0$
2. $\frac{dy}{dt} + y = 2\,, \quad y(0) = 1$
3. $\frac{dy}{dt} = \frac{2t+1}{2y-2}\ , \qquad y(0) = -1$