Discrete Mathematics

Second Stage

Information Technology Department

Lecture 2

Introduction

- Mathematics is divided into two major branches discrete mathematics and continuous mathematics.
- Discrete mathematics deals with only those real numbers which are multiples of same basic unit. If the basic unit is 1, then the discrete variable can assume only integral values.
- Thus in continuous mathematics, a number system is usually real numbers while for discrete mathematics it is the integer.

Introduction (Cont’d...)

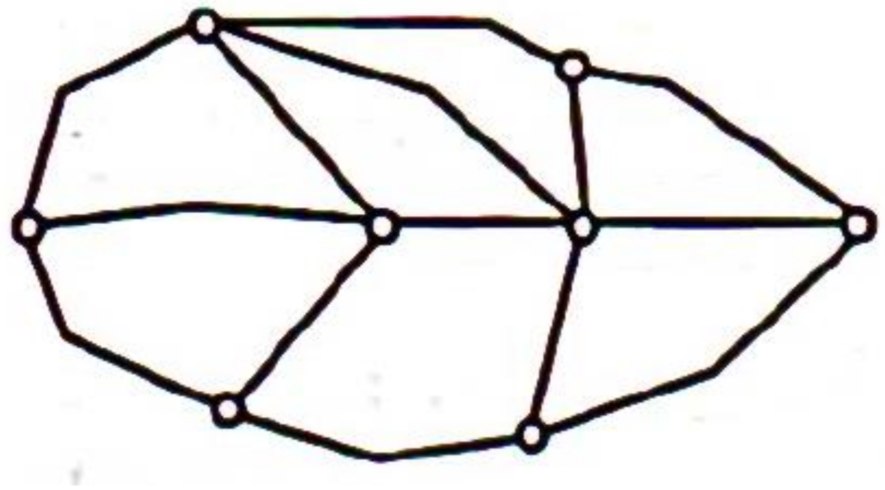
- Discrete Mathematics is mathematics that deals with discrete objects.
- Discrete objects are those which are separated from (not connected to/distinct from) each other.

Introduction (Cont'd...)

- Discrete mathematics does not support the notion of continuity.
- Discrete mathematics discusses languages used in mathematical reasoning, basic concepts, and their properties and relationships.

Introduction (Cont'd...)

- Examples of discrete mathematics:
  - Determine the **minimum number** of colors needed to color the regions in the diagram given below and explain why this number is minimum.



Propositional Calculus

Statements (or Propositions)

- Statements are kinds of sentences that we have to use to convey our thoughts to others.
- A sentence is a statement or proposition if it is possible to say whether what is conveyed by the sentence is true or false.
- Statements are logical entities.
- Sentences are grammatical entities.
- Not all sentences express statements and some sentences may express more than one statement.

Statements (or Propositions)

Examples

- Mars is a planet.
- 9>13.
- Y + 8 = 12
- There are 12 months in a year.

Statements (or Propositions) cont'd…

- A statement is a declarative sentence that is either true or false.
- Hence examples (1) and (4) *are statements.*
- In Example (3) ,the statement is true depends on the value of y.
- If Y is 4 the sentence is true, if $y \neq 4$ then the sentence is false.

Open Statement

- In example (3) , if we put $y = 4$, it becomes a true statement, if we take value of $y \neq 4$, it becomes false. Such statements are open statements.
- Thus if a mathematical sentence is neither true nor false it is called open sentence.
- An open statement is a sentence that contains one or more variables such that when certain values are substituted for the variables, we get statements.

Truth Table

- A table giving all possible truth values of a statement is called truth table.
- Statement has a definite truth value which is either true or false.
- True values are denoted by (T) and false values are denoted by (P).

Exercise

1. ***Which of the following statements are true and which are false ?***
   a) $9 < 12$
   b) $2 + 5 = 3 + 9$
2. ***What type of this sentence is***
   - $x + 8 = 17$?

Exercise (Cont'd…)

3. ***For what value of x following sentences will become true statements ?***
   a) $3x+9=15$
   b) $x+6=8$
   c) $x+1>5$
   d) $x+2<8$
   e) $5x\geq25$
   f) $5x\leq25$

Exercise (Cont'd…)

4. Which of the following statements are true?

a) $x+4=6$ *when* $x=2$

b) $x+4 \neq 6$ *when* $x=2$

c) $x + 5 \neq 8$ *when* $x = 3$

d) $2x + 4y = 14$ *when* $x = 1, Y = 3$

e) $3x + 5y = 11$ *when* $x = 0, y = 2$

Answers

1. a) True b) False

2. Open statement.

3. a) $x = 2$ b) $x = 2$ c) $x \geq 5$

   d) $x < 6$ or $x \leq 5$ e) $x \geq 5$ f) $x \leq 5$

4. a) True b) False c) False

   d) True e) False

Logical Connectives

- Every statement must be either true or false but not both.
- If two or more statements, they can be combined to produce a new statement.
- These new statements are called compound statements.
- To combine statements we use difference symbols.

Conjunction (‘^' or 'and')

- If p and q are statements, the compound statement 'p ^ q' is 'p and q' and is called 'p conjunction q' or 'p meet q'.
- ‘^' denotes 'and' and is known as conjunction.
- Examples
- Let us consider the statements.
  - p: 3 > 2
  - q :. 9 is an odd number,
  - then p ^ q is the statement
  - 3 > 2 and 9 is an odd number.

Truth table for ‘p ^ q’

p	q	p ∧ q
T	T	T
F	F	F
T	F	F
F	T	F

- ‘p ^ q’ is true only when both are true otherwise false.

Disjunction (V or 'or')

- When two or more statements are combined by the word 'or', the compound statement is known as 'disjunction'.
- The symbol 'p v q' is read as 'p or q' or 'p disjunction q' or 'p join q'.
- Example:
  - p: a is equal to 5.
  - q : b is equal to 7,
  - then P v q is the statement
  - a is equal to 5 or b is equal to 7.

Truth table for p v q

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

- 'p v q' is true if either p is true or q is true or both p and q are true and p v q is false if both p and q are false.

Remark

- The exclusive disjunction or exclusive 'OR' of two propositions p and q is the statement. Either p is true or q is true, but both are not true. Either p is true or q is true, but both are not true, we denote this by $p \oplus q$.

Truth table for $p \oplus q$ (exclusive 'OR')

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Negation (~P)

- Let P be any statement then negation of p is denoted by '~P' (or $\overline{P}$) is read as 'not P)
- If P is true then ~ P is false.
- If p is false then ~ P is true,
- **Truth table**

P	~ P
T	F
F	T

- ~P is a unary connective as only one statement is required to form negation.

Example

Example (1):

- If P is the statement ‘Aram is intelligent boy',
- then ˜ p is the statement ‘Aram is not intelligent boy.'

Example (2):

- If P is the statement 'I like to read‘.
- then ~ p is the statement ‘I don't like to read'

Homework

- Find truth table of the following proposition:

1. (~p ^ (~q ^ p)) v (q ^ p) v (p ^ p)
2. (p ^q) v (~p ^ q) v (p ^ ~q) v (~p ^ ~q)

Conditional Proposition

- If ***p*** *and* ***q*** *are proposition. the compound proposition" if p then q" denoted by p = > q is called* a **conditional proposition**
- Example:
- If it rains then I will carry an umbrella.
- Here:
  - *p : It rains* (antecedent)
  - *q : I will carry an umbrella (consequent)*

- The connective if .. .. then can also be read as follows.

1. *p Implies q.*
2. *p is sufficient for q.*
3. *p only if q.*
4. *q is necessary for p.*
5. *q if p.*
6. *q follows from p.*
7. *q is consequence of p.*

- The truth table for implication is given in Table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Note : The only circumstances under which the implication *p => q is false when p is true and q is* false.

- Example: Calculate truth table for:

(i) $p \vee \sim q \Rightarrow p$ (ii) $((\sim (p \wedge q) \vee r) \Rightarrow \sim p)$

I. Solution: The truth of the given compound statement is shown below.

p	q	$\sim q$	$p \vee \sim q$	$p \vee \sim q \Rightarrow p$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	F

II. Homework.....

Converse, Contrapositive and Inverse

- There are some related implication that can be formed from *p => q. If p => q is an implication.*
- *T*hen the converse of *p => q is the implication q = > p.*
- *The contrapositive of p => q is the implication ˜ q => ˜ p .*
- *The inverse of p => q is ˜ p => ˜ q.*

- The truth table of the four propositions follow:

p	q	Conditional $p \Rightarrow q$	Converse $q \Rightarrow p$	Inverse $\sim p \Rightarrow \sim q$	Contrapositive $\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

- Example:
- Consider the statement
  - *p : It rains.*
  - *q : The crops will grow*
- The implication (*p => q) states that.*
  - If it rains then the crops will grow.
- The converse (*q =>p) states that.*
  - *If the crops grow. then there has been rain.*
- The contrapositive *˜ q => ˜ p states that.*
  - *if the crops do not grow then there has been no rain.*
- The inverse *˜ p => ˜ q states that.*
  - *If it does not rain then the crops will not grow.*

- Example : Show that contrapositives are logically equivalent; that is : $\sim q \Rightarrow \sim p \equiv p \Rightarrow q$
- **Solution:**

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- Homework: Prove that if x^2 *is divisible by 4. then x is even.*

Biconditional Statement

- If *p and q are statement, then the compound statement p if and only if q. denoted by* $p \leftrightarrow q$ *is called a biconditional statement.*
- The truth table of $p \leftrightarrow q$ *is given:*

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Example : Show that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Solution:

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

- Homework: Show that $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$

Negation of Compound Statement

- Negation of Conjunction : the negation of the conjunction, would mean the negation of at least one of the two sub-statements.
- $\sim(p \wedge q) \equiv \sim p \vee \sim q$.
- Example : Write the negation of each of the following conjunctions: (2 +4 = 6 and 7< 12).
- *Solution: Let p : 2 + 4 = 6 and q: 7 < 12.*
- Then the conjunction *is given by "p ^ q".*
- Now : *~ p : 2 + 4 ≠ 6 and ~ q : 7 > 12.*
- The negation of *"p ^ q" is given by*
- *~(p ^ q) : 2 + 4 ≠ 6 or 7 > 12*

Negation of Disjunction

- The negation of a disjunction *p v q is the conjunction of the negation of p and the negation* of *q. we write:* $\sim(p \vee q) = \sim p \wedge \sim q$
- Example : Write the negation of each of the following disjunction: 9 is greater than 4 or 6 is less than 8.
- Solution: Let *p : 9 is greater than 4*
- *q : 6 is less than 8.*
- Then negation of *p v q is given by*
- *~(p v q) 9 is not greater than 4 and 6 is not less than 8.*

Negation of a Negation

- A negation of negation of a statement is the statement itself. Equivalently. we write~(~p) ≡ p.
- **Example :** Verify for the statement (*Roses are red*).
- *Solution:* The negation of *p is given by*
- ~ *p : Roses are not red.*
- Therefore. the negation of negation of~(~p):

Roses are red.

Negation of Implication

- If *p and q are two statements. Then* $\sim(p \Rightarrow q) \equiv p \wedge \sim q$
- *In order to prove the above equivalence. we prepare the following table.*

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

- **Example :** Write the negation of each of the following statements:
- If he studies then he will pass in the examination.
- Let *p : He Studies*
- *q : He will pass in the examination.*
- The given statement can be written as $p \Rightarrow q$. *The negation of* $p \Rightarrow q$ *is written as*
- $\sim (p \Rightarrow q) \equiv p \wedge \sim q$.
- **Result ?????**

Negation of Biconditional

- If *p and q are two statements. Then*

$$\sim(p \Leftrightarrow q) \equiv p \Leftrightarrow \sim q \equiv \sim p \Leftrightarrow q$$

- **Example** : Write the negation of each of the following statements :
- *He swims if and only if the water is warm.*
- Solution: *Let p : He swims and q = The water is warm.*
- The given statement can be written as $p \leftrightarrow q$. *The negation of* $p \leftrightarrow q$ *is written as*

$$\sim(p \Leftrightarrow q) \equiv p <=> \sim q \equiv \sim p \Leftrightarrow q$$

- He swims if and only if the water is not warm.

Derived Connective

- **NAND** : It means negation of conjunction of two statements. Assume *p and q be two prepos*itions.
- **NAND** of *p and q is a proposition which is false when both p and q art true otherwise* true.
- It is denoted by $p \uparrow q$.
- **Table: Truth table for NAND**

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

- **NOR:** It means negation of disjunction of two statements. Assume *p and q be two* propositions.
- **NOR** of ***p*** *and* ***q*** *is a proposition which is true when both p and q are false. Otherwise* false. It is denoted by $p \downarrow q$.
- **Table: Truth table for NOR**

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

- **XOR** (Exclusive OR): Assume *p and q be two proposition. The exclusive or (XOR) of p* and *q. denoted by p⊕q is the proposition that is true when exactly one of p and q is true but not* both and is false otherwise.

- **Table: Truth table for XOR**

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Tautologies and Contradictions

- A tautology is a proposition which is true for all truth values of its sub-propositions.
- In other words a proposition is a tautology if it is always **true for all assignments** of truth values.
- ~ (~P) ≡ P

Contradiction

- A proposition is a **contradiction** if it is always false for all assignments of **truth values**.
- **Remark :**
  - A proposition which is neither a tautology nor a contradiction is called a **contingency**.

Example

- **show *that p → p is a tautology.***
- **Solution:**

p	p	p → p
T	T	T
F	F	T

Since in $p \to p$ all the truth values are true (T), hence $p \to p$ is a tautology.

Exercise

1. *Show that p v (~p) is a tautology.*
2. *Show that ~(p^(~ p)) is a tautology.*
3. Verify that the proposition *p ^(q ^~ p) is a contradiction*

Functionally Complete Set of Connectives

- Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from the set is called a functionally complete set of connectives.
- **Thus** all the conditional and biconditional can be replaced by the three connectives ^, v, ~ .

$$p \Rightarrow q \equiv \sim p \vee q$$

$$p \Leftrightarrow q \equiv (\sim p \vee q) \wedge (p \wedge \sim q)$$

- **Example:** Write an equivalent expression for (p → q ^ r) v (r ↔ s) which contains neither the biconditional nor the conditional.
- **Solution:** First we replace the biconditional connective by its equivalent in the given expression. Then replace the conditional

$$(p \Rightarrow q \wedge r) \vee (r \Leftrightarrow s) \equiv (p \Rightarrow q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s))$$

$$\equiv (\sim p \vee q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s))$$

Normal Forms

- In logic, with the help of truth table we can compare if two statements are equivalent. **But when more statements or propositions are involved, then this method is not** practical.
- One method is to transform S_1 and S_2 to some standard form S'_1 and S'_2.

Disjunction Normal Forms

- A logical expression is said to be in disjunctive normal form if it is the sum of elementary products.
- In an logical expression, a **product** of the variables and their negations is called an elementary product. For example *p ^~ q ,~ p ^~ q ,~p ^ q are elementary products.*
- *A **sum** of the variables* and their negations is called an **elementary sum**. For example, *p v q, p v ~q,~p v~ q are* elementary sum

Procedure to obtain a disjunctive Normal Form of a given logical expression

1. Remove all → and ↔ by an equivalent expression containing the connectives v, ^, ~ *only.*
2. Eliminate ~ before sums and products by using the double negation.
3. Apply the distributive law until a sum of elementary product is obtained.

- **Example:** Obtain the disjunctive normal forms of the followings:

$(a)\ p \wedge (p \Rightarrow q)$ $\qquad$ $(b)\ p \vee (\sim p \Rightarrow (q \vee (q \Rightarrow \sim r)))$

- **Solution:**

a) $p \wedge (p \Rightarrow q) \equiv [p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)]$

b) homework…

Conjunctive Normal Form

- A logical expression is said to be in conjunctive normal form if it consists of a product of elementary sum.
- Example: Obtain a conjunctive normal form of the following: $[q \vee (p \wedge r)] \wedge \sim[(p \vee r) \wedge q]$
- **Solution:**

$$
\begin{aligned}
[q \vee (p \wedge r)] \wedge \sim[(p \vee r) \wedge q] &\equiv [q \vee (p \wedge r)] \wedge [\sim(p \vee r) \vee \sim q] \\
&\equiv [q \vee (p \wedge r)] \wedge [(\sim p \wedge \sim r) \vee \sim q] \\
&\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)
\end{aligned}
$$

Method of Proof

- A theorem is a proposition that can be proved to be true.
- An argument that establishes the **truth** of a **theorem** is called a **proof**.
- There are **many** different **types** of **proof**.
- In this section we shall look at some of the more common type.

Rule of Detachment or (Modus Ponens)

- If the statement in *p is assumed as true and also the statement* $p \rightarrow q$ is accepted as true, then, *q must be true.* Symbolically it is written in the following:

$$\begin{array}{r} p \Rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

- In this presentation of an argument, the assertions above the horizontal line are the hypotheses or premises; the assertion below the line is the conclusion.

Example

- If Sushma gets a first class with distinction in B.E. then she will get a good job easily.

Let p : Sushma gets a first class with distinction in B.E.

q : She will get a good job easily.

- The inference rule is

$$\begin{array}{l} p \rightarrow q \\ \quad p \\ \hline \therefore q \end{array}$$

Hence this form of argument is valid.

- To do so, we construct a truth table for the premises and conclusion.

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Law of Contraposition (or Modus Tollens)

- If the statement in p is assumed as false and also the statement p → q is accepted as true, then, q must be false.
- The form of the argument is:

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

i.e. [(p → q)^ (~q)→~p.

Truth table for [(p → q)^ (~q)→~p.

p	q	~ p	~ q	p → q	(p → q) ∧ (~ q)	(p → q) ∧ (~ q)] → ~ p
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Disjunctive Syllogism

- This rule states that "If $p \vee q$ is true and p is false then q is true."
- It is represented in the following form as

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore\ q \end{array}$$

- This argument is valid as $(p \vee q) \wedge \sim p \rightarrow q$ is a tautology.

p	q	~ p	(p ∨ q)	((p ∨ q) ∧ ~ p)	((p ∨ q) ∧ ~ p) → q
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Hypothetical Syllogism

- Whenever the two implications $p \rightarrow q$ *and* $q \rightarrow r$ *are accepted as true*. then the implication $p \rightarrow r$ *is accepted as true.*
- *Symbolically it can be represented as*

$$\begin{array}{r} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore \ p \Rightarrow r \end{array}$$

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Q. Test the validity of the argument. If a person is poor, he is unhappy. If a person is unhappy, he dies young. Therefore poor person dies young.

Ans. : Let p : Person is poor

q : Person is unhappy

r : Person dies young

In symbolic form argument is

$$\frac{\begin{array}{l} s_1 : p \to q \\ s_2 : q \to r \end{array}}{s : p \to r}$$

The above argument is the rule of hypothetical syllogism. Hence it is valid.

Quiz

Q. Express following statements in propositional form :

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi) If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey.