Dynamic Stability

1. Background

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t)$$

Solution:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{c}{m}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{k}{m}\right)x = 0$$

Solutions of this equation are generally of the form $x = Ae^{\lambda t}$

$$\lambda^2 + \left(\frac{c}{m}\right)\lambda + \left(\frac{k}{m}\right) = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_n\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_n^2 x = 0 \qquad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\lambda = \begin{cases} -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} & \text{for } \zeta > 1 \\ -\zeta\omega_n & \text{for } \zeta = 1 \\ -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} & \text{for } \zeta < 1 \end{cases}$$

Overdamped System

For cases in which $\zeta > 1$, the characteristic equation has two (distinct) real roots, and the solution takes the form:

$$x = a_1e^{\lambda_1 t} + a_2e^{\lambda_2 t} \qquad \lambda_1 = -\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) \qquad \lambda_2 = -\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right)$$

The constants a1 and a2 are determined from the initial conditions

$$x(0) = a_1 + a_2 \qquad \dot{x}(0) = a_1\lambda_1 + a_2\lambda_2 \qquad \text{in matrix form,} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix}$$

Critically Damped System: When the damping ratio $\zeta = 1$, $\lambda_1 = \lambda_2 = -\omega_n$

The constants a_1 and a_2 are again determined from the initial conditions

$$x = (a_1 + a_2t)\,e^{-\omega_n t} \qquad \begin{aligned} x(0) &= a_1 \\ \dot{x}(0) &= a_1\lambda_1 + a_2 \end{aligned}$$

Underdamped System; When the damping ratio ζ < 1,

$$\lambda_1 = \omega_n \left(-\zeta + i\sqrt{1-\zeta^2}\right) \qquad \lambda_2 = \omega_n \left(-\zeta - i\sqrt{1-\zeta^2}\right)$$

$$x = e^{-\zeta\omega_n t}\left[a_1 \cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + a_2 \sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$$

$$x(0) = a_1 \qquad \dot{x}(0) = -\zeta\omega_n a_1 + \omega_n\sqrt{1-\zeta^2}a_2$$

in matrix form $\begin{pmatrix} 1 & 0 \\ -\zeta\omega_n & \omega_n\sqrt{1-\zeta^2} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix}$

The period is given by: $T = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

time to damp to $1/n$ times the initial amplitude is given by[1] $t_{1/n} = \frac{\ln n}{\omega_n\zeta}$

2. **<u>Longitudinal Motions</u>**

Recall that the linearized equations describing small longitudinal perturbations from a longitudinal equilibrium state can be written

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - X_u\right]u + g_0\cos\Theta_0\theta - X_w w = X_{\delta_e}\delta_e + X_{\delta_T}\delta_T$$

$$-Z_u u + \left[(1 - Z_{\dot{w}})\frac{\mathrm{d}}{\mathrm{d}t} - Z_w\right]w - [u_0 + Z_q]\,q + g_0\sin\Theta_0\theta = Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T$$

$$-M_u u - \left[M_{\dot{w}}\frac{\mathrm{d}}{\mathrm{d}t} + M_w\right]w + \left[\frac{\mathrm{d}}{\mathrm{d}t} - M_q\right]q = M_{\delta_e}\delta_e + M_{\delta_T}\delta_T$$

If we introduce the longitudinal state variable vector $\mathbf{x} = [u \quad w \quad q \quad \theta]^T$

and the longitudinal control vector $\eta = [\delta_e \quad \delta_T]^T$

And it is common to neglect $Z_{\dot{w}}$ with respect to unity and to neglect Zq relative to u0, Hence represent in state space form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta$$

$$\mathbf{A} = \begin{pmatrix} X_u & X_w & 0 & -g_0\cos\Theta_0 \\ Z_u & Z_w & u_0 & -g_0\sin\Theta_0 \\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + u_0 M_{\dot{w}} & -M_{\dot{w}}g_0\sin\Theta_0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} X_{\delta_e} & X_{\delta_T} \\ Z_{\delta_e} & Z_{\delta_T} \\ M_{\delta_e} + M_{\dot{w}}Z_{\delta_e} & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T} \\ 0 & 0 \end{pmatrix}$$

The various dimensional stability derivatives appearing in above are related to their dimensionless aerodynamic coefficient counterparts in following Table;

Variable	X	Z	M
u	$X_u = \frac{QS}{mu_0}[2\mathbf{C}_{X0} + \mathbf{C}_{Xu}]$	$Z_u = \frac{QS}{mu_0}[2\mathbf{C}_{Z0} + \mathbf{C}_{Zu}]$	$M_u = \frac{QS\bar{c}}{I_y u_0}\mathbf{C}_{mu}$
w	$X_w = \frac{QS}{mu_0}\mathbf{C}_{X\alpha}$	$Z_w = \frac{QS}{mu_0}\mathbf{C}_{Z\alpha}$	$M_w = \frac{QS\bar{c}}{I_y u_0}\mathbf{C}_{m\alpha}$
$\dot{w}$	$X_{\dot{w}} = 0$	$Z_{\dot{w}} = \frac{QS\bar{c}}{2mu_0^2}\mathbf{C}_{Z\dot{\alpha}}$	$M_{\dot{w}} = \frac{QS\bar{c}^2}{2I_y u_0^2}\mathbf{C}_{m\dot{\alpha}}$
q	$X_q = 0$	$Z_q = \frac{QS\bar{c}}{2mu_0}\mathbf{C}_{Zq}$	$M_q = \frac{QS\bar{c}^2}{2I_y u_0}\mathbf{C}_{mq}$

Example:

For example, We illustrate this response using the stability derivatives for an aircraft (the Boeing 747) at its Mach 0.25 power approach configuration at standard sea-level conditions.

$$V = 279.1 \text{ ft/sec}, \quad \rho = 0.002377 \text{ slug/ft}^3$$
$$S = 5,500. \text{ ft}^2, \quad \bar{c} = 27.3 \text{ ft}$$
$$W = 564,032. \text{ lb}, \quad I_y = 32.3 \times 10^6 \text{ slug-ft}^2$$

and the relevant aerodynamic coefficients are

$$\begin{array}{llll} \mathbf{C}_L = 1.108, & \mathbf{C}_D = 0.102, & \Theta_0 = 0 & \\ \mathbf{C}_{L\alpha} = 5.70, & \mathbf{C}_{L\dot{\alpha}} = 6.7, & \mathbf{C}_{Lq} = 5.4, & \mathbf{C}_{LM} = 0 \\ \mathbf{C}_{D\alpha} = 0.66, & & & \\ \mathbf{C}_{m\alpha} = -1.26, & \mathbf{C}_{m\dot{\alpha}} = -3.2, & \mathbf{C}_{mq} = -20.8, & \mathbf{C}_{mM} = 0 \end{array}$$

These values correspond to the following dimensional stability derivatives

$$\begin{array}{llll} X_u = -0.0212, & X_w = 0.0466 & & \\ Z_u = -0.2306, & Z_w = -0.6038, & Z_{\dot{w}} = -0.0341, & Z_q = -7.674 \\ M_u = 0.0, & M_w = -0.0019, & M_{\dot{w}} = -0.0002, & M_q = -0.4381 \end{array}$$

and the plant matrix is

$$\mathbf{A} = \begin{pmatrix} -0.0212 & 0.0466 & 0.000 & -32.174 \\ -0.2229 & -0.5839 & 262.472 & 0.0 \\ 0.0001 & -0.0018 & -0.5015 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{pmatrix}$$

The characteristic equation is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \lambda^4 + 1.1066\lambda^3 + 0.7994\lambda^2 + 0.0225\lambda + 0.0139 = 0$$

and its roots are $\lambda_{\text{sp}} = -0.5515 \pm \imath\, 0.6880 \quad \lambda_{\text{ph}} = -0.00178 \pm \imath\, 0.1339$

Where $\boldsymbol{\lambda = \eta + i\omega_d = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}}$

Hence $\zeta = \dfrac{1}{\sqrt{1+\left(\frac{\omega d}{\eta}\right)^2}}$ **and** $\omega_n = \dfrac{\eta}{\zeta}$

$$\zeta_{sp} = \frac{1}{\sqrt{1+\left(\frac{\omega d}{\eta}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{0.6880}{0.5515}\right)^2}} = \mathbf{0.6255}, \quad \omega_{nsp} = \frac{\eta}{\zeta} = \frac{0.5515}{0.6255} = \mathbf{0.882\ rad/sec}$$

$$\zeta_{ph} = \frac{1}{\sqrt{1+\left(\frac{\omega d}{\eta}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{0.1339}{0.00178}\right)^2}} = \mathbf{0.0133}, \quad \omega_{nph} = \frac{\eta}{\zeta} = \frac{0.00178}{0.0133} = \mathbf{0.134\ rad/sec}$$

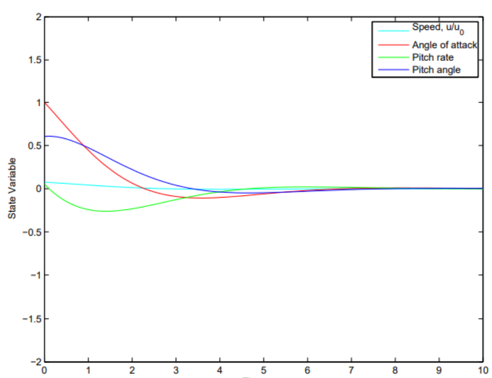
The period of the two modes:

$$T_{\text{sp}} = \frac{2\pi}{\omega_{n_{\text{sp}}}\sqrt{1-\zeta_{\text{sp}}^2}} = 9.13 \text{ sec} \qquad T_{\text{ph}} = \frac{2\pi}{\omega_{n_{\text{ph}}}\sqrt{1-\zeta_{\text{ph}}^2}} = 46.9 \text{ sec}$$

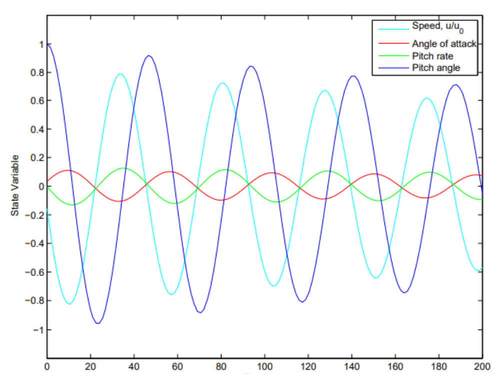
and the numbers of cycles to damp to half amplitude of the respective modes are:

$$N_{1/2_{\text{sp}}} = \frac{\ln 2}{2\pi}\frac{\sqrt{1-\zeta_{\text{sp}}^2}}{\zeta_{\text{sp}}} = \frac{\ln 2}{2\pi}\frac{\sqrt{1-(0.6255)^2}}{0.6255} = 0.1376$$

Figure below illustrates the short period and phugoid responses for the A/P under these conditions. These show the time histories of the state variables following an initial perturbation that is chosen to excite only the (a) short period mode or the (b) phugoid mode, respectively.



(a) Short period



(b) Phugoid

Approximation to Short Period Mode

A useful approximation for the mode can thus be developed by setting u = 0 and solving

$$(1 - Z_{\dot{w}})\,\dot{w} = Z_w w + (u_0 + Z_q)\,q$$
$$-M_{\dot{w}}\dot{w} + \dot{q} = M_w w + M_q q$$

which can be written in state-space form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} w \\ q \end{pmatrix} = \begin{pmatrix} \frac{Z_w}{1-Z_{\dot{w}}} & \frac{u_0+Z_q}{1-Z_{\dot{w}}} \\ M_w + \frac{M_{\dot{w}}Z_w}{1-Z_{\dot{w}}} & M_q + M_{\dot{w}}\frac{u_0+Z_q}{1-Z_{\dot{w}}} \end{pmatrix}\begin{pmatrix} w \\ q \end{pmatrix}$$

Since $\frac{Z_q}{u_0} = \frac{QS\bar{c}}{2mu_0^2}\mathbf{C}_{Z\,q} = -\frac{\eta V_H a_t}{\mu}$ **and** $Z_{\dot{w}} = \frac{QS\bar{c}}{2mu_0^2}\mathbf{C}_{Z\,\dot{\alpha}} = -\frac{\eta V_H a_t}{\mu}\frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha}$

it is consistent with the level of our approximation to neglect Zq relative to u0 and $Z\dot{\boldsymbol{\omega}}$. With respect to one, Thus:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} w \\ q \end{pmatrix} = \begin{pmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 \end{pmatrix}\begin{pmatrix} w \\ q \end{pmatrix}$$

The characteristic equation for the simplified plant matrix of the above eq.is:

$$\lambda^2 - (Z_w + M_q + u_0 M_{\dot{w}})\,\lambda + Z_w M_q - u_0 M_w = 0$$

if the derivatives with respect to ω are expressed as derivatives with respect to α,

$$\lambda^2 - \left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)\lambda - M_\alpha + \frac{Z_\alpha M_q}{u_0} = 0$$

$$\omega_n = \sqrt{-M_\alpha + \frac{Z_\alpha M_q}{u_0}} \qquad \zeta = -\frac{M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}}{2\omega_n}$$

<u>Example</u>

For the example considered in the preceding sections of the a/p in powered approach we find

$$\omega_n = \sqrt{0.54 + \frac{(-168.5)(-.4381)}{279.1}}\ \mathrm{sec}^{-1} = 0.897\ \mathrm{sec}^{-1}$$
$$\zeta = -\frac{-.4381 - .056 + \frac{(-168.5)}{279.1}}{2(0.897)} = 0.612$$

When these numbers are compared to ωn = 0.882 rad/sec and ζ = 0.6255 from the more complete analysis (of the full fourth-order system), we see that the approximate analysis over predicts the undamped natural frequency by only about 1 per cent, and under predicts the damping ratio by less than 2 per cent.

Approximation to Phugoid Mode

Since the phugoid mode typically proceeds at nearly constant angle of attack, and the motion is so slow that the pitch rate q is very small, we can approximate the behavior of the mode by writing only the X- and Z-force equations

$$\dot{u} = X_u u + X_w w - g_0 \cos\Theta_0 \theta$$
$$(1 - Z_{\dot{w}})\,\dot{w} = Z_u u + Z_w w + (u_0 + Z_q)\,q - g_0 \sin\Theta_0 \theta$$

which, upon setting ω = $\dot{\omega}$= 0, can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} X_u & -g_0 \cos\Theta_0 \\ -\frac{Z_u}{u_0+Z_q} & \frac{g_0 \sin\Theta_0}{u_0+Z_q} \end{pmatrix}\begin{pmatrix} u \\ \theta \end{pmatrix}$$

Since, Zq is typically very small relative to the speed u0, it is consistent with our neglect of $\dot{q}$ and $\dot{\omega}$ also to neglect Zq relative to u0. Also, we will consider only the case of level flight for the initial equilibrium, so Θ0 = 0, and Eq. becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} X_u & -g_0 \\ -\frac{Z_u}{u_0} & 0 \end{pmatrix}\begin{pmatrix} u \\ \theta \end{pmatrix}$$

Charac. Eq is . $\lambda^2 - X_u\lambda - \frac{g_0}{u_0} Z_u = 0$, and $\omega_n = \sqrt{-\frac{g_0}{u_0} Z_u}$ $\zeta = \frac{-X_u}{2\omega_n}$

It is useful to express these results in terms of dimensionless aerodynamic coefficients.

$$Z_u = -\frac{QS}{mu_0}\left[2\mathbf{C}_{L0} + \mathbf{M}\mathbf{C}_{L\mathbf{M}}\right]$$

for the case of a constant-thrust propulsive system,

$$X_u = -\frac{QS}{mu_0}\left[2\mathbf{C}_{D0} + \mathbf{M}\mathbf{C}_{D\mathbf{M}}\right]$$

if we further neglect compressibility effects, we have

$$\omega_n = \sqrt{2}\frac{g_0}{u_0} \qquad \zeta = \frac{1}{\sqrt{2}}\frac{\mathbf{C}_{D0}}{\mathbf{C}_{L0}}$$

Example:

For the example for the A/P in previous examples, we find

$$\omega_n = \sqrt{2}\frac{32.174 \text{ ft/sec}^2}{279.1 \text{ ft/sec}} = 0.163 \text{ sec}^{-1} \qquad \zeta = \frac{1}{\sqrt{2}}\frac{0.102}{1.108} = 0.0651$$

When these numbers are compared to $\omega_n = 0.134$ sec^{-1} and $\zeta = 0.0133$ from the more complete analysis (of the full fourth-order system), we see that the approximate analysis over predicts the undamped natural frequency by about 20 per cent, and over predicts the damping ratio by a factor of almost 5.

LATERAL/DIRECTIONALMOTIONS

Recall that the linearized equations describing small lateral/directional perturbations from a longitudinal equilibrium state can be written

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - Y_v\right] v - Y_p p + [u_0 - Y_r]\, r - g_0 \cos\Theta_0 \phi = Y_{\delta_r}\delta_r$$
$$-L_v v + \left[\frac{\mathrm{d}}{\mathrm{d}t} - L_p\right] p - \left[\frac{I_{xz}}{I_x}\frac{\mathrm{d}}{\mathrm{d}t} + L_r\right] r = L_{\delta_r}\delta_r + L_{\delta_a}\delta_a$$
$$-N_v v - \left[\frac{I_{xz}}{I_z}\frac{\mathrm{d}}{\mathrm{d}t} + N_p\right] p + \left[\frac{\mathrm{d}}{\mathrm{d}t} - N_r\right] r = N_{\delta_r}\delta_r + N_{\delta_a}\delta_a$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta$$

$$\mathbf{A} = \begin{pmatrix} Y_v & Y_p & g_0\cos\Theta_0 & Y_r - u_0 \\ L_v & L_p & 0 & L_r \\ 0 & 1 & 0 & 0 \\ N_v & N_p & 0 & N_r \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} Y_{\delta_r} & 0 \\ L_{\delta_r} & L_{\delta_a} \\ 0 & 0 \\ N_{\delta_r} & N_{\delta_a} \end{pmatrix}$$

$$\mathbf{x} = [v \quad p \quad \phi \quad r]^T \qquad \eta = [\delta_r \quad \delta_a]^T$$

This is the approximate form of the linearized equations for lateral/directional motions as they appear inmany texts

Variable	Y	L	N
v	$Y_v = \frac{QS}{mu_0}\mathbf{C}_{y\beta}$	$L_v = \frac{QSb}{I_x u_0}\mathbf{C}_{l\beta}$	$N_v = \frac{QSb}{I_z u_0}\mathbf{C}_{n\beta}$
p	$Y_p = \frac{QSb}{2mu_0}\mathbf{C}_{yp}$	$L_p = \frac{QSb^2}{2I_x u_0}\mathbf{C}_{lp}$	$N_p = \frac{QSb^2}{2I_z u_0}\mathbf{C}_{np}$
r	$Y_r = \frac{QSb}{2mu_0}\mathbf{C}_{yr}$	$L_r = \frac{QSb^2}{2I_x u_0}\mathbf{C}_{lr}$	$N_r = \frac{QSb^2}{2I_z u_0}\mathbf{C}_{nr}$

Table: Relation of dimensional stability derivatives for lateral/directional motions to dimension less derivatives of aerodynamic coefficients.

Weillustrate this response again using the stability derivatives for the Boeing 747 aircraft at its Mach 0.25 powered approach configuration at standard sea-level conditions.

For the lateral/directional response we need the following vehicle parameters:

$$W = 564,032.\ \text{lbf}, \qquad b = 195.7\ \text{ft}$$
$$I_x = 14.3 \times 10^6\ \text{slug ft}^2, \quad I_z = 45.3 \times 10^6\ \text{slug ft}^2, \quad I_{xz} = -2.23 \times 10^6\ \text{slug ft}^2$$

and the aerodynamic derivatives

$$\begin{array}{lll} \mathbf{C}_{y\beta} = -.96 & \mathbf{C}_{y_p} = 0.0 & \mathbf{C}_{y_r} = 0.0 \\ \mathbf{C}_{l\beta} = -.221 & \mathbf{C}_{lp} = -.45 & \mathbf{C}_{lr} = 0.101 \\ \mathbf{C}_{n\beta} = 0.15 & \mathbf{C}_{np} = -.121 & \mathbf{C}_{nr} = -.30 \end{array}$$

These values correspond to the following dimensional stability derivatives

$$\begin{array}{lll} Y_v = -0.0999, & Y_p = 0.0, & Y_r = 0.0 \\ L_v = -0.0055, & L_p = -1.0994, & L_r = 0.2468 \\ N_v = 0.0012, & N_p = -.0933, & N_r = -.2314 \end{array}$$

Using these values, the plant matrix is found to be

$$\mathbf{A} = \begin{pmatrix} -0.0999 & 0.0000 & 32.174 & -279.10 \\ -0.0057 & -1.0932 & 0.0 & 0.2850 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0015 & -.0395 & 0.0 & -.2454 \end{pmatrix}$$

The characteristic equation is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \lambda^4 + 1.4385\lambda^3 + 0.8222\lambda^2 + 0.7232\lambda + 0.0319 = 0$$

and its roots are

$$\begin{aligned} \lambda_{\text{DR}} &= -.08066 \pm \imath\, 0.7433 \\ \lambda_{\text{roll}} &= -1.2308 \\ \lambda_{\text{spiral}} &= -.04641 \end{aligned}$$

where, as suggested by the subscripts, the first pair of roots corresponds to the Dutch Roll mode, and the real roots corresponds to the rolling and spiral modes, respectively.

The undamped natural frequency and damping ratio of the Dutch Roll mode is thus given by

$$\zeta_{\text{DR}} = \sqrt{\frac{1}{1 + \left(\frac{\eta}{\xi}\right)^2_{\text{DR}}}} = \sqrt{\frac{1}{1 + \left(\frac{0.7433}{0.08066}\right)^2}} = 0.1079$$

$$\omega_{n_{\mathrm{DR}}} = \frac{0.08066}{0.1079} = 0.7477 \ \mathrm{sec}^{-1}$$

The period of the Dutch Roll mode is

$$T_{\mathrm{DR}} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{0.7477\sqrt{1-0.1079^2}} = 8.45 \ \mathrm{sec}$$

Approximation to Rolling Mode

It has been seen that the rolling mode typically corresponds to almost pure roll. Thus, it is reasonable to neglect all equations except the rolling moment equation, and all perturbations except p. We thus approximate the rolling mode by the single first-order equation

$$\dot{p} = \frac{L_p + i_x N_p}{1 - i_x i_z} p$$

for which the characteristic value is $\lambda = \dfrac{L_p + i_x N_p}{1 - i_x i_z}$

For our example of the Boeing 747 in powered approach at M = 0.25,

$$\lambda = \frac{-1.0994 + (-.1559)(-.0933)}{1 - (-.1559)(-.0492)} \ \mathrm{sec}^{-1} = -1.093 \ \mathrm{sec}^{-1}$$

which is a bit more than 10 per cent less than the value of-1.2308 from the analysis for the full fourth-order system.

Approximation to Spiral Mode

The spiral mode consists of a slow rolling/yawing motion for which the sideslip is relatively small. The roll rate is quite small compared to the yaw rate, so a reasonable approximation is to set

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0 = \frac{L_v + i_x N_v}{1 - i_x i_z} v + \frac{L_r + i_x N_r}{1 - i_x i_z} r$$

$$v \approx -\frac{L_r + i_x N_r}{L_v + i_x N_v} r$$

Since ix and iz are generally very small, this can be approximated as

$$v \approx -\frac{L_r}{L_v} r$$

$$\text{The yaw equation} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{N_v + i_z L_v}{1 - i_x i_z} v + \frac{N_r + i_z L_r}{1 - i_x i_z} r$$

Neglect of the product of inertia terms can then be written

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left(N_r - \frac{L_r N_v}{L_v} \right) r$$

For our example of the Boeing 747 in powered approach at M = 0.25,

$$\lambda = -.2314 - \frac{0.2468}{-.0055}(0.0012) = -.178$$

<u>Approximation to Dutch Roll Mode</u>

The most useful approximations require neglecting either the roll component or simplifying the sideslip component by assuming the vehicle c.g. travels in a straight line. This latter approximation means that $\psi = -\beta$, or

$$r = -\frac{\dot{v}}{u_0}$$

The roll and yaw moment equations (neglecting the product of inertia terms ix and iz) can then be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} v \\ p \\ r \end{pmatrix} = \begin{pmatrix} 0 & 0 & -u_0 \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{pmatrix} \begin{pmatrix} v \\ p \\ r \end{pmatrix}$$

The characteristic equation for this system is

$$\lambda^3 - (L_p + N_r)\,\lambda^2 + (L_p N_r + u_0 N_v - L_r N_p)\,\lambda + u_0\,(L_v N_p - L_p N_v) = 0$$

$$\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

Thus, the undamped natural frequency is given by

$$\omega_n^2 = \frac{u_0\,(L_p N_v - L_v N_p)}{L_p + N_r}$$

The damping ratio is seen to be proportional to

$$2\zeta\omega_n = \frac{-L_p N_r - u_0 N_v + L_r N_p}{L_p + N_r} + \frac{u_0\,(-L_v N_p + L_p N_v)}{(L_p + N_r)^2}$$

$$2\zeta\omega_n \approx -N_r\left(1+\frac{u_0}{L_p^2}N_v\right)+\frac{N_p}{L_p}\left(L_r-\frac{u_0}{L_p}L_v\right)$$

For our example of the Boeing747 in powered approach at M=0.25,

$$\omega_n = \left[\frac{(279.1)\left[(-1.0994)(0.0012)-(-.0055)(-.0933)\right]}{-1.0994-.2314}\ \mathrm{sec}^{-2}\right]^{1/2} = 0.620\ \mathrm{sec}^{-1}$$

and

$$\zeta = -\frac{\frac{(-1.0994)(-.2314)-(.2468)(-.0933)+(279.1)(0.0012)}{-1.0994-.2314}+(279.1)\frac{(-.0055)(-.0933)-(-1.0994)(0.0012)}{(-1.0994-.2314)^2}}{2(0.620)}$$

$$= 0.138$$

Summary of Lateral/Directional Modes

1. A *rolling* mode that usually is heavily damped, whose time to damp to half amplitude is determined largely by the roll damping L_p;
2. A *spiral* mode that usually is only lightly damped, or may even be unstable. Dihedral effect is an important stabilizing influence, while weathercock stability is destabilizing, for this mode; and
3. A lightly damped oscillatory, intermediate frequency *Dutch Roll* mode, which consists of a coordinated yawing, rolling, sideslipping motion. For this mode, dihedral effect is generally destabilizing, while weathercock stability is stabilizing.