EoMs for Translation

$$\left.\frac{d\left(m\vec{V}\right)}{dt}\right|_{abs} = \sum \vec{F}_{ext}$$

$$\vec{a}_{b_{abs}} = \left.\frac{\mathrm{d}\vec{V}_b}{\mathrm{d}t}\right|_{Oxyz} + \vec{\omega}_b \times \vec{V}_b$$

$$\vec{V}_b = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad \vec{\omega}_b = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

so we have

$$\vec{a}_{b_{abs}} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{vmatrix} i & j & k \\ P & Q & R \\ U & V & W \end{vmatrix} = \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

we have defined our **absolute acceleration terms in body axes**, which means we can define the LHS of Equation

$$\left.\frac{d\left(m\vec{V}\right)}{dt}\right|_{abs} = m\begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

$$\sum \vec{F}_b = \vec{F}_{G_b} + \vec{F}_{A_b} + \vec{F}_{P_b}$$

We presume the aircraft has one or more propulsions providing thrust, T, along a vector defined in the **x/z** plane, at angle $\boldsymbol{\theta_T}$ to **x**. We include an additional term for side wash effects due to propulsion, $\mathbf{F_{Ty}}$

$$\vec{F}_{T_b} = \begin{bmatrix} F_{Tx} \\ F_{Ty} \\ F_{Tz} \end{bmatrix} = \begin{bmatrix} T \cdot \cos\theta_T \\ F_{Ty} \\ -T \cdot \sin\theta_T \end{bmatrix}$$

equations of motion for translation:

$$m \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix} = \begin{matrix} -mg\sin\theta - D\cos\alpha + L\sin\alpha + T\cos\theta_T \\ mg\sin\phi\cos\theta + F_{A_Y} + F_{T_Y} \\ mg\cos\phi\cos\theta - D\sin\alpha - L\cos\alpha - T\sin\theta_T \end{matrix}$$

EoMs for Rotation

Newton's second law: *the angular acceleration is proportional to the net torque and inversely proportional to the moment of inertia*. *angular momentum* is defined as:

$$\vec{H} = \vec{r} \times \left(m\vec{V}\right) \qquad \left.\frac{d\vec{H}}{dt}\right|_{abs} = \vec{M}$$

$$\vec{H} = \begin{matrix} H_x \\ H_y \\ H_z \end{matrix} = \begin{matrix} P \cdot I_{xx} - Q \cdot I_{xy} - R \cdot I_{xz} \\ Q \cdot I_{yy} - R \cdot I_{yz} - P \cdot I_{xy} \\ R \cdot I_{zz} - P \cdot I_{xz} - Q \cdot I_{yz} \end{matrix}$$

$$= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= [\,I\,]\,\vec{\omega}$$

Simplification of the Inertia Tensor via symmetry

Most aircraft are symmetric about a longitudinal/vertical plane, and will also tend to have the principal yaw tensor axis aligned withZb. This means we can assume that: **Ixy = Iyx = Izy =Iyz = 0**

$$\vec{H} = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{zx} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= [\,I\,]\,\vec{\omega}$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} \cdot P - I_{xz} \cdot R \\ I_{yy} \cdot Q \\ -I_{xz} \cdot P + I_{zz} \cdot R \end{bmatrix}$$

A bsolute rate of change of angular momentum

$$\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}_{abs} = \frac{\mathrm{d}\vec{H}}{\mathrm{d}t}_{Oxyz} + \vec{\omega} \times \vec{H}_{Oxyz}$$

$$= \frac{\mathrm{d}\,[\,I\,]\,\vec{\omega}}{\mathrm{d}t}_{Oxyz} + \vec{\omega} \times ([\,I\,]\,\vec{\omega})$$

$$= \frac{\mathrm{d}\,[\,I\,]}{\mathrm{d}t}\vec{\omega} + [\,I\,]\,\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ P & Q & R \\ (I_{xx} \cdot P - I_{xz} \cdot R) & (I_{yy} \cdot Q) & (-I_{xz} \cdot P + I_{zz} \cdot R) \end{vmatrix}$$

The products and moments of inertia are constant in time, hence the time rate of change of [I]cancels to zero

$$\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}_{abs} = \frac{\mathrm{d}\,[\,I\,]}{\mathrm{d}t} \cdot \vec{\omega} + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \cdot [\,I\,] \quad + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ P & Q & R \\ (I_{xx} \cdot P - I_{xz} \cdot R) & (I_{yy} \cdot Q) & (-I_{xz} \cdot P + I_{zz} \cdot R) \end{vmatrix}$$

$$= \begin{bmatrix} I_{xx} \cdot \dot{P} - I_{xz} \cdot \dot{R} \\ I_{yy} \cdot \dot{Q} \\ I_{zz} \cdot \dot{r} - I_{xz} \cdot \dot{P} \end{bmatrix} + \begin{bmatrix} Q\,(I_{zz} \cdot R - I_{xz} \cdot P) - R\,(I_{yy} \cdot Q) \\ R\,(I_{xx} \cdot P - I_{xz} \cdot R) - P\,(I_{zz} \cdot R - I_{xz} \cdot P) \\ P\,(I_{yy} \cdot Q) - Q\,(I_{xx} \cdot P - I_{xz} \cdot R) \end{bmatrix}$$

collecting terms of the moments and products of inertia yields

$$\begin{bmatrix} \dot{H}_x \\ \dot{H}_y \\ \dot{H}_z \end{bmatrix}_{abs} = \begin{bmatrix} \dot{P} \cdot I_{xx} & + & Q \cdot R\,(I_{zz} - I_{yy}) & - & \left(\dot{R} + P \cdot Q\right) I_{xz} \\ \dot{Q} \cdot I_{yy} & + & P \cdot R\,(I_{xx} - I_{zz}) & + & (P^2 - R^2)\, I_{xz} \\ \dot{R} \cdot I_{zz} & + & P \cdot Q\,(I_{yy} - I_{xx}) & + & \left(Q \cdot R - \dot{P}\right) I_{xz} \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

Summary of the equations of motion#

With the translational (Eq. (29)) and rotational (Eq. (33)) defined, and the relationship between the Euler angles and the body rates (Eqs. `eq:eulerratetoomega` and (18)), fully unconstrained flight in 6DoF can be described.

In total twelve equations have been derived (body/attitude rates are only three but the matrix is tough to invert, so I'll include both versions below) that can describe aircraft position and attitude in a Newtonian framework:

Translational Motion:

$$m\begin{bmatrix}\dot{U}+QW-RV\\ \dot{V}+RU-PW\\ \dot{W}+PV-QU\end{bmatrix}=\begin{matrix}-mg\sin\theta-D\cos\alpha+L\sin\alpha+T\cos\theta_T\\ mg\sin\phi\cos\theta+F_{A_Y}+F_{T_Y}\\ mg\cos\phi\cos\theta-D\sin\alpha-L\cos\alpha-T\sin\theta_T\end{matrix}$$

Body angular rate due to an attitude rate:

$$\begin{bmatrix}P\\ Q\\ R\end{bmatrix}=\begin{bmatrix}1 & 0 & -\sin\theta\\ 0 & \cos\phi & \sin\phi\cos\theta\\ 0 & -\sin\phi & \cos\phi\cos\theta\end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}$$

Angular Motion:

$$\begin{bmatrix}\dot{P}\cdot I_{xx} & + & Q\cdot R\,(I_{zz}-I_{yy}) & - & \left(\dot{R}+P\cdot Q\right)I_{xz}\\ \dot{Q}\cdot I_{yy} & + & P\cdot R\,(I_{xx}-I_{zz}) & + & (P^2-R^2)\,I_{xz}\\ \dot{R}\cdot I_{zz} & + & P\cdot Q\,(I_{yy}-I_{xx}) & + & \left(Q\cdot R-\dot{P}\right)I_{xz}\end{bmatrix}=\begin{bmatrix}L\\ M\\ N\end{bmatrix}$$

Attitude rate due to a body rate

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}=\begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta}\end{bmatrix}\begin{bmatrix}P\\ Q\\ R\end{bmatrix}$$

Stability Derivatives

Reduced derivative force and moment perturbations:

$$\Delta X = \left.\frac{\partial X}{\partial u}\right|_0 u + \left.\frac{\partial X}{\partial w}\right|_0 w$$

$$\Delta Y = \left.\frac{\partial Y}{\partial v}\right|_0 v + \left.\frac{\partial Y}{\partial \delta_r}\right|_0 \delta_r$$

$$\Delta Z = \left.\frac{\partial Z}{\partial u}\right|_0 u + \left.\frac{\partial Z}{\partial w}\right|_0 w + \left.\frac{\partial Z}{\partial \delta_e}\right|_0 \delta_e$$

$$\Delta L = \left.\frac{\partial L}{\partial v}\right|_0 v + \left.\frac{\partial L}{\partial p}\right|_0 p + \left.\frac{\partial L}{\partial r}\right|_0 r + \left.\frac{\partial L}{\partial \delta_r}\right|_0 \delta_r + \left.\frac{\partial L}{\partial \delta_a}\right|_0 \delta_a$$

$$\Delta M = \left.\frac{\partial M}{\partial u}\right|_0 u + \left.\frac{\partial M}{\partial w}\right|_0 w + \left.\frac{\partial M}{\partial \dot{w}}\right|_0 \dot{w} + \left.\frac{\partial M}{\partial q}\right|_0 q + \left.\frac{\partial M}{\partial \delta_a}\right|_0 \delta_a$$

$$\Delta N = \left.\frac{\partial N}{\partial v}\right|_0 v + \left.\frac{\partial N}{\partial p}\right|_0 p + \left.\frac{\partial N}{\partial r}\right|_0 r + \left.\frac{\partial N}{\partial \delta_r}\right|_0 \delta_r + \left.\frac{\partial N}{\partial \delta_a}\right|_0 \delta_a$$

The **linear force and moment perturbations**, may be substituted into the **linear equations of motion**, to have:

$$m\dot{u} = \left.\frac{\partial X}{\partial u}\right|_0 u + \left.\frac{\partial X}{\partial w}\right|_0 w - mg \cdot \cos\theta_0 \cdot \theta'$$

$$m\dot{v} = \left.\frac{\partial Y}{\partial v}\right|_0 v - mU_0 r + mg \cdot cos\theta_0 \cdot \phi' \left.\frac{\partial Y}{\partial \delta_r}\right|_0 \delta_r$$

$$m\dot{w} = \left.\frac{\partial Z}{\partial u}\right|_0 u + \left.\frac{\partial Z}{\partial w}\right|_0 w + mU_0 q - mg \cdot \sin\Theta_0 \cdot \theta' + \left.\frac{\partial Z}{\partial \delta_e}\right|_0 \delta_e$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = \frac{\partial L}{\partial v}\bigg|_0 v + \frac{\partial L}{\partial p}\bigg|_0 p + \frac{\partial L}{\partial r}\bigg|_0 r + \frac{\partial L}{\partial \delta_r}\bigg|_0 \delta_r + \frac{\partial L}{\partial \delta_a}\bigg|_0 \delta_a$$

$$I_{yy}\dot{q} = \frac{\partial M}{\partial u}\bigg|_0 u + \frac{\partial M}{\partial w}\bigg|_0 w + \frac{\partial M}{\partial \dot{w}}\bigg|_0 \dot{w} + \frac{\partial M}{\partial q}\bigg|_0 q + \frac{\partial M}{\partial \delta_e}\bigg|_0 \delta_e$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = \frac{\partial N}{\partial v}\bigg|_0 v + \frac{\partial N}{\partial p}\bigg|_0 p + \frac{\partial N}{\partial r}\bigg|_0 r + \frac{\partial N}{\partial \delta_r}\bigg|_0 \delta_r + \frac{\partial N}{\partial \delta_a}\bigg|_0 \delta_a$$

Concise Form of Dimensional EoMs

$$m\dot{u} = \frac{\partial X}{\partial u}\bigg|_0 u + \frac{\partial X}{\partial w}\bigg|_0 w - mg \cdot \cos\theta_0 \cdot \theta'$$

$$m\dot{w} = \frac{\partial Z}{\partial u}\bigg|_0 u + \frac{\partial Z}{\partial w}\bigg|_0 w + mU_0 q - mg \cdot \sin\Theta_0 \cdot \theta' + \frac{\partial Z}{\partial \delta_e}\bigg|_0 \delta_e$$

$$I_{yy}\dot{q} = \frac{\partial M}{\partial u}\bigg|_0 u + \frac{\partial M}{\partial w}\bigg|_0 w + \frac{\partial M}{\partial \dot{w}}\bigg|_0 \dot{w} + \frac{\partial M}{\partial q}\bigg|_0 q + \frac{\partial M}{\partial \delta_e}\bigg|_0 \delta_e$$

The X-Force equation (*forward speed equation*):

$$\dot{u} = X_u u + X_w w - g \cdot \cos\theta_0 \theta' \qquad X_u \triangleq \frac{1}{m}\frac{\partial X}{\partial u}\bigg|_0, X_w \triangleq \frac{1}{m}\frac{\partial X}{\partial w}\bigg|_0$$

The Z-Force equation (*heave equation*):

$$\dot{w} = Z_u u + Z_w w + U_e q - g \cdot \sin\theta_0 \cdot \theta' + Z_{\delta_e}\delta_e$$

$$Z_u \triangleq \frac{1}{m}\frac{\partial Z}{\partial u}\bigg|_0, Z_w \triangleq \frac{1}{m}\frac{\partial Z}{\partial w}\bigg|_0, Z_{\delta_e} \triangleq \frac{1}{m}\frac{\partial Z}{\partial \delta_e}\bigg|_0$$

The M-Moment equation (*pitching moment equation*):

$$\dot{q} = M_u^* u + M_w^* w + M_{\dot{\omega}}^* \omega + M_q^* q + M_{\delta_e}^* \delta_e$$

$$M_u^* \triangleq M_u + M_{\dot{w}} Z_u \qquad M_w^* \triangleq M_w + M_{\dot{w}} Z_w \qquad M_q^* \triangleq M_q + M_{\dot{w}} U_0$$

$$M^*_{\delta_e} \triangleq M_{\delta_e} + M_{\dot{w}} Z_{\delta_e} \qquad M^*_{\theta} \triangleq -M_{\dot{w}} g \sin\theta_0$$

The linearised Euler pitch equation (*pitch rate kinematic equation*):

q is pitch rate, = **θ'**

Now Equations can be written in matrix form to give the **linearised equation of longitudinal motion in concise form**:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g\cdot\cos\theta_0 \\ Z_u & Z_w & U_0 & -g\cdot\sin\theta_0 \\ M^*_u & M^*_w & M^*_q & M^*_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ Z_{\delta_e} \\ M^*_{\delta_e} \\ 0 \end{bmatrix} [\delta_e]$$

Lateral/Directional EoMs

The asymmetric or lateral/directional EoMs may be similarly manipulated.

$$m\dot{v} = \left.\frac{\partial Y}{\partial v}\right|_0 v - mU_0 r + mg\cdot cos\theta_0 \cdot \phi' \left.\frac{\partial Y}{\partial \delta_r}\right|_0 \delta_r$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = \left.\frac{\partial L}{\partial v}\right|_0 v + \left.\frac{\partial L}{\partial p}\right|_0 p + \left.\frac{\partial L}{\partial r}\right|_0 r + \left.\frac{\partial L}{\partial \delta_r}\right|_0 \delta_r + \left.\frac{\partial L}{\partial \delta_a}\right|_0 \delta_a$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = \left.\frac{\partial N}{\partial v}\right|_0 v + \left.\frac{\partial N}{\partial p}\right|_0 p + \left.\frac{\partial N}{\partial r}\right|_0 r + \left.\frac{\partial N}{\partial \delta_r}\right|_0 \delta_r + \left.\frac{\partial N}{\partial \delta_a}\right|_0 \delta_a$$

The Y-Force equation (*sideslip equation*)

$$\dot{v} = Y_v v - U_e r + g\cdot\cos\theta_0 \phi + Y_{\delta_r}\delta_r \qquad \text{where} \qquad Y_v \triangleq \frac{1}{m}\frac{\partial Y}{\partial v},\ Y_{\delta_r} \triangleq \frac{1}{m}\frac{\partial Y}{\partial \delta_r}$$

The two moment equations are coupled, so need to be decoupled. Dividing Equation by the rolling moment of inertia, **Ixx**:

$$\dot{p} = \frac{I_{xz}}{I_{xx}}\dot{r} + L_v v + L_p p + L_r r + L_{\delta_r}\delta_r + L_{\delta_a}\delta_a \qquad L_v \triangleq \frac{1}{I_{xx}}\frac{\partial L}{\partial v},\ L_p \triangleq \frac{1}{I_{xx}}\frac{\partial L}{\partial p},$$

the same can be performed , dividing by the yawing moment of inertia **Izz**:

$$\dot{r} = \frac{I_{xz}}{I_{zz}}\dot{p} + N_v v + N_p p + N_r r + N_{\delta_r}\delta_r + N_{\delta_a}\delta_a$$

Lastly, the yaw rate and body rate kinematic equations give:

$r = \dot{\psi}\cos\theta_0 \quad \dot{\psi} = r \cdot \sec\theta_0 \quad p = \dot{\phi} - \dot{\psi}\sin\theta_0 \quad \dot{\phi} = p + r \cdot \tan\theta_0$

So finally, the *linearised lateral/directional equations of motion* may be expressed in *state-space form*

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} Y_v & 0 & -U_0 & g \cdot \cos\theta_0 & 0 \\ L_v^* & L_p^* & L_r^* & 0 & 0 \\ N_v^* & N_p^* & N_r^* & 0 & 0 \\ 0 & 1 & \tan\theta_0 & 0 & 0 \\ 0 & 0 & \sec\theta_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} Y_{\delta_r} & 0 \\ L_{\delta_r}^* & L_{\delta_a}^* \\ N_{\delta_r}^* & N_{\delta_a}^* \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix}$$

The two sets of matrix equations, , are known as *the equations of motion in concise form*.

These are in **state space form**

$$\dot{\vec{x}} = \boldsymbol{A}\vec{x} + \boldsymbol{B}\vec{u}$$

where $\vec{x} = \text{the state vector (n)}$ $\vec{u} = \text{the control matrix (m)}$

$\boldsymbol{A} = \text{the system matrix (n by n)}$ $\boldsymbol{B} = \text{the control matrix (n by m)}$