Tishk International University
Faculty of Administrative Sciences and Economics

MATHEMATICS
FOR ECONOMICS AND BUSINESS

BUS 143
Part 6

I Grade- Fall

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Matrices

A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)

We talk about one **matrix**, or several **matrices**.

There are many things we can do with them ...

Adding

To add two matrices: add the numbers in the matching positions:

These are the calculations:

3+4=7	8+0=8
4+1=5	6−9=−3

The two matrices must be the same size, i.e. the rows must match in size, and the columns must match in size.

> Example: a matrix with **3 rows** and **5 columns** can be added to another matrix of **3 rows** and **5 columns**.
>
> But it could not be added to a matrix with **3 rows** and **4 columns** (the columns don't match in size)

Negative

The negative of a matrix is also simple:

These are the calculations:

−(2)=−2	−(−4)=+4
−(7)=−7	−(10)=−10

Subtracting

To subtract two matrices: subtract the numbers in the matching positions:

These are the calculations:

3−4=−1	8−0=8
4−1=3	6−(−9)=15

Note: subtracting is actually defined as the ***addition*** *of a negative matrix: A + (−B)*

Multiply by a Constant

We can multiply a matrix by a **constant** *(the value 2 in this case)*:

These are the calculations:

2×4=8	2×0=0
2×1=2	2×−9=−18

We call the constant a **scalar**, so officially this is called "scalar multiplication".

Multiplying by Another Matrix

To **multiply two matrices together** is a bit more difficult ... read Multiplying Matrices to learn how.

Dividing

And what about division? Well we **don't** actually divide matrices, we do it this way:

$$A/B = A \times (1/B) = A \times B^{-1}$$

where $\mathbf{B}^{-1}$ means the "inverse" of B.

So we don't divide, instead we **multiply by an inverse**.

And there are special ways to find the Inverse, learn more at Inverse of a Matrix.

Transposing

To "transpose" a matrix, swap the rows and columns.

We put a "T" in the top right-hand corner to mean transpose:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 4 & -9 \\ 24 & 8 \end{bmatrix}$$

Notation

A matrix is usually shown by a **capital letter** (such as A, or B)

Each entry (or "element") is shown by a **lower case letter** with a "subscript" of **row,column**:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

Rows and Columns

So which is the row and which is the column?

- Rows go **left-right**
- Columns go **up-down**

To remember that rows come before columns use the word **"arc"**:

$$a_{r,c}$$

Example:

$$B = \begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

Here are some sample entries:

$b_{1,1} = 6$ *(the entry at row 1, column 1 is 6)*

$b_{1,3} = 24$ *(the entry at row 1, column 3 is 24)*

$b_{2,3} = 8$ *(the entry at row 2, column 3 is 8)*

What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

A $\begin{bmatrix} -1 & -8 \\ -1 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & -8 \\ -1 & 0 \end{bmatrix}$

C $\begin{bmatrix} 1 & -8 \\ -7 & 0 \end{bmatrix}$

D $\begin{bmatrix} 1 & 8 \\ -1 & 0 \end{bmatrix}$

To add two matrices, just add the numbers in the matching positions.

These are the calculations:

$2 + (-1) = 1$ $\quad -3 + (-5) = -8$

$-4 + 3 = -1$ $\quad 2 + (-2) = 0$

What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

A $\begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 \\ -7 & 4 \end{bmatrix}$

C $\begin{bmatrix} 3 & -8 \\ -7 & 4 \end{bmatrix}$

D $\begin{bmatrix} 3 & 2 \\ -7 & 0 \end{bmatrix}$

To subtract two matrices, just subtract the numbers in the matching positions:

These are the calculations:

2 - (-1) = 3   -3 - (-5) = 2

-4 - 3 = -7   2 - (-2) = 4

What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

A $\begin{bmatrix} 2 & -1 & 6 \\ -6 & -2 & 9 \end{bmatrix}$

B $\begin{bmatrix} 2 & -1 & 6 \\ -6 & 2 & 9 \end{bmatrix}$

C $\begin{bmatrix} 2 & -1 & 6 \\ 4 & -2 & 9 \end{bmatrix}$

D $\begin{bmatrix} 2 & -9 & 6 \\ -6 & -2 & 9 \end{bmatrix}$

To add two matrices, just add the numbers in the matching positions:

These are the calculations

$3 - (-1) = 2$, $-5 + 4 = -1$, $4 + 2 = 6$

$-1 + (-5) = -6$, $4 + (-2) = 2$, $6 + 3 = 9$

What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

A $\begin{bmatrix} 4 & -9 & 2 \\ 4 & 6 & 3 \end{bmatrix}$

B $\begin{bmatrix} 4 & -9 & 2 \\ -6 & 6 & 3 \end{bmatrix}$

C $\begin{bmatrix} 4 & -9 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

D $\begin{bmatrix} 2 & -9 & 2 \\ 4 & 6 & 3 \end{bmatrix}$

What is the transpose of the matrix $\begin{bmatrix} -3 & 1 & 4 \\ 5 & 2 & -1 \end{bmatrix}$?

A $\begin{bmatrix} 3 & -1 & -4 \\ -5 & -2 & 1 \end{bmatrix}$

B $\begin{bmatrix} 5 & 2 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

C $\begin{bmatrix} -3 & 5 \\ 1 & 2 \\ 4 & -1 \end{bmatrix}$

D $\begin{bmatrix} 5 & -3 \\ 2 & 1 \\ -1 & 4 \end{bmatrix}$

To "transpose" a matrix, just swap the rows and columns.

Therefore $\begin{bmatrix} -3 & 1 & 4 \\ 5 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 \\ 1 & 2 \\ 4 & -1 \end{bmatrix}$

What is the transpose of the matrix $\begin{bmatrix} 2 & -5 & 6 \\ -1 & 2 & -4 \\ -3 & -1 & 0 \end{bmatrix}$?

A $\begin{bmatrix} -5 & 2 & -1 \\ 6 & -4 & 0 \\ 2 & -1 & -3 \end{bmatrix}$

B $\begin{bmatrix} 6 & -4 & 0 \\ 2 & -1 & -3 \\ -5 & 2 & -1 \end{bmatrix}$

C $\begin{bmatrix} 2 & -1 & -3 \\ -5 & 2 & -1 \\ 6 & -4 & 0 \end{bmatrix}$

D $\begin{bmatrix} -2 & 5 & -6 \\ 1 & -2 & 4 \\ 3 & 1 & 0 \end{bmatrix}$

If $A = \begin{bmatrix} -3 & 1 \\ -2 & 4 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 0 & -2 \\ -2 & 4 \end{bmatrix}$ then what is $3A - 2B$?

A $\begin{bmatrix} -1 & -3 \\ -6 & 8 \\ 11 & 5 \end{bmatrix}$

B $\begin{bmatrix} -1 & 9 \\ -6 & 8 \\ 11 & -11 \end{bmatrix}$

C $\begin{bmatrix} -17 & 9 \\ -6 & 16 \\ 19 & -11 \end{bmatrix}$

D $\begin{bmatrix} -17 & -3 \\ 6 & 16 \\ 11 & -11 \end{bmatrix}$

To find 3A, multiply all the entries of A by 3:

$$\text{Therefore } 3A = \begin{bmatrix} 3\times(-3) & 3\times 1 \\ 3\times(-2) & 3\times 4 \\ 3\times 5 & 3\times(-1) \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ -6 & 12 \\ 15 & -3 \end{bmatrix}$$

To find 2B, multiply all the entries of B by 2:

$$\text{Therefore } 2B = \begin{bmatrix} 2\times 4 & 2\times(-3) \\ 2\times 0 & 2\times(-2) \\ 2\times(-2) & 2\times 4 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 0 & -4 \\ -4 & 8 \end{bmatrix}$$

$$\text{Therefore } 3A - 2B = \begin{bmatrix} -9 & 3 \\ -6 & 12 \\ 15 & -3 \end{bmatrix} - \begin{bmatrix} 8 & -6 \\ 0 & -4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} -17 & 9 \\ -6 & 16 \\ 19 & -11 \end{bmatrix}$$

If $A = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 3 & -2 \end{bmatrix}$ then what is $2A + B^T$?

A $\begin{bmatrix} 0 & 16 \\ 10 & -10 \end{bmatrix}$

B $\begin{bmatrix} 2 & 13 \\ 6 & -8 \end{bmatrix}$

C $\begin{bmatrix} 2 & 6 \\ 13 & -8 \end{bmatrix}$

D $\begin{bmatrix} 2 & 5 \\ 14 & -8 \end{bmatrix}$

First find 2A by multiplying all the elements by 2:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 4 & 2 \\ 10 & -6 \end{bmatrix}$$

Next find B^T by swapping the rows and columns:

$$B = \begin{bmatrix} -2 & 4 \\ 3 & -2 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} -2 & 3 \\ 4 & -2 \end{bmatrix}$$

Now add 2A to B^T:

$$2A + B^T = \begin{bmatrix} 4 & 2 \\ 10 & -6 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 14 & -8 \end{bmatrix}$$

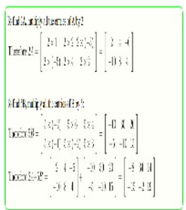
If $A = \begin{bmatrix} 1 & 2 & -3 \\ -5 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 6 & 4 \\ -1 & -2 & 3 \end{bmatrix}$ then what is $2A + 5B$?

A $\begin{bmatrix} -8 & 34 & 14 \\ -5 & -2 & 19 \end{bmatrix}$

B $\begin{bmatrix} -8 & 34 & 14 \\ -15 & -2 & 19 \end{bmatrix}$

C $\begin{bmatrix} 12 & 34 & 14 \\ -15 & -2 & 19 \end{bmatrix}$

D $\begin{bmatrix} 12 & -26 & -26 \\ -5 & 18 & -11 \end{bmatrix}$



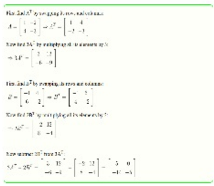
If $A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 6 & -2 \end{bmatrix}$ then what is $3A^T - 2B^T$?

A $\begin{bmatrix} 1 & 24 \\ 2 & -13 \end{bmatrix}$

B $\begin{bmatrix} 5 & 24 \\ -14 & -5 \end{bmatrix}$

C $\begin{bmatrix} 5 & 0 \\ -14 & -13 \end{bmatrix}$

D $\begin{bmatrix} 5 & 0 \\ -14 & -5 \end{bmatrix}$



How to Multiply Matrices

A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)

To multiply a matrix by a single number is easy:

2×4=8

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

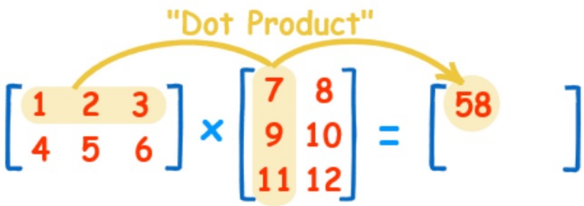
2×4=8	2×0=0
2×1=2	2×-9=-18

We call the number ("2" in this case) a **scalar**, so this is called "scalar multiplication".

Multiplying a Matrix by Another Matrix

But to multiply a matrix **by another matrix** we need to do the " dot product " of rows and columns ... what does that mean? Let us see with an example:

To work out the answer for the **1st row** and **1st column**:



The "Dot Product" is where we **multiply matching members**, then sum up:

$$(1, 2, 3) \bullet (7, 9, 11) = 1\times7 + 2\times9 + 3\times11$$
$$= 58$$

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

Want to see another example? Here it is for the 1st row and **2nd column**:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

$$(1, 2, 3) \bullet (8, 10, 12) = 1{\times}8 + 2{\times}10 + 3{\times}12 \\ = 64$$

We can do the same thing for the **2nd row** and **1st column**:

$$(4, 5, 6) \bullet (7, 9, 11) = 4{\times}7 + 5{\times}9 + 6{\times}11 \\ = 139$$

And for the **2nd row** and **2nd column**:

$$(4, 5, 6) \bullet (8, 10, 12) = 4{\times}8 + 5{\times}10 + 6{\times}12 \\ = 154$$

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix} \checkmark$$

DONE!

Example: The local shop sells 3 types of pies.

- Apple pies cost **$3** each
- Cherry pies cost **$4** each
- Blueberry pies cost **$2** each

And this is how many they sold in 4 days:

	Mon	Tue	Wed	Thu
Apple	**13**	**9**	**7**	**15**
Cherry	**8**	**7**	**4**	**6**
Blueberry	**6**	**4**	**0**	**3**

Now think about this ... the **value of sales** for Monday is calculated this way:

→ Apple pie value + Cherry pie value + Blueberry pie value

→ $3×13 + $4×8 + $2×6 = $83

So it is, in fact, the "dot product" of prices and how many were sold:

($3, $4, $2) • (13, 8, 6) = $3×13 + $4×8 + $2×6
= $83

We **match** the price to how many sold, **multiply** each, then **sum** the result.

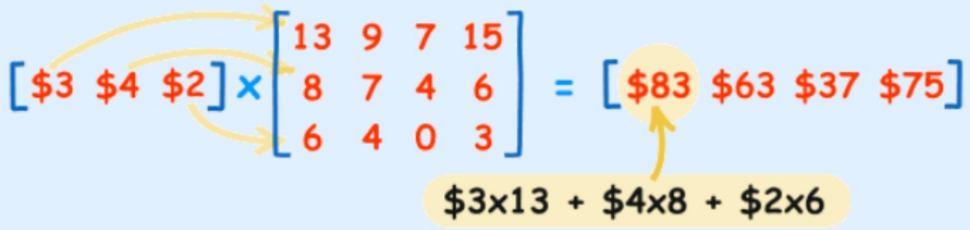
In other words:

- The sales for Monday were: Apple pies: **$3×13=$39**, Cherry pies: **$4×8=$32**, and Blueberry pies: **$2×6=$12**. Together that is $39 + $32 + $12 = **$83**
- And for Tueday: **$3×9 + $4×7 + $2×4 = $63**
- And for Wednesday: **$3×7 + $4×4 + $2×0 = $37**
- And for Thursday: **$3×15 + $4×6 + $2×3 = $75**

So it is important to match each price to each quantity.

Now you know why we use the "dot product".

And here is the full result in Matrix form:



They sold **$83** worth of pies on Monday, **$63** on Tuesday, etc.

(You can put those values into the Matrix Calculator to see if they work.)

Rows and Columns

To show how many rows and columns a matrix has we often write **rows×columns**.

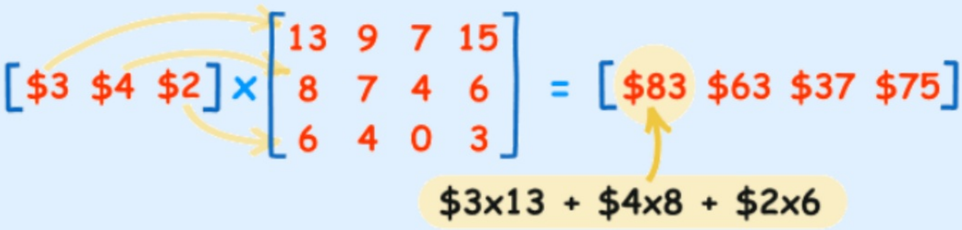
Example: This matrix is **2×3** (2 rows by 3 columns):

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

When we do multiplication:

- The number of **columns of the 1st matrix** must equal the number of **rows of the 2nd matrix**.
- And the result will have the same number of **rows as the 1st matrix**, and the same number of **columns as the 2nd matrix**.

Example:



In that example we multiplied a 1×3 matrix by a 3×4 matrix (note the 3s are the same), and the result was a 1×4 matrix.

In General:

To multiply an **m×n** matrix by an **n×p** matrix, the **n**s must be the same, and the result is an **m×p** matrix.

$m \times n \;\times\; n \times p \;\rightarrow\; m \times p$

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×3 Identity Matrix

- It is "square" (has same number of rows as columns)
- It can be large or small (2×2, 100×100, ... whatever)
- It has **1**s on the diagonal and **0**s everywhere else
- Its symbol is the capital letter **I**

It is a **special matrix**, because when we multiply by it, the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Order of Multiplication

In arithmetic we are used to:

$$3 \times 5 = 5 \times 3$$

(The Commutative Law of Multiplication)

But this is **not** generally true for matrices (matrix multiplication is **not commutative**):

$$AB \neq BA$$

When we change the order of multiplication, the answer is (usually) **different**.

Example:

See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{x} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \text{x} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

The answers are different!

A is a 3 × 2 matrix

B is a 2 × 3 matrix

C is a 2 × 2 matrix

D is a 3 × 3 matrix

Which of the following products does not exist?

A AB

B AC

C BD

D CD

When we do multiplication of matrices, the number of columns of the 1st matrix must equal the number of rows of the 2nd matrix.

The number of columns of A (2) and the number of rows of B (2) are equal, so the product AB exists.

The number of columns of A (2) and the number of rows of C (2) are equal, so the product AC exists.

The number of columns of B (3) and the number of rows of D (3) are equal, so the product BD exists.

The number of columns of C (2) and the number of rows of D (3) are not equal, so the product CD does not exist.

If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$,then what is AB ?

A $\begin{bmatrix} 6 & 3 \\ 4 & 10 \end{bmatrix}$

B $\begin{bmatrix} 7 & 14 \\ 10 & 22 \end{bmatrix}$

C $\begin{bmatrix} 9 & 8 \\ 14 & 14 \end{bmatrix}$

D $\begin{bmatrix} 18 & 8 \\ 23 & 11 \end{bmatrix}$

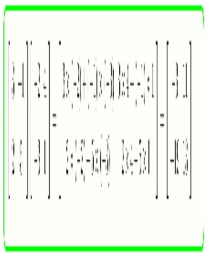
If $X = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$,then what is XY ?

A $\begin{bmatrix} -3 & 11 \\ -19 & 13 \end{bmatrix}$

B $\begin{bmatrix} -10 & -10 \\ 16 & -1 \end{bmatrix}$

C $\begin{bmatrix} -6 & -4 \\ -6 & 5 \end{bmatrix}$

D $\begin{bmatrix} 2 & 22 \\ -7 & 8 \end{bmatrix}$



If $A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$,then what is AB ?

A $\begin{bmatrix} 6 & 1 & -6 \\ 0 & 16 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & -8 \\ -8 & 16 \end{bmatrix}$

C $\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 \\ -8 & 16 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2+(-1)\times(-1)-2\times(-3) & 3\times 0+(-1)\times 4+2\times 2 \\ -2\times 2+4\times(-1)+0\times(-3) & -2\times 0+4\times 4+0\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 16 \end{bmatrix}$$

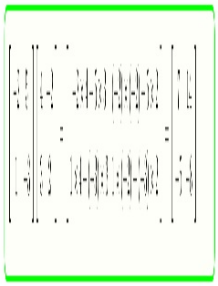
If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$,then what is PQ ?

A $\begin{bmatrix} 7 & 14 \\ -5 & -8 \end{bmatrix}$

B $\begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$

C $\begin{bmatrix} -8 & -10 \\ 3 & -6 \end{bmatrix}$

D $\begin{bmatrix} -10 & -4 \\ 26 & 9 \end{bmatrix}$



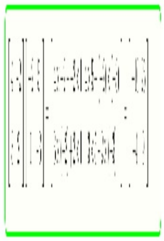
If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$,then what is QP ?

A $\begin{bmatrix} 7 & 14 \\ -5 & -8 \end{bmatrix}$

B $\begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$

C $\begin{bmatrix} -8 & -10 \\ 3 & -6 \end{bmatrix}$

D $\begin{bmatrix} -10 & -4 \\ 26 & 9 \end{bmatrix}$



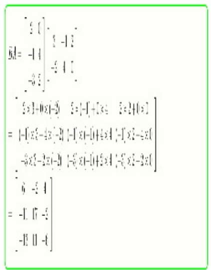
If $A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$, then what is BA ?

A $\begin{bmatrix} 1 & 0 \\ -8 & 16 \end{bmatrix}$

B $\begin{bmatrix} 6 & 1 & -6 \\ 0 & 16 & 0 \end{bmatrix}$

C $\begin{bmatrix} 6 & 0 \\ 1 & 16 \\ -6 & 0 \end{bmatrix}$

D $\begin{bmatrix} 6 & -2 & 4 \\ -11 & 17 & -2 \\ -13 & 11 & -6 \end{bmatrix}$



If $A = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $AB = BA$ then what is the value of x?

A x = 0

B x = 1

C x = 2

D x = 3

$$AB = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\times 1 + x\times 1 & 1\times 1 + x \times 2 \\ 2\times 1 + 3\times 1 & 2\times 1 + 3\times 2 \end{bmatrix} = \begin{bmatrix} 1+x & 1+2x \\ 5 & 8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 1\times 2 & 1\times x + 1 \times 3 \\ 1\times 1 + 2\times 2 & 1\times x + 2\times 3 \end{bmatrix} = \begin{bmatrix} 3 & x+3 \\ 5 & x+6 \end{bmatrix}$$

If AB = BA, then the numbers in each cell of the two matrices must be equal

⇒ 1 + x = 3, so x=2

⇒ 1 + 2x = x + 3 ⇒ 2x = x + 2 ⇒ x = 2

⇒ 5 = 5 and

⇒ 8 = x + 6, so x=2

These equations are all satisfied by x = 2

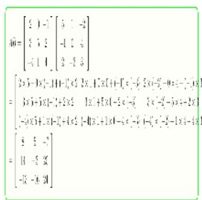
If $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix}$, then what is AB ?

A $\begin{bmatrix} 8 & 5 & -7 \\ 14 & -3 & 20 \\ -13 & -16 & 24 \end{bmatrix}$

B $\begin{bmatrix} 10 & 0 & 2 \\ -3 & 0 & 8 \\ -8 & -3 & 12 \end{bmatrix}$

C $\begin{bmatrix} 21 & 3 & -11 \\ -18 & 4 & 17 \\ -17 & -12 & 4 \end{bmatrix}$

D $\begin{bmatrix} -1 & 14 & -4 \\ -3 & -3 & 23 \\ 1 & -13 & 22 \end{bmatrix}$



If $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix}$, then what is BA ?

A $\begin{bmatrix} 10 & 0 & 2 \\ -3 & 0 & 8 \\ -8 & -3 & 12 \end{bmatrix}$

B $\begin{bmatrix} 8 & 5 & -7 \\ 14 & -3 & 20 \\ -13 & -16 & 24 \end{bmatrix}$

C $\begin{bmatrix} -1 & -3 & 1 \\ 14 & -3 & -13 \\ -4 & 23 & 22 \end{bmatrix}$

$$BA = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 2+1\times 3+(-2)\times(-4) & 5\times 0+1\times 5+(-2)\times 1 & 5\times(-1)+1\times 2+(-2)\times 4 \\ (-1)\times 2+0\times 3+4\times(-4) & (-1)\times 0+0\times 5+4\times 1 & (-1)\times(-1)+0\times 2+4\times 4 \\ 2\times 2+(-3)\times 3+3\times(-4) & 2\times 0+(-3)\times 5+3\times 1 & 2\times(-1)+(-3)\times 2+3\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 3 & -11 \\ -18 & 4 & 17 \\ -17 & -12 & 4 \end{bmatrix}$$

If $A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$, then what is AB ?

A $\begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix}$

B $\begin{bmatrix} 6 & 2 & 0 \\ 0 & 5 & 12 \\ -4 & 8 & 2 \end{bmatrix}$

C $\begin{bmatrix} 5 & 27 & -14 \\ 3 & 29 & -14 \\ -3 & -10 & 6 \end{bmatrix}$

D $\begin{bmatrix} 8 & 29 & -15 \\ 1 & 17 & -16 \\ -10 & -24 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2+(-2)\times 3+5\times 1 & 3\times(-1)+(-2)\times(-5)+5\times 4 & 3\times 0+(-2)\times 2+5\times(- \\ 0\times 2+(-1)\times 3+6\times 1 & 0\times(-1)+(-1)\times(-5)+6\times 4 & 0\times 0+(-1)\times 2+6\times(- \\ -4\times 2+2\times 3+(-1)\times 1 & -4\times(-1)+2\times(-5)+(-1)\times 4 & -4\times 0+2\times 2+(-1)\times(\end{bmatrix}$$

$$= \begin{bmatrix} 5 & 27 & -14 \\ 3 & 29 & -14 \\ -3 & -10 & 6 \end{bmatrix}$$

If $A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$, then what is BA ?

A $\begin{bmatrix} 6 & 2 & 0 \\ 0 & 5 & 12 \\ -4 & 8 & 2 \end{bmatrix}$

B $\begin{bmatrix} 8 & 1 & -10 \\ 29 & 17 & -24 \\ -15 & -16 & 6 \end{bmatrix}$

C $\begin{bmatrix} 5 & 27 & -14 \\ 3 & 29 & -14 \\ -3 & -10 & 6 \end{bmatrix}$

D $\begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix}$

$$BA = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times3+(-1)\times0+0\times(-4) & 2\times(-2)+(-1)\times(-1)+0\times2 & 2\times5+(-1)\times6+0\times(-1) \\ 3\times3+(-5)\times0+2\times(-4) & 3\times(-2)+(-5)\times(-1)+2\times2 & 3\times5+(-5)\times6+2\times(-1) \\ 1\times3+4\times0+(-2)\times(-4) & 1\times(-2)+4\times(-1)+(-2)\times2 & 1\times5+4\times6+(-2)\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix}$$

Determinant of a Matrix

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

A Matrix is an array of numbers:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 2 Columns)

The determinant of that matrix is (calculations are explained later):

$3\times6 - 8\times4 = 18 - 32 = \mathbf{-14}$

What is it for?

The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

Symbol

The **symbol** for determinant is two vertical lines either side.

Example:

|A| means the determinant of the matrix **A**

(Exactly the same symbol as absolute value.)

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)

Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$\begin{aligned} |B| &= 4\times8 - 6\times3 \\ &= 32-18 \\ &= 14 \end{aligned}$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

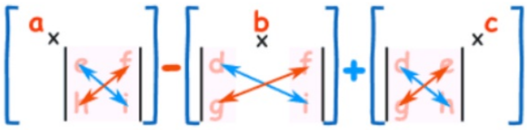
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:



To work out the determinant of a **3×3** matrix:

- Multiply **a** by the **determinant of the 2×2 matrix** that is **not in a**'s row or column.
- Likewise for **b**, and for **c**
- Sum them up, but remember the minus in front of the **b**

As a formula *(remember the vertical bars || mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"The determinant of A equals a times the determinant of ... etc"

Example:

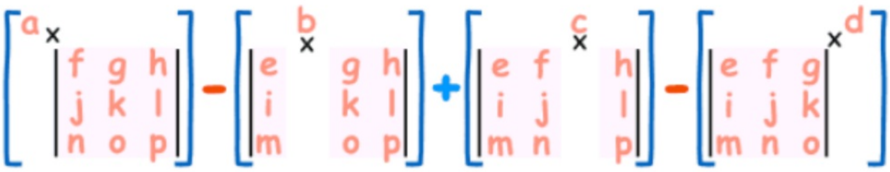
$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |C| &= 6\times(-2\times7 - 5\times8) - 1\times(4\times7 - 5\times2) + 1\times(4\times8 - (-2\times2)) \\ &= 6\times(-54) - 1\times(18) + 1\times(36) \\ &= \mathbf{-306} \end{aligned}$$

For 4×4 Matrices and Higher

The pattern continues for 4×4 matrices:

- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the **+−+−** pattern (+a... −b... +c... −d...). This is important to remember.

The pattern continues for 5×5 matrices and higher. Usually best to use a Matrix Calculator for those!

Not The Only Way

This method of calculation is called the "Laplace expansion" and I like it because the pattern is easy to remember. But there are other methods (just so you know).

Summary

- For a 2×2 matrix the determinant is **ad - bc**
- For a 3×3 matrix multiply **a** by the **determinant of the 2×2 matrix** that is **not** in **a**'s row or column, likewise for **b** and **c**, but remember that **b** has a negative sign!
- The pattern continues for larger matrices: multiply **a** by the **determinant of the matrix** that is **not** in **a**'s row or column, continue like this across the whole row, but remember the + − + − pattern.

What is the determinant of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$?

A 11

B 1

C -11

D -1

Generally, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant $|A| = ad - bc$

In this case $a = 2$, $b = 5$, $c = 1$ and $d = -3$

So $|A| = 2 \times (-3) - 5 \times 1 = -6 - 5 = -11$

What is the determinant of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$?

A -13

B -7

C 7

D 13

Generally, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant $|A| = ad - bc$

In this case a = 3, b = -5, c = 2 and d = 1

So $|A| = [3 \times 1] - [(-5) \times 2] = 3 - (-10) = 3 + 10 = 13$

What is the determinant of the matrix $A = \begin{bmatrix} -2 & 4 \\ -6 & 2 \end{bmatrix}$?

A -28

B -20

C 20

D 28

Generally, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the Determinant $|A| = ad - bc$

In this case a = -2, b = 4, c = -6 and d = 2

So $|A| = (-2 \times 2) - (4 \times -6) = -4 - (-24) = -4 + 24 = 20$

What is the determinant of the matrix $A = \begin{bmatrix} -3 & -1 \\ 4 & -5 \end{bmatrix}$?

A -19

B -11

C 11

D 19

Generally, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant $|A| = ad - bc$

In this case $a = -3$, $b = -1$, $c = 4$ and $d = -5$

So $|A| = (-3 \times (-5)) - (-1 \times 4) = 15 - (-4) = 15 + 4 = 19$

What is the determinant of the matrix $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 4 \\ -3 & 1 & 3 \end{bmatrix}$?

A 70	B -44
C -60	D -74

Generally for the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$

In this case a = 3, b = 0, c = -1, d = 2, e = -5, f = 4, g = -3, h = 1 and i = 3

So |A|

[illegible]

[illegible]

[illegible]

[illegible]

= -44

What is the determinant of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -1 \\ -5 & 3 & -2 \end{bmatrix}$?

A -63

B -19

C -15

D 59

Generally, for the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

The determinant $|A| = a(ei - fh) - b(di - fg) - c(dh - eg)$

In this case a = 2, b = -3, c = 1, d = 4, e = 2, f = -1, g = -5, h = 3 and i = -2

So |A|

$= 2[2 \times (-2) - (-1) \times 3] - (-3)[4 \times (-2) - (-1) \times (-5)] + 1[4 \times 3 - 2 \times (-5)]$

$= 2[(-4) - (-3)] + 3[(-8) - 5] + 1[12 - (-10)]$

$= 2 \times (-1) + 3 \times (-13) + 1 \times 22$

$= -2 - 39 + 22$

$= -19$

What is the determinant of the matrix $A = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix}$?

A -55

B 43

C 55

D 65

Generally, for the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

The determinant $|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$

In this case $a = 5, b = 1, c = -2, d = -1, e = 0, f = 4, g = 2, h = -3$ and $i = 3$

So $|A|$

$= 5[0 \times 3 - 4 \times (-3)] - 1[(-1) \times 3 - 4 \times 2] + (-2)[(-1) \times (-3) - 0 \times 2]$

$= 5[0 - (-12)] - 1[(-3) - 8] - 2[3 - 0]$

$= 5 \times 12 - 1 \times (-11) - 2 \times 3$

$= 60 + 11 - 6$

$= 65$

What is the determinant of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$?

A 13

B 21

C 53

D 59

Generally, for the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

The determinant $|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$

In this case $a = 2, b = 0, c = -1, d = 3, e = 5, f = 2, g = -4, h = 1$ and $i = 4$

So A

$= 2[5 \times 4 - 2 \times 1] - 0[3 \times 4 - 2 \times (-4)] + (-1)[3 \times 1 - 5 \times (-4)]$

$= 2[20 - 2] - 0 - 1[3 - (-20)]$

$= 2 \times 18 - 0 - 1 \times 23$

$= 36 - 23$

$= 13$

What is the determinant of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$?

A -44

B -4

C 8

D 12

Generally, for the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

The determinant $|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$

In this case a = 2, b = -1, c = 0, d = 3, e = -5, f = 2, g = 1, h = 4 and i = -2

So |A|

$= 2[(-5)\times(-2) - 2\times 4] - (-1)[3\times(-2) - 2\times 1] + 0[3\times 4 - (-5)\times 1]$

$= 2[10 - 8] - 1[(-6) - 2] + 0$

$= 2\times 2 - 1\times(-8) + 0$

$= 4 - 8$

$= -4$

Inverse of a Matrix

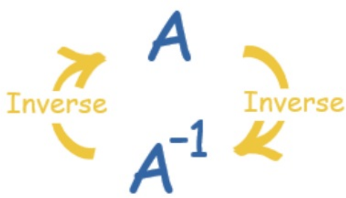
Please read our Introduction to Matrices first.

What is the Inverse of a Matrix?

This is the reciprocal of a **number**:

Reciprocal of a Number

The **Inverse of a Matrix** is the **same idea** but we write it A^{-1}



Why not $^1/_A$? Because we don't divide by a matrix! And anyway $^1/_8$ can also be written 8^{-1}

And there are other similarities:

- When we **multiply a number** by its **reciprocal** we get **1**

$$8 \times (^1/_8) = \mathbf{1}$$

- When we **multiply a matrix** by its **inverse** we get the **Identity Matrix** (which is like "1" for matrices):

$$A \times A^{-1} = \mathbf{I}$$

- Same thing when the inverse comes first:

$$(^1/_8) \times 8 = \mathbf{1}$$

$$A^{-1} \times A = \mathbf{I}$$

Identity Matrix

We just mentioned the "Identity Matrix". It is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- Its symbol is the capital letter **I**.

The Identity Matrix can be 2×2 in size, or 3×3, 4×4, etc ...

Definition

Here is the definition:

The inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Sometimes there is no inverse at all.

2x2 Matrix

OK, how do we calculate the inverse?

Well, for a 2x2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4{\times}0.6 + 7{\times}{-0.2} & 4{\times}{-0.7} + 7{\times}0.4 \\ 2{\times}0.6 + 6{\times}{-0.2} & 2{\times}{-0.7} + 6{\times}0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And, hey!, we end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Why don't you have a go at multiplying these? See if you also get the Identity Matrix:

$$\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Why Do We Need an Inverse?

Because with matrices we **don't divide**! Seriously, there is no concept of dividing by a matrix.

But we can **multiply by an inverse**, which achieves the same thing.

Imagine we can't divide by numbers ...

... and someone asks "How do I share 10 apples with 2 people?"

But we can take the **reciprocal** of 2 (which is 0.5), so we answer:

$$\mathbf{10 \times 0.5 = 5}$$

They get 5 apples each.

The same thing can be done with matrices:

Say we want to find matrix X, and we know matrix A and B:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

In that example we were very careful to get the multiplications correct, because with matrices the order of multiplication matters. AB is almost never equal to BA.

A Real Life Example: Bus and Train

A group took a trip on a **bus**, at $3 per child and $3.20 per adult for a total of $118.40.

They took the **train** back at $3.50 per child and $3.60 per adult for a total of $135.20.

How many children, and how many adults?

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\overset{\text{Child}\quad\text{Adult}}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \overset{\text{Bus}\quad\text{Train}}{\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}} = \overset{\text{Bus}\quad\text{Train}}{\begin{bmatrix} 118.4 & 135.2 \end{bmatrix}}$$

This is just like the example above:

$$XA = B$$

So to solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3\text{x}3.6-3.5\text{x}3.2} \begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

Now we have the inverse we can solve using:

$$X = BA^{-1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 118.4{\times}{-9} + 135.2{\times}8 & 118.4{\times}8.75 + 135.2{\times}{-7.5} \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \end{bmatrix}$$

There were 16 children and 22 adults!

The answer almost appears like magic. But it is based on good mathematics.

Calculations like that (but using much larger matrices) help Engineers design buildings, are used in video games and computer animations to make things look 3-dimensional, and many other places.

It is also a way to solve Systems of Linear Equations.

The calculations are done by computer, but the people must understand the formulas.

Order is Important

Say that we are trying to find "X" in this case:

$$AX = B$$

This is different to the example above! X is now **after** A.

With matrices the order of multiplication usually changes the answer. Do not assume that AB = BA, it is almost never true.

So how do we solve this one? Using the same method, but put A^{-1} in front:

$$A^{-1}AX = A^{-1}B$$

And we know that $A^{-1}A = I$, so:

$$IX = A^{-1}B$$

We can remove I:

$$X = A^{-1}B$$

And we have our answer (assuming we can calculate A^{-1})

Why don't we try our bus and train example, but with the data set up that way around.

It can be done that way, but we must be careful how we set it up.

This is what it looks like as AX = B:

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

It looks so neat! I think I prefer it like this.

Also note how the rows and columns are swapped over ("Transposed") compared to the previous example.

To solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3\times3.6-3.2\times3.5} \begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}$$

It is like the inverse we got before, but
Transposed (rows and columns swapped over).

Now we can solve using:

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix} \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

$$= \begin{bmatrix} -9 \times 118.4 + 8 \times 135.2 \\ 8.75 \times 118.4 - 7.5 \times 135.2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 22 \end{bmatrix}$$

Same answer: 16 children and 22 adults.

So matrices are powerful things, but they do need to be set up correctly!

The Inverse May Not Exist

First of all, to have an inverse the matrix must be "square" (same number of rows and columns).

But also the **determinant cannot be zero** (or we end up dividing by zero). How about this:

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \frac{1}{3\times8-4\times6}\begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{24-24}\begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

24-24? That equals 0, and **1/0 is undefined**.
We cannot go any further! This Matrix has no Inverse.

Such a matrix is called "Singular", which only happens when the determinant is zero.

And it makes sense ... look at the numbers: the second row is just double the first row, and does **not add any new information**.

And the determinant lets us know this fact.

(Imagine in our bus and train example that the prices on the train were all exactly 50% higher than the bus: so now we can't figure out any differences between adults and children. There needs to be something to set them apart.)

Conclusion

- The inverse of A is A^{-1} only when $A \times A^{-1} = A^{-1} \times A = \mathbf{I}$
- To find the inverse of a 2x2 matrix: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).
- Sometimes there is no inverse at all

What is the inverse of the matrix

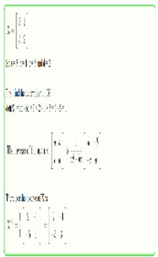
$$X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}?$$

A $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

B $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$

C $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

D $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$



What is the inverse of the matrix

$$X = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}?$$

A $\begin{bmatrix} 1 & -4 \\ -2 & 7 \end{bmatrix}$

B $\begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$

C $\begin{bmatrix} -4 & 1 \\ 7 & -2 \end{bmatrix}$

D

$$X = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$$

So a = 7, b = 4, c = 2 and d = 1

First find the determinant of X:
det(X) = ad - bc = 7 × 1 - 4 × 2 = 7 - 8 = -1

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -4 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$$

What is the inverse of the matrix

$$X = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} ?$$

A $\begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$

B $\begin{bmatrix} -3.5 & 1.5 \\ 2 & -1 \end{bmatrix}$

C $\begin{bmatrix} 3.5 & -1.5 \\ -2 & 1 \end{bmatrix}$

D $\begin{bmatrix} -1.5 & 3.5 \\ 1 & -2 \end{bmatrix}$

$$X = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

So $a = 2$, $b = 3$, $c = 4$ and $d = 7$

First find the determinant of X:

$\det(X) = ad - bc = 2 \times 7 - 3 \times 4 = 14 - 12 = 2$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3.5 & -1.5 \\ -2 & 1 \end{bmatrix}$$

What is the inverse of the matrix

$$Y = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}?$$

A $\begin{bmatrix} 0.75 & 0.5 \\ 1 & 1 \end{bmatrix}$

B $\begin{bmatrix} -0.75 & -0.5 \\ -1 & -1 \end{bmatrix}$

C $\begin{bmatrix} 0.5 & 0.75 \\ 1 & 1 \end{bmatrix}$

D $\begin{bmatrix} -0.5 & -0.75 \\ -1 & -1 \end{bmatrix}$

$$Y = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}$$

So a = 4, b = -3, c = -4 and d = 2

First find the determinant of Y.

det(Y) = ad - bc = 4 × 2 - (-3) × (-4) = 8 - 12 = -4

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of Y is

$$Y^{-1} = \frac{1}{-4}\begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} -0.5 & -0.75 \\ -1 & -1 \end{bmatrix}$$

What is the inverse of the matrix

$$Z = \begin{bmatrix} -8 & 4 \\ 6 & -5 \end{bmatrix} ?$$

A $\begin{bmatrix} \frac{5}{16} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{2} \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{4} & -\frac{5}{16} \\ -\frac{1}{2} & -\frac{3}{8} \end{bmatrix}$

C $\begin{bmatrix} -\frac{5}{16} & -\frac{1}{4} \\ -\frac{3}{8} & -\frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} \frac{1}{4} & \frac{5}{16} \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}$

$$Z = \begin{bmatrix} -8 & 4 \\ 6 & -5 \end{bmatrix}$$

So $a = -8$, $b = 4$, $c = 6$ and $d = -5$

First find the determinant of Z

$\det(Z) = ad - bc = (-8)\times(-5) - 4\times 6 = 40 - 24 = 16$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of Z is

$$Z^{-1} = \frac{1}{16}\begin{bmatrix} -5 & -4 \\ -6 & -8 \end{bmatrix} = \begin{bmatrix} -\frac{5}{16} & -\frac{1}{4} \\ -\frac{3}{8} & -\frac{1}{2} \end{bmatrix}$$

Which one of the following matrices is singular?

A $P = \begin{bmatrix} -6 & 9 \\ 2 & 3 \end{bmatrix}$

B $Q = \begin{bmatrix} -6 & 9 \\ -2 & -3 \end{bmatrix}$

C $R = \begin{bmatrix} 6 & 9 \\ 2 & -3 \end{bmatrix}$

D $S = \begin{bmatrix} -6 & 9 \\ 2 & -3 \end{bmatrix}$

A singular matrix will have a determinant of 0

$\det(P) = -6 \times 3 - 9 \times 2 = -18 - 18 = -36$

$\det(Q) = -6 \times (-3) - 9 \times (-2) = 18 + 18 = 36$

$\det(R) = 6 \times (-3) - 9 \times 2 = -18 - 18 = -36$

$\det(S) = -6 \times (-3) - 9 \times 2 = 18 - 18 = 0$

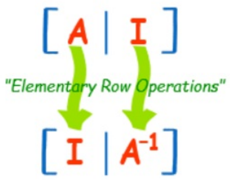
$\det(S) = 0$, so S is singular.

Inverse of a Matrix
using Elementary Row Operations

Also called the Gauss-Jordan method.

This is a fun way to find the Inverse of a Matrix:

Play around with the rows (adding, multiplying or swapping) until we make Matrix **A** into the Identity Matrix **I**



And by ALSO doing the changes to an Identity Matrix it magically turns into the Inverse!

The **"Elementary Row Operations"** are simple things like adding rows, multiplying and swapping ... but let's see with an example:

Example: find the Inverse of "A":

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix A, and write it down with an Identity Matrix I next to it:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

(This is called the "Augmented Matrix")

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

Now we do our best to turn "A" (the Matrix on the left) into an Identity Matrix. The goal is to make Matrix A have **1**s on the diagonal and **0**s elsewhere (an Identity Matrix) ... and the right hand side comes along for the ride, with every operation being done on it as well.

But we can only do these **"Elementary Row Operations"**:

- **swap** rows
- **multiply** or divide each element in a a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

(A, I)

Start with **A** next to **I**

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Add

Add row 2 to row 1,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Divide by 5

then divide row 1 by 5,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Subtract x 2

Then take 2 times the first row, and subtract it from the second row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Multiply by $-\frac{1}{2}$

Multiply second row by -1/2,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right]$$

Swap

Now swap the second and third row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right]$$

Subtract

(I, A^{-1})

Last, subtract the third row from the second row,

And we are done!

And matrix **A** has been made into an Identity Matrix ...

... and at the same time an Identity Matrix got made into $\mathbf{A}^{-1}$

$$A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

DONE! Like magic, and just as fun as solving any puzzle.

And note: there is no "right way" to do this, just keep playing around until we succeed!

Larger Matrices

We can do this with larger matrices, for example, try this 4x4 matrix:

$$B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Start Like this:

$$\left[\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

See if you can do it yourself (I would begin by dividing the first row by 4, but you do it your way).

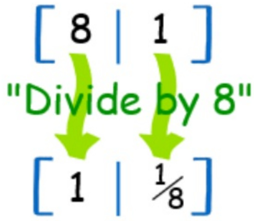
You can check your answer using the Matrix Calculator (use the "inv(A)" button).

Why it Works

I like to think of it this way:

- when we turn "8" into "1" by dividing by 8,
- and do the same thing to "1", it turns into "1/8"

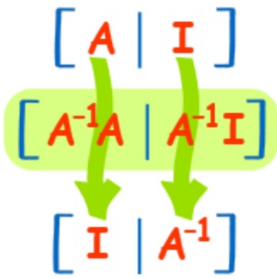
And "1/8" is the (multiplicative) **inverse of 8**



Or, more technically:

The **total effect of all the row operations** is the same as **multiplying by $\mathbf{A}^{-1}$**

So $\mathbf{A}$ becomes $\mathbf{I}$ (because $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$)
And $\mathbf{I}$ becomes $\mathbf{A}^{-1}$ (because $\mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$)



Find the inverse of the matrix

$$A = \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} 2 & 1 \\ -3\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$

B $\begin{bmatrix} 2 & 1 \\ 3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$

C $\begin{bmatrix} -2 & -1 \\ 3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} 2 & -1 \\ -3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$

This is just one of several possible methods:

Start with A next to I

$$\begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix} \bigg| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add half the second row to the first one

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 7 & 4 \end{bmatrix} \bigg| \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Subtract 14 times the first row from the second row

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 4 \end{bmatrix} \bigg| \begin{bmatrix} 1 & \frac{1}{2} \\ -14 & -6 \end{bmatrix}$$

Multiply the first row by 2

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \bigg| \begin{bmatrix} 2 & 1 \\ -14 & -6 \end{bmatrix}$$

Divide the second row by 4

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bigg| \begin{bmatrix} 2 & 1 \\ -3\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$$

Therefore $A^{-1} = \begin{bmatrix} 2 & 1 \\ -3\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$

Find the inverse of the matrix

$$B = \begin{bmatrix} 5 & -1 \\ 15 & -2 \end{bmatrix}$$

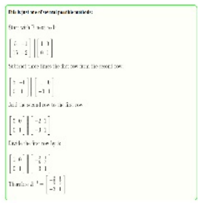
using Elementary Row Operations

A $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -3 & 1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ 3 & -1 \end{bmatrix}$

C $\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ -3 & 1 \end{bmatrix}$

D $\begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ -3 & -1 \end{bmatrix}$



Find the inverse of the matrix

$$D = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} -2 & -2\frac{1}{2} \\ 1 & 1\frac{1}{2} \end{bmatrix}$

B $\begin{bmatrix} 2 & -2\frac{1}{2} \\ 1 & -1\frac{1}{2} \end{bmatrix}$

C $\begin{bmatrix} -2 & 2\frac{1}{2} \\ -1 & 1\frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} 2 & 2\frac{1}{2} \\ -1 & -1\frac{1}{2} \end{bmatrix}$

This is just one of several possible methods:

Start with D next to I:

$$\begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add one and a quarter times the second row to the first row:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -2 & -4 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & 1\frac{1}{4} \\ 0 & 1 \end{bmatrix}$$

Multiply the first row by 2:

$$\begin{bmatrix} 1 & 0 \\ -2 & -4 \end{bmatrix} \Bigg| \begin{bmatrix} 2 & 2\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Add 2 times the first row to the second row:

$$\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \Bigg| \begin{bmatrix} 2 & 2\frac{1}{2} \\ 4 & 6 \end{bmatrix}$$

Divide the second row by -4.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} 2 & 2\frac{1}{2} \\ -1 & -1\frac{1}{2} \end{bmatrix}$$

Therefore $D^{-1} = \begin{bmatrix} 2 & 2\frac{1}{2} \\ -1 & -1\frac{1}{2} \end{bmatrix}$

Find the inverse of the matrix

$$E = \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} 0.25 & 0.125 \\ -0.5 & -0.75 \end{bmatrix}$

B $\begin{bmatrix} -0.25 & -0.125 \\ 0.5 & 0.75 \end{bmatrix}$

C $\begin{bmatrix} 0.25 & 0.125 \\ 0.5 & 0.75 \end{bmatrix}$

D $\begin{bmatrix} 0.25 & -0.125 \\ -0.5 & 0.75 \end{bmatrix}$

This is just one of several possible methods:

Start with E next to I:

$$\begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix} \Bigg\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add one half times the second row to the first row:

$$\begin{bmatrix} 4 & 0 \\ -4 & 2 \end{bmatrix} \Bigg\| \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Add the first row to the second row:

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \Bigg\| \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1\frac{1}{2} \end{bmatrix}$$

Divide the first row by 4:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Bigg\| \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ 1 & 1\frac{1}{2} \end{bmatrix}$$

Divide the second row by 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Bigg\| \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Therefore $E^{-1} = \begin{bmatrix} 0.25 & 0.125 \\ 0.5 & 0.75 \end{bmatrix}$

Find the inverse of the matrix

$$F = \begin{bmatrix} 4 & -3 \\ 2.5 & -2 \end{bmatrix}$$

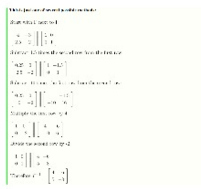
using Elementary Row Operations

A $\begin{bmatrix} -4 & 6 \\ 5 & -8 \end{bmatrix}$

B $\begin{bmatrix} 4 & -6 \\ 5 & -8 \end{bmatrix}$

C $\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$

D $\begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$



Find the inverse of the matrix

$$C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

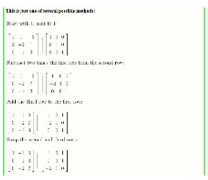
using Elementary Row Operations

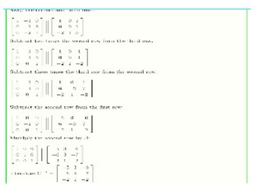
A $\begin{bmatrix} -5 & -3 & -6 \\ -6 & -3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$

B $\begin{bmatrix} -5 & 3 & -6 \\ 6 & 3 & 7 \\ -2 & 1 & 2 \end{bmatrix}$

C $\begin{bmatrix} 5 & 3 & 6 \\ -6 & 3 & -7 \\ 2 & 1 & -2 \end{bmatrix}$

D $\begin{bmatrix} -5 & 3 & -6 \\ -6 & 3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$





Find the inverse of the matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ -3 & 0 & 1 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} -0.75 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ -2.25 & 0.75 & 1.75 \end{bmatrix}$

B $\begin{bmatrix} 0.75 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 2.25 & 0.75 & 1.75 \end{bmatrix}$

C $\begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0.25 \\ 2.25 & 0.75 & 1.75 \end{bmatrix}$

D $\begin{bmatrix} 0.75 & 0.25 & 0.25 \\ 2.25 & 0.75 & 1.75 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$

Find the inverse of the matrix

$$G = \begin{bmatrix} 3 & 3 & 2 \\ -2 & 1 & 5 \\ 4 & -3 & -12 \end{bmatrix}$$

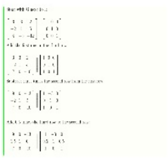
using Elementary Row Operations

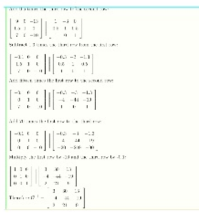
A $\begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$

B $\begin{bmatrix} 3 & 30 & 13 \\ 4 & 44 & 19 \\ 2 & 21 & 9 \end{bmatrix}$

C $\begin{bmatrix} -3 & -30 & -13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$

D $\begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ -2 & -21 & -9 \end{bmatrix}$





Find the inverse of the matrix

$$H = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

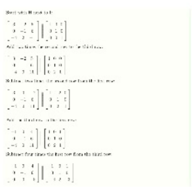
using Elementary Row Operations

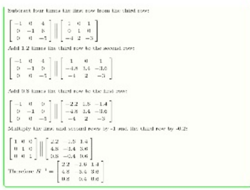
A $\begin{bmatrix} -2.2 & 1.6 & -1.4 \\ 4.8 & -3.4 & 3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

B $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ -4.8 & 3.4 & -3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

C $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ 4.8 & -3.4 & 3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

D $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ 4.8 & -3.4 & 3.6 \\ -0.8 & 0.4 & -0.6 \end{bmatrix}$





Find the inverse of the matrix

$$J = \begin{bmatrix} -2 & -2 & 1 \\ -4 & -8 & 4 \\ -1 & 5 & 0 \end{bmatrix}$$

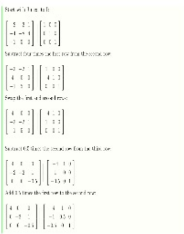
using Elementary Row Operations

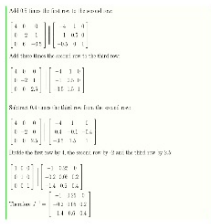
A $\begin{bmatrix} 1 & -0.25 & 0 \\ -0.2 & 0.05 & 0.2 \\ -1.4 & 0.6 & 0.4 \end{bmatrix}$

B $\begin{bmatrix} -1 & 0.25 & 0 \\ 0.2 & -0.05 & -0.2 \\ -1.4 & 0.6 & 0.4 \end{bmatrix}$

C $\begin{bmatrix} -1 & 0.25 & 0 \\ -0.2 & 0.05 & 0.2 \\ 1.4 & -0.6 & -0.4 \end{bmatrix}$

D $\begin{bmatrix} -1 & 0.25 & 0 \\ -0.2 & 0.05 & 0.2 \\ -1.4 & 0.6 & 0.4 \end{bmatrix}$





Inverse of a Matrix using Minors, Cofactors and Adjugate

(Note: also check out Matrix Inverse by Row Operations and the Matrix Calculator.)

We can calculate the Inverse of a Matrix by:

- Step 1: calculating the Matrix of Minors,
- Step 2: then turn that into the Matrix of Cofactors,
- Step 3: then the Adjugate, and
- Step 4: multiply that by 1/Determinant.

But it is best explained by working through an example!

Example: find the Inverse of A:

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

It needs 4 steps. It is all simple arithmetic but there is a lot of it, so try not to make a mistake!

Step 1: Matrix of Minors

The first step is to create a "Matrix of Minors". This step has the most calculations.

For each element of the matrix:

- ignore the values on the current row and column
- calculate the determinant of the remaining values

Put those determinants into a matrix (the "Matrix of Minors")

Determinant

For a 2×2 matrix (2 rows and 2 columns) the determinant is easy: **ad-bc**

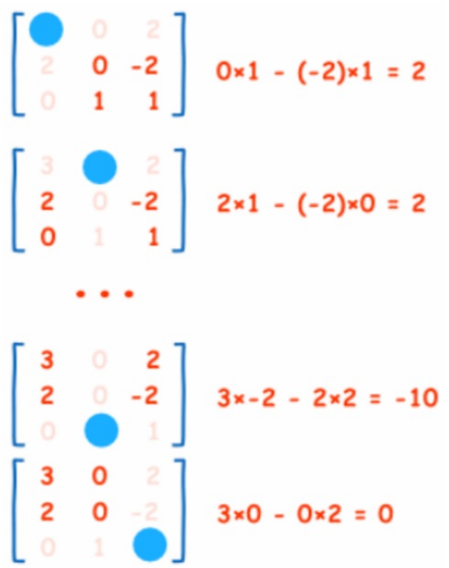
Think of a cross:

- Blue means positive (+ad),
- Red means negative (-bc)

(It gets harder for a 3×3 matrix, etc)

The Calculations

Here are the first two, and last two, calculations of the "**Matrix of Minors**" (notice how I ignore the values in the current row and columns, and calculate the determinant using the remaining values):



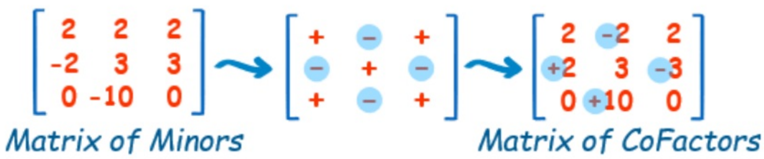
And here is the calculation for the whole matrix:

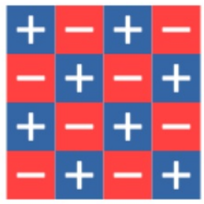
$$\begin{bmatrix} 0\times1 - (-2)\times1 & 2\times1 - (-2)\times0 & 2\times1 - 0\times0 \\ 0\times1 - 2\times1 & 3\times1 - 2\times0 & 3\times1 - 0\times0 \\ 0\times(-2) - 2\times0 & 3\times(-2) - 2\times2 & 3\times0 - 0\times2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

Matrix of Minors

Step 2: Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, we need to change the sign of alternate cells, like this:



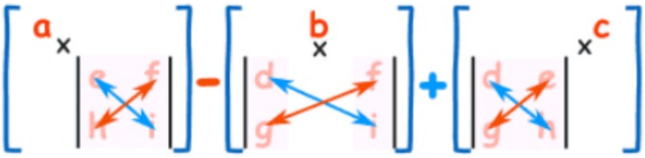


Step 3: Adjugate (also called Adjoint)

Now "Transpose" all elements of the previous matrix... in other words swap their positions over the diagonal (the diagonal stays the same):

Step 4: Multiply by 1/Determinant

Now find the determinant of the original matrix. This isn't too hard, because we already calculated the determinants of the smaller parts when we did "Matrix of Minors".



In practice we can just multiply each of the top row elements by the cofactor for the same location:

Elements of top row: 3, 0, 2
Cofactors for top row: 2, −2, 2

Determinant = 3×2 + 0×(−2) + 2×2 = **10**

(Just for fun: try this for any other row or column, they should also get 10.)

And now multiply the Adjugate by 1/Determinant:

$$A^{-1} = \frac{1}{10}\underbrace{\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}}_{\text{Adjugate}} = \underbrace{\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}}_{\text{Inverse}}$$

And we are done!

Larger Matrices

It is exactly the same steps for larger matrices (such as a 4×4, 5×5, etc), but wow! there is a lot of calculation involved.

For a 4×4 Matrix we have to calculate 16 3×3 determinants. So it is often easier to use computers (such as the Matrix Calculator.)

Conclusion

- For each element, calculate the **determinant of the values not on the row or column**, to make the Matrix of Minors
- Apply a **checkerboard** of minuses to make the Matrix of Cofactors
- **Transpose** to make the Adjugate
- Multiply by **1/Determinant** to make the Inverse

Find the inverse of the matrix

$$C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

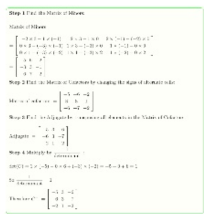
using Minors, Cofactors and Adjugate

A $\begin{bmatrix} -5 & -3 & -6 \\ 6 & 3 & 7 \\ -2 & -1 & -2 \end{bmatrix}$

B $\begin{bmatrix} 5 & 3 & 6 \\ -6 & -3 & -7 \\ 2 & 1 & 2 \end{bmatrix}$

C $\begin{bmatrix} -5 & 3 & -6 \\ -6 & 3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$

D $\begin{bmatrix} -5 & -6 & -2 \\ 3 & 3 & 1 \\ -6 & -7 & -2 \end{bmatrix}$


Step 1 Find the Matrix of Minors

Matrix of Minors

Step 2 Find the Matrix of Cofactors by changing the signs of alternate cells

Step 4 Multiply by

Find the inverse of the matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ -3 & 0 & 1 \end{bmatrix}$$

Step 1 Find the Matrix of Minors:

Matrix of Minors

$$= \begin{bmatrix} 3\times 1-(-1)\times 0 & 1\times 1-(-1)\times(-3) & 1\times 0-3\times(-3) \\ -1\times 1-0\times 0 & 2\times 1-0\times(-3) & 2\times 0-(-1)\times(-3) \\ (-1)\times(-1)-0\times 3 & 2\times(-1)-0\times 1 & 2\times 3-(-1)\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 9 \\ -1 & 2 & -3 \\ 1 & -2 & 7 \end{bmatrix}$$

Step 2 Find the Matrix of Cofactors by changing the signs of alternate cells:

$$\text{Matrix of cofactors} = \begin{bmatrix} 3 & 2 & 9 \\ 1 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

Step 3 Find the Adjugate by transposing all elements in the Matrix of Cofactors:

$$\text{Adjugate} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 9 & 3 & 7 \end{bmatrix}$$

Step 4 Multiply by $\frac{1}{\text{determinant}}$:

$\det(M) = 2\times 3-(-1)\times(-2)+0\times 9 = 6-2+0 = 4$

So $\frac{1}{\text{determinant}} = \frac{1}{4}$

Therefore $M^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 9 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 2.25 & 0.75 & 1.75 \end{bmatrix}$

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -2 & 3 & -4 \\ -3 & 1 & -4 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} -1.6 & 0.6 & -1.8 \\ 0.8 & 0.2 & 0.4 \\ -1.4 & 0.4 & -1.2 \end{bmatrix}$

B $\begin{bmatrix} -1.6 & 0.6 & -1.8 \\ 0.8 & 0.2 & 0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

C $\begin{bmatrix} 1.6 & -0.6 & 1.8 \\ 0.8 & 0.2 & 0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

D $\begin{bmatrix} -1.6 & 0.6 & -1.8 \\ -0.8 & -0.2 & -0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

Step 1: Find the Matrix of Minors

Matrix of Minors

$$= \begin{bmatrix} 3\times(-4)-(-4)\times 1 & -2\times(-4)-(-4)\times(-3) & -2\times 1-3\times(-3) \\ 0\times(-4)-3\times 1 & 2\times(-4)-3\times(-3) & 2\times 1-0\times(-3) \\ 0\times(-4)-3\times 3 & 2\times(-4)-3\times(-2) & 2\times 3-0\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -4 & 7 \\ -3 & 1 & 2 \\ -9 & -2 & 6 \end{bmatrix}$$

Step 2: Find the Matrix of Cofactors by changing the signs of alternate cells:

$$\text{Matrix of cofactors} = \begin{bmatrix} -8 & 4 & 7 \\ 3 & 1 & -2 \\ -9 & 2 & 6 \end{bmatrix}$$

Step 3: Find the Adjugate by transposing all elements in the Matrix of Cofactors:

$$\text{Adjugate} = \begin{bmatrix} -8 & 3 & -9 \\ 4 & 1 & 2 \\ 7 & -2 & 6 \end{bmatrix}$$

Step 4: Multiply by $\frac{1}{\text{determinant}}$:

$\det(A) = 2\times(-8) - 0\times(-4) + 3\times 7 = -16 - 0 + 21 = 5$

So $\frac{1}{\text{determinant}} = \frac{1}{5} = 0.2$

Therefore $A^{-1} = \begin{bmatrix} -1.6 & 0.6 & -1.8 \\ 0.8 & 0.2 & 0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

Find the inverse of the matrix

$$B = \begin{bmatrix} -3 & 1 & -6 \\ 2 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 4 & -4.5 & 2 \\ -1 & 1.5 & 0 \\ -2 & 2.5 & -1 \end{bmatrix}$

B $\begin{bmatrix} -4 & -4.5 & -2 \\ 1 & 1.5 & 0 \\ 2 & 2.5 & 1 \end{bmatrix}$

C $\begin{bmatrix} -4 & -4.5 & 2 \\ 1 & 1.5 & 0 \\ 2 & 2.5 & -1 \end{bmatrix}$

D $\begin{bmatrix} -4 & 4.5 & 2 \\ 1 & -1.5 & 0 \\ 2 & -2.5 & -1 \end{bmatrix}$

Find the inverse of the matrix

$$D = \begin{bmatrix} 6 & -3 & -2 \\ -5 & -1 & 3 \\ -2 & 4 & 0 \end{bmatrix}$$

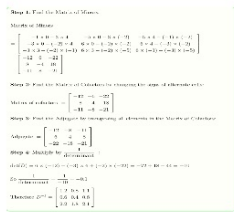
using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 1.2 & 0.8 & 1.1 \\ 0.6 & 0.4 & 0.8 \\ 2.2 & 1.8 & 2.1 \end{bmatrix}$

B $\begin{bmatrix} 1.2 & 0.8 & 1.1 \\ -0.6 & -0.4 & -0.8 \\ 2.2 & 1.8 & 2.1 \end{bmatrix}$

C $\begin{bmatrix} 1.2 & 0.8 & 1.1 \\ 0.6 & 0.4 & 0.8 \\ -2.2 & -1.8 & -2.1 \end{bmatrix}$

D $\begin{bmatrix} -1.2 & -0.8 & -1.1 \\ 0.6 & 0.4 & 0.8 \\ 2.2 & 1.8 & 2.1 \end{bmatrix}$


Step 1: Find the Matrix of Minors

Matrix of Minors

[illegible]

Step 2: [illegible]

Step 3: [illegible]

Step 4: Multiply by $\frac{1}{\text{determinant}}$

[illegible]

Therefore $D^{-1} =$ [illegible]

Find the inverse of the matrix

$$E = \begin{bmatrix} 5 & -2 & 3 \\ -2 & 6 & -1 \\ 3 & -4 & 2 \end{bmatrix}$$

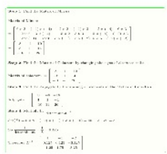
using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 1 & -1 & 2 \\ 0.125 & 0.125 & 0.125 \\ -1.25 & 1.75 & -3.25 \end{bmatrix}$

B $\begin{bmatrix} 1 & -1 & -2 \\ 0.125 & 0.125 & -0.125 \\ -1.25 & 1.75 & 3.25 \end{bmatrix}$

C $\begin{bmatrix} -1 & -1 & -2 \\ -0.125 & 0.125 & -0.125 \\ 1.25 & 1.75 & 3.25 \end{bmatrix}$

D $\begin{bmatrix} 1 & 1 & -2 \\ 0.125 & -0.125 & -0.125 \\ -1.25 & -1.75 & 3.25 \end{bmatrix}$



Find the inverse of the matrix

$$F = \begin{bmatrix} -3 & 7 & -5 \\ 4 & -1 & 12 \\ -2 & 9 & -1 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\frac{1}{11}\begin{bmatrix} 107 & -38 & 79 \\ -20 & 7 & 16 \\ -34 & 13 & 25 \end{bmatrix}$

B $\frac{1}{11}\begin{bmatrix} -107 & 38 & 79 \\ 20 & -7 & 16 \\ -34 & 13 & 25 \end{bmatrix}$

C $\frac{1}{11}\begin{bmatrix} -107 & -38 & -79 \\ 20 & -7 & 16 \\ -34 & -13 & -25 \end{bmatrix}$

D $\frac{1}{11}\begin{bmatrix} -107 & -38 & 79 \\ -20 & -7 & 16 \\ 34 & 13 & -25 \end{bmatrix}$

Step 1: Find the Matrix of Minors:

Matrix of Minors

Step 2: Find the Matrix of Cofactors by changing the signs of alternate cells:

Step 3: Find the Adjugate by transposing all elements in the Matrix of Cofactors:

Step 4: Multiply by $\frac{1}{\text{determinant}}$:

$\det(F) = -3 \times (-107) - 7 \times 20 + (-5) \times 34 = 321 - 140 - 170 = 11$

So $\frac{1}{\text{determinant}} = \frac{1}{11}$

Therefore $F^{-1} = \frac{1}{11}\begin{bmatrix} -107 & -38 & 79 \\ -20 & -7 & 16 \\ 34 & 13 & -25 \end{bmatrix}$

Find the inverse of the matrix

Step 1: Find the Matrix of Minors:

Matrix of Minors

$$= \begin{bmatrix} 1\times(-12)-5\times(-3) & -2\times(-12)-5\times 4 & -2\times(-3)-1\times 4 \\ 3\times(-12)-2\times(-3) & 3\times(-12)-2\times 4 & 3\times(-3)-3\times 4 \\ 3\times 5-2\times 1 & 3\times 5-2\times(-2) & 3\times 1-3\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ -30 & -44 & -21 \\ 13 & 19 & 9 \end{bmatrix}$$

Step 2: Find the Matrix of Cofactors by changing the signs of alternate cells:

$$\text{Matrix of cofactors } = \begin{bmatrix} 3 & -4 & 2 \\ 30 & -44 & 21 \\ 13 & -19 & 9 \end{bmatrix}$$

Step 3: Find the Adjugate by transposing all elements in the Matrix of Cofactors:

$$\text{Adjugate } = \begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$$

Step 4: Multiply by $\frac{1}{\text{determinant}}$:

$\det(G) = 3\times 3 - 3\times 4 + 2\times 2 = 9 - 12 + 4 = 1$

So $\frac{1}{\text{determinant}} = \frac{1}{1} = 1$

$$\text{Therefore } G^{-1} = \begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$$

Find the inverse of the matrix

$$H = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 2.2 & 1.6 & 1.4 \\ 4.8 & 3.4 & 3.6 \\ 0.8 & 0.4 & 0.6 \end{bmatrix}$

B $\begin{bmatrix} 2.2 & -1.6 & -1.4 \\ 4.8 & -3.4 & -3.6 \\ 0.8 & -0.4 & -0.6 \end{bmatrix}$

C $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ 4.8 & -3.4 & 3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

D $\begin{bmatrix} -2.2 & -1.6 & 1.4 \\ -4.8 & -3.4 & 3.6 \\ -0.8 & -0.4 & 0.6 \end{bmatrix}$

Find the inverse of the matrix

$$J = \begin{bmatrix} -2 & -2 & 1 \\ -4 & -8 & 4 \\ -1 & 5 & 0 \end{bmatrix}$$

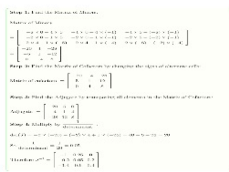
using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 1 & 0.25 & 0 \\ -0.2 & -0.05 & 0.2 \\ -1.4 & 0.6 & -0.4 \end{bmatrix}$

B $\begin{bmatrix} -1 & -0.25 & 0 \\ -0.2 & 0.05 & -0.2 \\ 1.4 & 0.6 & 0.4 \end{bmatrix}$

C $\begin{bmatrix} -1 & 0.25 & 0 \\ 0.2 & 0.05 & 0.2 \\ -1.4 & -0.6 & 0.4 \end{bmatrix}$

D $\begin{bmatrix} -1 & 0.25 & 0 \\ -0.2 & 0.05 & 0.2 \\ -1.4 & 0.6 & 0.4 \end{bmatrix}$



Solving Systems of Linear Equations Using Matrices

Hi there! This page is only going to make sense when you know a little about Systems of Linear Equations and Matrices, so please go and learn about those if you don't know them already!

The Example

One of the last examples on Systems of Linear Equations was this one:

Example: Solve

- $x + y + z = 6$
- $2y + 5z = -4$
- $2x + 5y - z = 27$

We then went on to solve it using "elimination" ... but we can solve it using Matrices!

Using Matrices makes life easier because we can use a computer program (such as the Matrix Calculator) to do all the "number crunching".

But first we need to write the question in Matrix form.

In Matrix Form?

OK. A Matrix is an array of numbers, right?

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

Well, think about the equations:

$$\begin{aligned} x + y + z &= 6 \\ 2y + 5z &= -4 \\ 2x + 5y - z &= 27 \end{aligned}$$

They could be turned into a table of numbers like this:

1	1	1	=	6
0	2	5	=	−4
2	5	−1	=	27

We could even separate the numbers before and after the "=" into:

1	1	1		6
0	2	5	and	−4
2	5	−1		27

Now it looks like we have 2 Matrices.

Now it looks like we have 2 Matrices.

In fact we have a third one, which is [x y z]:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

Why does [x y z] go there? Because when we Multiply Matrices the left side becomes:

"Dot Product"

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y + z \\ 2y + 5z \\ 2x + 5y - z \end{bmatrix}$$

Which is the original left side of our equations above (you might like to check that).

The Matrix Solution

We can write this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

like this:

$$AX = B$$

where

- A is the 3x3 matrix of x, y and z **coefficients**
- X is **x, y and z**, and
- B is **6, −4 and 27**

Then (as shown on the Inverse of a Matrix page) the solution is this:

$$X = A^{-1}B$$

What does that mean?

It means that we can find the values of x, y and z (the X matrix) by multiplying the **inverse of the A matrix** by the **B matrix**.

So let's go ahead and do that.

First, we need to find the **inverse of the A matrix** (assuming it exists!)

Using the Matrix Calculator we get this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix}^{-1} = \frac{1}{-21}\begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}$$

(I left the 1/determinant outside the matrix to make the numbers simpler)

Then multiply A^{-1} by B (we can use the Matrix Calculator again):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-21}\begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21}\begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

And we are done! The solution is:

$$x = 5,$$
$$y = 3,$$
$$z = -2$$

Just like on the Systems of Linear Equations page.

Quite neat and elegant, and the human does the thinking while the computer does the calculating.

Just For Fun ... Do It Again!

For fun (and to help you learn), let us do this all again, but put matrix "X" first.

I want to show you this way, because many people think the solution above is so neat it must be the only way.

So we will solve it like this:

$$XA = B$$

And because of the way that matrices are multiplied we need to set up the matrices differently now. The rows and columns have to be switched over ("transposed"):

"Dot Product"

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix} = \begin{bmatrix} x+y+z & 2y+5z & 2x+5y-z \end{bmatrix}$$

And $XA = B$ looks like this:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 27 \end{bmatrix}$$

The Matrix Solution

Then (also shown on the Inverse of a Matrix page) the solution is this:

$$X = BA^{-1}$$

This is what we get for A^{-1}:

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix}^{-1} = \frac{1}{-21}\begin{bmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{bmatrix}$$

In fact it is just like the Inverse we got before, but Transposed (rows and columns swapped over).

Next we multiply B by A^{-1}:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \frac{1}{-21}\begin{bmatrix} 6 & -4 & 27 \end{bmatrix}\begin{bmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{bmatrix}$$

$$= \frac{1}{-21}\begin{bmatrix} -105 & -63 & 42 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & -2 \end{bmatrix}$$

And the solution is the same:

$$x = 5, y = 3 \text{ and } z = -2$$

It didn't look as neat as the previous solution, but it does show us that there is more than one way to set up and solve matrix equations. Just be careful about the rows and columns!

Use matrices to solve the equations :

3x + y = 3

4x - 3y = 17

A x = -2 and y = 3

B x = 2 and y = -3

he pair of linear equations as a matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \end{bmatrix}$$

the inverse of the matrix:

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

= 1, c = 4 and d = -3

erminant of X:

- bc = 3 × (-3) - 1 × 4 = -9 - 4 = -13

se of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{-13}\begin{bmatrix} -3 & -1 \\ -4 & 3 \end{bmatrix}$$

Multiply both sides of the matrix equation by X^{-1}:

$$\frac{1}{-13}\begin{bmatrix} -3 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -3 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \end{bmatrix}$$

Therefore: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -26 \\ 39 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

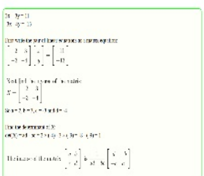
$\Rightarrow x = 2$ and $y = -3$

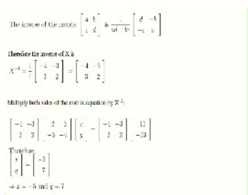
Use matrices to solve the equations:

$2x + 3y = 11$

$-3x - 4y = -13$

A $x = 1$ and $y = 3$	B $x = -2$ and $y = 5$
C $x = -5$ and $y = 7$	D $x = 5$ and $y = -7$





Use matrices to solve the equations:
2x − 5y = 31
-3x + 7y = -45

A $x = 1$ and $y = -6$

B $x = 3$ and $y = -5$

C $x = 8$ and $y = -3$

D $x = 13$ and $y = -1$

2x − 5y = 31
-3x + 7y = -45

First, write the pair of linear equations as a matrix equation

$$\begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 31 \\ -45 \end{bmatrix}$$

Now find the inverse of the matrix:

$$M = \begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}$$

so a = 2, b = -5, c = -3 and d = 7

Find the determinant of M:

det(M) = ad − bc = 2 × 7 − (-5) × (-3) = 14 − 15 = -1

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of M is

$$M^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ -3 & -2 \end{bmatrix}$$

Multiply both sides of the matrix equation by M^{-1}:

[illegible]

Therefore:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$\Rightarrow x = 8$ and $y = -3$

Use matrices to solve the equations :

$3x - y = -11$

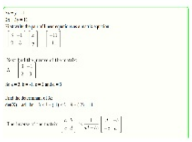
$2x + 3y = 11$

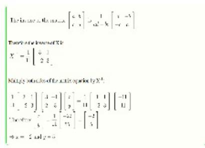
A $x = -2$ and $y = -5$

B $x = -2$ and $y = 5$

C $x = 2$ and $y = 5$

D $x = 3$ and $y = 20$





Use matrices to solve the equations :

$5x + 2y = 1$

$2x - y = 13$

A x = 5 and y = -12

B x = -3 and y = -7

C x = 3 and y = 7

D x = 3 and y = -7

$5x + 2y = 1$

$2x - y = 13$

First write the pair of linear equations as a matrix equation:

$$\begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

Next find the inverse of the matrix:

$$X = \begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix}$$

So a = 5, b = 2, c = 2 and d = -1

Find the determinant of X:

det(X) = ad - bc = 5 × (-1) - 2 × 2 = -5 - 4 = -9

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{-9}\begin{bmatrix} -1 & -2 \\ -2 & 5 \end{bmatrix}$$

Multiply both sides of the matrix equation by X^{-1}:

$$\frac{1}{-9}\begin{bmatrix} -1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-9}\begin{bmatrix} -1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

Therefore: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-9}\begin{bmatrix} -27 \\ 63 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

$\Rightarrow x = 3$ and $y = -7$

Use matrices to solve the equations:

$3x + 2y = -9$

$-5x - 7y = -7$

A $x = -7$ and $y = 6$

B $x = 7$ and $y = -6$

C $x = 3$ and $y = -9$

D $x = 7$ and $y = 6$

Claire and Dale shopped at the same store.

Claire bought 5 kg of apples and 2 kg of bananas and paid altogether $22
Dale bought 4 kg of apples and 6 kg of bananas and paid altogether $33

Use matrices to find the cost of 1 kg of bananas

A $3

B $3.50

C $4

D $4.50

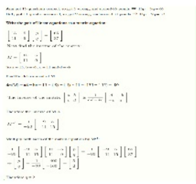
Ann and Billy both entered a quiz. The quiz had twenty questions and points were allocated as follows:

- p points were added for each correctly answered question.
- q points were deducted for each incorrect (or unanswered) question.

Ann got 15 questions correct and scored 65 points.
Billy got 11 questions correct and scored 37 points.

Use matrices to find the value of q?

A $q = 2$	B $q = 3$
C $q = 4$	D $q = 5$



A store sells books and CD's. All the books have the same price and all the CD's have the same price, but different from that of the books.

Diane and Eric both shopped at the store.

Diane bought 5 books and 3 CD's and paid altogether \$90
Eric bought 2 books and 8 CD's and paid altogether \$138

Use matrices to find the cost of one CD.

A \$9

B \$11

C \$13

D \$15

Use matrices to solve the equations :

$x + y - z = -3$

$2x - 3y + 4z = 23$

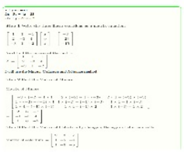
$-3x + y - 2z = -15$

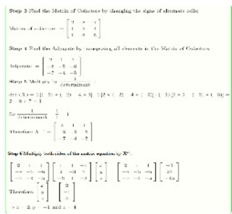
A $x = 2$, $y = -1$ and $z = 4$

B $x = 1$, $y = -2$ and $z = 2$

C $x = 2$, $y = -9$ and $z = -4$

D $x = -2$, $y = -1$ and $z = 0$




Step 3 Find the Matrix of Cofactors by changing the signs of alternate cells:

Step 4 Find the Adjugate by transposing all elements in the Matrix of Cofactors

$\Rightarrow x = 2, y = -1$ and $z = 4$

FINISH LINE