

Tishk International University
Faculty of Administrative Sciences and
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MATHEMATICS

FOR ECONOMICS AND BUSINESS

BUS 143
Part 6

I Grade- Fall

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Matrices

A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)



We talk about one **matrix**, or several **matrices**.

There are many things we can do with them ...

Adding

To add two matrices: add the numbers in the matching positions:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

These are the calculations:

$3+4=7$	$8+0=8$
$4+1=5$	$6-9=-3$

The two matrices must be the same size, i.e. the rows must match in size, and the columns must match in size.

Example: a matrix with **3 rows** and **5 columns** can be added to another matrix of **3 rows** and **5 columns**.

But it could not be added to a matrix with **3 rows** and **4 columns** (the columns don't match in size)

Negative

The negative of a matrix is also simple:


$$-\begin{bmatrix} 2 & -4 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -7 & -10 \end{bmatrix}$$

These are the calculations:

$-(2)=-2$	$-(-4)=+4$
$-(7)=-7$	$-(10)=-10$

Subtracting

To subtract two matrices: subtract the numbers in the matching positions:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

These are the calculations:

$$3-4=-1$$

$$8-0=8$$

$$4-1=3$$

$$6-(-9)=15$$

*Note: subtracting is actually defined as the **addition** of a negative matrix: $A + (-B)$*

Multiply by a Constant

We can multiply a matrix by a **constant** (the value 2 in this case):

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

$2 \times 4 = 8$	$2 \times 0 = 0$
$2 \times 1 = 2$	$2 \times -9 = -18$

We call the constant a **scalar**, so officially this is called "scalar multiplication".

Multiplying by Another Matrix

To **multiply two matrices together** is a bit more difficult ... read [Multiplying Matrices](#) to learn how.

Dividing

And what about division? Well we **don't** actually divide matrices, we do it this way:

$$A/B = A \times (1/B) = A \times B^{-1}$$

where B^{-1} means the "inverse" of B.

So we don't divide, instead we **multiply by an inverse**.

And there are special ways to find the Inverse, learn more at [Inverse of a Matrix](#).

Transposing

To "transpose" a matrix, swap the rows and columns.

We put a "T" in the top right-hand corner to mean transpose:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 4 & -9 \\ 24 & 8 \end{bmatrix}$$

Notation

A matrix is usually shown by a **capital letter** (such as A, or B)

Each entry (or "element") is shown by a **lower case letter** with a "subscript" of **row,column**:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$



Columns go
up and down

Rows and Columns

So which is the row and which is the column?

- Rows go **left-right**
- Columns go **up-down**

To remember that rows come before columns use the word "**arc**":

$a_{r,c}$

Example:

$$B = \begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

Here are some sample entries:

$$b_{1,1} = 6 \text{ (the entry at row 1, column 1 is 6)}$$

$$b_{1,3} = 24 \text{ (the entry at row 1, column 3 is 24)}$$

$$b_{2,3} = 8 \text{ (the entry at row 2, column 3 is 8)}$$

What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

A $\begin{bmatrix} -1 & -8 \\ -1 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & -8 \\ -1 & 0 \end{bmatrix}$

C $\begin{bmatrix} 1 & -8 \\ -7 & 0 \end{bmatrix}$

D $\begin{bmatrix} 1 & 8 \\ -1 & 0 \end{bmatrix}$

What numbers will be in the empty space?

The correct answers

$$\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -8 \\ -1 & 0 \end{bmatrix}$$

What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

A $\begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 \\ -7 & 4 \end{bmatrix}$

C $\begin{bmatrix} 3 & -8 \\ -7 & 4 \end{bmatrix}$

D $\begin{bmatrix} 3 & 2 \\ -7 & 0 \end{bmatrix}$

To subtract two matrices, just subtract the numbers in the matching positions.

Here are the calculations:

$$2 - (-1) = 3 \quad -3 - (-5) = 2$$

$$-4 - 3 = -7 \quad 2 - (-2) = 4$$

What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

A $\begin{bmatrix} 2 & -1 & 6 \\ -6 & -2 & 9 \end{bmatrix}$

B $\begin{bmatrix} 2 & -1 & 6 \\ -6 & 2 & 9 \end{bmatrix}$

C $\begin{bmatrix} 2 & -1 & 6 \\ 4 & -2 & 9 \end{bmatrix}$

D $\begin{bmatrix} 2 & -9 & 6 \\ -6 & -2 & 9 \end{bmatrix}$

To add two matrices, you add the numbers in the matching positions.

Here are the calculations:

$$3 + (-1) = 2, \quad -5 + 4 = -1, \quad 4 + 2 = 6$$

$$-1 + (-5) = -6, \quad 4 + (-2) = 2, \quad 6 + 3 = 9$$

What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

A $\begin{bmatrix} 4 & -9 & 2 \\ 4 & 6 & 3 \end{bmatrix}$

B $\begin{bmatrix} 4 & -9 & 2 \\ -6 & 6 & 3 \end{bmatrix}$

C $\begin{bmatrix} 4 & -9 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

D $\begin{bmatrix} 2 & -9 & 2 \\ 4 & 6 & 3 \end{bmatrix}$



What is the transpose of the matrix $\begin{bmatrix} -3 & 1 & 4 \\ 5 & 2 & -1 \end{bmatrix}$?

A $\begin{bmatrix} 3 & -1 & -4 \\ -5 & -2 & 1 \end{bmatrix}$

B $\begin{bmatrix} 5 & 2 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

C $\begin{bmatrix} -3 & 5 \\ 1 & 2 \\ 4 & -1 \end{bmatrix}$

D $\begin{bmatrix} 5 & -3 \\ 2 & 1 \\ -1 & 4 \end{bmatrix}$

Find the inverse of the matrix.

$$\text{Matrix } \begin{bmatrix} -4 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix}$$

If $A = \begin{bmatrix} -3 & 1 \\ -2 & 4 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 0 & -2 \\ -2 & 4 \end{bmatrix}$ then what is $3A - 2B$?

A $\begin{bmatrix} -1 & -3 \\ -6 & 8 \\ 11 & 5 \end{bmatrix}$

B $\begin{bmatrix} -1 & 9 \\ -6 & 8 \\ 11 & -11 \end{bmatrix}$

C $\begin{bmatrix} -17 & 9 \\ -6 & 16 \\ 19 & -11 \end{bmatrix}$

D $\begin{bmatrix} -17 & -3 \\ 6 & 16 \\ 11 & -11 \end{bmatrix}$

What is $3A + 4B$?
 Therefore $3A + 4B = \begin{bmatrix} 1+5 & 1+1 \\ 1+2 & 1+4 \\ 2+3 & 2+2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \\ 5 & 4 \end{bmatrix}$

What is $2B + 3A$?
 Therefore $2B + 3A = \begin{bmatrix} 1+4 & 1+5 \\ 1+2 & 1+4 \\ 2+3 & 2+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 3 & 5 \\ 5 & 4 \end{bmatrix}$

Therefore $3A - 2B$ =
 $\begin{bmatrix} -9 & 3 \\ -6 & 8 \\ 5 & -4 \end{bmatrix} - \begin{bmatrix} 8 & -6 \\ 0 & -4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} -17 & 9 \\ -6 & 16 \\ 19 & -11 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 3 & -2 \end{bmatrix}$ then what is $2A + B^T$?

A $\begin{bmatrix} 0 & 16 \\ 10 & -10 \end{bmatrix}$

B $\begin{bmatrix} 2 & 13 \\ 6 & -8 \end{bmatrix}$

C $\begin{bmatrix} 2 & 6 \\ 13 & -8 \end{bmatrix}$

D $\begin{bmatrix} 2 & 5 \\ 14 & -8 \end{bmatrix}$

For the following problem:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Find the following:

$$B^T = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$$

Now find:

$$2A + B^T = 2 \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 4+1 & 2+0 \\ 10+2 & -6-2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 12 & -8 \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & 2 & -3 \\ -5 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 6 & 4 \\ -1 & -2 & 3 \end{bmatrix}$ then what is $2A + 5B$?

A $\begin{bmatrix} -8 & 34 & 14 \\ -5 & -2 & 19 \end{bmatrix}$

B $\begin{bmatrix} -8 & 34 & 14 \\ -15 & -2 & 19 \end{bmatrix}$

C $\begin{bmatrix} 12 & 34 & 14 \\ -15 & -2 & 19 \end{bmatrix}$

D $\begin{bmatrix} 12 & -26 & -26 \\ -5 & 18 & -11 \end{bmatrix}$

Working for (A):

$$2A = 2 \begin{bmatrix} 1 & 2 & -3 \\ -5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -6 \\ -10 & 8 & 4 \end{bmatrix}$$
Working for (B):

$$5B = 5 \begin{bmatrix} -2 & 6 & 4 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 30 & 20 \\ -5 & -10 & 15 \end{bmatrix}$$
Working for (C):

$$2A + 5B = \begin{bmatrix} 2 & 4 & -6 \\ -10 & 8 & 4 \end{bmatrix} + \begin{bmatrix} -10 & 30 & 20 \\ -5 & -10 & 15 \end{bmatrix} = \begin{bmatrix} -8 & 34 & 14 \\ -15 & -2 & 19 \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 6 & -2 \end{bmatrix}$ then what is $3A^T - 2B^T$?

A $\begin{bmatrix} 1 & 24 \\ 2 & -13 \end{bmatrix}$

B $\begin{bmatrix} 5 & 24 \\ -14 & -5 \end{bmatrix}$

C $\begin{bmatrix} 5 & 0 \\ -14 & -13 \end{bmatrix}$

D $\begin{bmatrix} 5 & 0 \\ -14 & -5 \end{bmatrix}$

For 2×2 systems given in matrix form:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For the S.C. to be satisfied in matrix form:

$$\Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

For 2×2 systems in row and column:

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \det A^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

For the S.C. to be satisfied in row and column:

$$\Rightarrow \det A^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = 0$$

For vector \vec{v} and matrix A :

$$\vec{v}^T \cdot A \vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

How to Multiply Matrices

A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)

To multiply a matrix by a single number is easy:

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

$2 \times 4 = 8$	$2 \times 0 = 0$
$2 \times 1 = 2$	$2 \times -9 = -18$

We call the number ("2" in this case) a **scalar**, so this is called "**scalar multiplication**".

Multiplying a Matrix by Another Matrix

But to multiply a matrix **by another matrix** we need to do the "[dot product](#)" of rows and columns ... what does that mean? Let us see with an example:

To work out the answer for the **1st row** and **1st column**:

The diagram shows the multiplication of two matrices. The first matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and the second matrix is $\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$. A yellow curved arrow labeled "Dot Product" connects the first row of the first matrix (1, 2, 3) to the first column of the second matrix (7, 9, 11). The result of the dot product is shown as 58 in a yellow circle, which is the first element of the resulting matrix $\begin{bmatrix} 58 & \dots \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \dots \end{bmatrix}$$

The "Dot Product" is where we **multiply matching members**, then sum up:

$$(1, 2, 3) \bullet (7, 9, 11) = 1 \times 7 + 2 \times 9 + 3 \times 11 \\ = 58$$

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

Want to see another example? Here it is for the **1st row** and **2nd column**:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$$(1, 2, 3) \cdot (8, 10, 12) = 1 \times 8 + 2 \times 10 + 3 \times 12 \\ = 64$$

We can do the same thing for the **2nd row** and **1st column**:

$$(4, 5, 6) \cdot (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 \\ = 139$$

And for the **2nd row** and **2nd column**:

$$(4, 5, 6) \cdot (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 \\ = 154$$

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix} \quad \checkmark$$

DONE!

Example: The local shop sells 3 types of pies.

- Apple pies cost **\$3** each
- Cherry pies cost **\$4** each
- Blueberry pies cost **\$2** each

And this is how many they sold in 4 days:

	Mon	Tue	Wed	Thu
<i>Apple</i>	13	9	7	15
<i>Cherry</i>	8	7	4	6
<i>Blueberry</i>	6	4	0	3

Now think about this ... the **value of sales** for Monday is calculated this way:

➡ Apple pie value + Cherry pie value + Blueberry pie value

➡ $\$3 \times 13 + \$4 \times 8 + \$2 \times 6 = \83

So it is, in fact, the "dot product" of prices and how many were sold:

$$(\$3, \$4, \$2) \bullet (13, 8, 6) = \$3 \times 13 + \$4 \times 8 + \$2 \times 6 \\ = \$83$$

We **match** the price to how many sold, **multiply** each, then **sum** the result.

In other words:

- The sales for Monday were: Apple pies: $\$3 \times 13 = \39 , Cherry pies: $\$4 \times 8 = \32 , and Blueberry pies: $\$2 \times 6 = \12 . Together that is $\$39 + \$32 + \$12 = \83
- And for Tuesday: $\$3 \times 9 + \$4 \times 7 + \$2 \times 4 = \63
- And for Wednesday: $\$3 \times 7 + \$4 \times 4 + \$2 \times 0 = \37
- And for Thursday: $\$3 \times 15 + \$4 \times 6 + \$2 \times 3 = \75

So it is important to match each price to each quantity.

Now you know why we use the "dot product".

And here is the full result in Matrix form:

The diagram illustrates the matrix multiplication $\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \$83 & \$63 & \$37 & \$75 \end{bmatrix}$. Yellow arrows show the dot product for the first row: $\$3 \times 13$, $\$4 \times 8$, and $\$2 \times 6$ are highlighted in a yellow box with the equation $\$3 \times 13 + \$4 \times 8 + \$2 \times 6$ below it. An arrow points from this box to the $\$83$ element in the resulting vector.

They sold **\$83** worth of pies on Monday, **\$63** on Tuesday, etc.

(You can put those values into the [Matrix Calculator](#) to see if they work.)

Rows and Columns

To show how many rows and columns a matrix has we often write **rows**×**columns**.

Example: This matrix is **2×3** (2 rows by 3 columns):

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

When we do multiplication:

- The number of **columns of the 1st matrix** must equal the number of **rows of the 2nd matrix**.
- And the result will have the same number of **rows as the 1st matrix**, and the same number of **columns as the 2nd matrix**.

Example:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \$83 & \$63 & \$37 & \$75 \end{bmatrix}$$
$$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$$

In that example we multiplied a 1×3 matrix by a 3×4 matrix (note the 3s are the same), and the result was a 1×4 matrix.

In General:

To multiply an $m \times n$ matrix by an $n \times p$ matrix, the n s must be the same, and the result is an $m \times p$ matrix.

$$m \times n \times n \times p \rightarrow m \times p$$

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×3 Identity Matrix

- It is "square" (has same number of rows as columns)
- It can be large or small (2×2, 100×100, ... whatever)
- It has **1**s on the diagonal and **0**s everywhere else
- Its symbol is the capital letter **I**

It is a **special matrix**, because when we multiply by it, the original is unchanged:

$$\mathbf{A} \times \mathbf{I} = \mathbf{A}$$

$$\mathbf{I} \times \mathbf{A} = \mathbf{A}$$

Order of Multiplication

In arithmetic we are used to:

$$3 \times 5 = 5 \times 3$$

(The Commutative Law of Multiplication)

But this is **not** generally true for matrices (matrix multiplication is **not commutative**):

$$AB \neq BA$$

When we change the order of multiplication, the answer is (usually) **different**.

Example:

See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

The answers are different!

A is a 3×2 matrix

B is a 2×3 matrix

C is a 2×2 matrix

D is a 3×3 matrix

Which of the following products does not exist?

A AB

B AC

C BD

D CD

When we do multiplication of matrices, the number of columns of the 1st matrix must equal the number of rows of the 2nd matrix.

The number of columns of A (2) and the number of rows of B (2) are equal, so the product AB exists.

The number of columns of A (2) and the number of rows of C (2) are equal, so the product AC exists.

The number of columns of B (3) and the number of rows of D (3) are equal, so the product BD exists.

The number of columns of C (2) and the number of rows of D (3) are not equal, so the product CD does not exist.

If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, then what is AB ?

A $\begin{bmatrix} 6 & 3 \\ 4 & 10 \end{bmatrix}$

B $\begin{bmatrix} 7 & 14 \\ 10 & 22 \end{bmatrix}$

C $\begin{bmatrix} 9 & 8 \\ 14 & 14 \end{bmatrix}$

D $\begin{bmatrix} 18 & 8 \\ 23 & 11 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

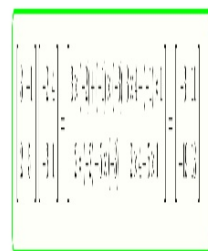
If $X = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$, then what is XY ?

A $\begin{bmatrix} -3 & 11 \\ -19 & 13 \end{bmatrix}$

B $\begin{bmatrix} -10 & -10 \\ 16 & -1 \end{bmatrix}$

C $\begin{bmatrix} -6 & -4 \\ -6 & 5 \end{bmatrix}$

D $\begin{bmatrix} 2 & 22 \\ -7 & 8 \end{bmatrix}$



If $A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$, then what is AB ?

A $\begin{bmatrix} 6 & 1 & -6 \\ 0 & 16 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & -8 \\ -8 & 16 \end{bmatrix}$

C $\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 \\ -8 & 16 \end{bmatrix}$

$$\begin{aligned} & \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3(2)+(-1)(-1)+2(-3) & 3(0)+(-1)(4)+2(2) \\ -2(2)+4(-1)+0(-3) & -2(0)+4(4)+0(2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -8 & 16 \end{bmatrix} \end{aligned}$$

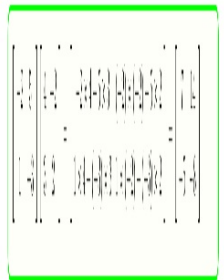
If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$, then what is PQ ?

A $\begin{bmatrix} 7 & 14 \\ -5 & -8 \end{bmatrix}$

B $\begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$

C $\begin{bmatrix} -8 & -10 \\ 3 & -6 \end{bmatrix}$

D $\begin{bmatrix} -10 & -4 \\ 26 & 9 \end{bmatrix}$



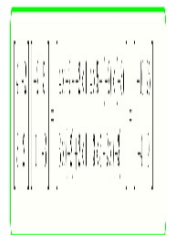
If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$, then what is QP ?

A $\begin{bmatrix} 7 & 14 \\ -5 & -8 \end{bmatrix}$

B $\begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$

C $\begin{bmatrix} -8 & -10 \\ 3 & -6 \end{bmatrix}$

D $\begin{bmatrix} -10 & -4 \\ 26 & 9 \end{bmatrix}$



If $A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$, then what is BA ?

A $\begin{bmatrix} 1 & 0 \\ -8 & 16 \end{bmatrix}$

B $\begin{bmatrix} 6 & 1 & -6 \\ 0 & 16 & 0 \end{bmatrix}$

C $\begin{bmatrix} 6 & 0 \\ 1 & 16 \\ -6 & 0 \end{bmatrix}$

D $\begin{bmatrix} 6 & -2 & 4 \\ -11 & 17 & -2 \\ -13 & 11 & -6 \end{bmatrix}$

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 & 1 \cdot 4 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 0 \cdot 4 + 1 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 4 + 0 \cdot 0 + 1 \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

If $A = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $AB = BA$ then what is the value of x ?

A $x = 0$

B $x = 1$

C $x = 2$

D $x = 3$

$$AB = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + x \times 1 & 1 \times 1 + x \times 2 \\ 2 \times 1 + 3 \times 1 & 2 \times 1 + 3 \times 2 \end{bmatrix} = \begin{bmatrix} 1 + x & 1 + 2x \\ 5 & 8 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 1 \times 2 & 1 \times x + 1 \times 3 \\ 1 \times 1 + 2 \times 2 & 1 \times x + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 3 & x + 3 \\ 5 & x + 6 \end{bmatrix}$$

If $AB = BA$, then the numbers in each cell of the two matrices must be equal

$$\Rightarrow 1 + x = 3, \text{ so } x = 2$$

$$\Rightarrow 1 + 2x = x + 3 \Rightarrow 2x = x + 2 \Rightarrow x = 2$$

$$\Rightarrow 5 = 5 \text{ and}$$

$$\Rightarrow 8 = x + 6, \text{ so } x = 2$$

These equations are all satisfied by $x = 2$

If $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix}$, then what is AB ?

A $\begin{bmatrix} 8 & 5 & -7 \\ 14 & -3 & 20 \\ -13 & -16 & 24 \end{bmatrix}$

B $\begin{bmatrix} 10 & 0 & 2 \\ -3 & 0 & 8 \\ -8 & -3 & 12 \end{bmatrix}$

C $\begin{bmatrix} 21 & 3 & -11 \\ -18 & 4 & 17 \\ -17 & -12 & 4 \end{bmatrix}$

D $\begin{bmatrix} -1 & 14 & -4 \\ -3 & -3 & 23 \\ 1 & -13 & 22 \end{bmatrix}$

$$\begin{aligned} & \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} (5)(2)+(1)(3)+(-2)(-4) & (5)(0)+(1)(5)+(-2)(1) & (5)(-1)+(1)(2)+(-2)(4) \\ (-1)(2)+(0)(3)+(4)(-4) & (-1)(0)+(0)(5)+(4)(1) & (-1)(-1)+(0)(2)+(4)(4) \\ (2)(2)+(-3)(3)+(3)(-4) & (2)(0)+(-3)(5)+(3)(1) & (2)(-1)+(-3)(2)+(3)(4) \end{bmatrix} \\ &= \begin{bmatrix} 25 & 5 & -11 \\ -18 & 4 & 17 \\ -17 & -12 & 4 \end{bmatrix} \end{aligned}$$

If $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix}$, then what is BA ?

A $\begin{bmatrix} 10 & 0 & 2 \\ -3 & 0 & 8 \\ -8 & -3 & 12 \end{bmatrix}$

B $\begin{bmatrix} 8 & 5 & -7 \\ 14 & -3 & 20 \\ -13 & -16 & 24 \end{bmatrix}$

C $\begin{bmatrix} -1 & -3 & 1 \\ 14 & -3 & -13 \\ -4 & 23 & 22 \end{bmatrix}$

$$\begin{aligned}
 BA &= \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 5 \times 2 + 1 \times 3 + (-2) \times (-4) & 5 \times 0 + 1 \times 5 + (-2) \times 1 & 5 \times (-1) + 1 \times 2 + (-2) \times 4 \\ (-1) \times 2 + 0 \times 3 + 4 \times (-4) & (-1) \times 0 + 0 \times 5 + 4 \times 1 & (-1) \times (-1) + 0 \times 2 + 4 \times 4 \\ 2 \times 2 + (-3) \times 3 + 3 \times (-4) & 2 \times 0 + (-3) \times 5 + 3 \times 1 & 2 \times (-1) + (-3) \times 2 + 3 \times 4 \end{bmatrix} \\
 &= \begin{bmatrix} 21 & 3 & -11 \\ -18 & 4 & 17 \\ -17 & -12 & 4 \end{bmatrix}
 \end{aligned}$$

If $A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$, then what is AB ?

A $\begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix}$

B $\begin{bmatrix} 6 & 2 & 0 \\ 0 & 5 & 12 \\ -4 & 8 & 2 \end{bmatrix}$

C $\begin{bmatrix} 5 & 27 & -14 \\ 3 & 29 & -14 \\ -3 & -10 & 6 \end{bmatrix}$

D $\begin{bmatrix} 8 & 29 & -15 \\ 1 & 17 & -16 \\ -10 & -24 & 6 \end{bmatrix}$

$$\begin{aligned} AB &= \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \times 2 + (-2) \times 3 + 5 \times 1 & 3 \times (-1) + (-2) \times (-5) + 5 \times 4 & 3 \times 0 + (-2) \times 2 + 5 \times (-2) \\ 0 \times 2 + (-1) \times 3 + 6 \times 1 & 0 \times (-1) + (-1) \times (-5) + 6 \times 4 & 0 \times 0 + (-1) \times 2 + 6 \times (-2) \\ (-4) \times 2 + 2 \times 3 + (-1) \times 1 & (-4) \times (-1) + 2 \times (-5) + (-1) \times 4 & (-4) \times 0 + 2 \times 2 + (-1) \times (-2) \end{bmatrix} \\ &= \begin{bmatrix} 5 & 27 & -14 \\ 3 & 29 & -14 \\ -3 & -10 & 6 \end{bmatrix} \end{aligned}$$

If $A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$, then what is BA ?

A $\begin{bmatrix} 6 & 2 & 0 \\ 0 & 5 & 12 \\ -4 & 8 & 2 \end{bmatrix}$

B $\begin{bmatrix} 8 & 1 & -10 \\ 29 & 17 & -24 \\ -15 & -16 & 6 \end{bmatrix}$

C $\begin{bmatrix} 5 & 27 & -14 \\ 3 & 29 & -14 \\ -3 & -10 & 6 \end{bmatrix}$

D $\begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix}$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2(3) + (-1)(0) + 0(-4) & 2(-2) + (-1)(-1) + 0(2) & 2(5) + (-1)(6) + 0(-1) \\ 3(3) + (-5)(0) + 2(-4) & 3(-2) + (-5)(-1) + 2(2) & 3(5) + (-5)(6) + 2(-1) \\ 1(3) + 4(0) + (-2)(-4) & 1(-2) + 4(-1) + (-2)(2) & 1(5) + 4(6) + (-2)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 6 & -3 & 4 \\ 1 & 3 & -17 \\ 11 & -10 & 31 \end{bmatrix} \end{aligned}$$

Determinant of a Matrix

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

A **Matrix** is an array of numbers:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 2 Columns)

The determinant of that matrix is (calculations are explained later):

$$\Rightarrow 3 \times 6 - 8 \times 4 = 18 - 32 = -14$$

What is it for?

The determinant tells us things about the matrix that are useful in [systems of linear equations](#), helps us find the [inverse of a matrix](#), is useful in calculus and more.

Symbol

The **symbol** for determinant is two vertical lines either side.

Example:

$|\mathbf{A}|$ means the determinant of the matrix \mathbf{A}

(Exactly the same symbol as [absolute value](#).)

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)



Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$\begin{aligned} |B| &= 4 \times 8 - 6 \times 3 \\ &= 32 - 18 \\ &= 14 \end{aligned}$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

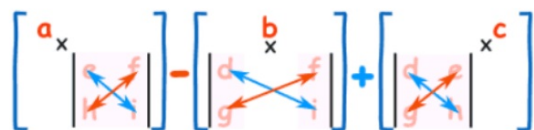
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:


$$\left[a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right]$$

To work out the determinant of a **3×3** matrix:

- Multiply **a** by the **determinant of the 2×2 matrix** that is **not in a's** row or column.
- Likewise for **b**, and for **c**
- Sum them up, but remember the minus in front of the **b**

As a formula (*remember the vertical bars || mean "determinant of"*):

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"The determinant of A equals a times the determinant of ... etc"

Example:

$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |C| &= 6 \times (-2 \times 7 - 5 \times 8) - 1 \times (4 \times 7 - 5 \times 2) + 1 \times (4 \times 8 - (-2 \times 2)) \\ &= 6 \times (-54) - 1 \times (18) + 1 \times (36) \\ &= \mathbf{-306} \end{aligned}$$

For 4×4 Matrices and Higher

The pattern continues for 4×4 matrices:

- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,

$$\left[\begin{array}{c|ccc} a & x & & & \\ \hline & f & g & h & \\ & j & k & l & \\ & n & o & p & \end{array} \right] - \left[\begin{array}{c|ccc} & & b & x & \\ \hline e & & g & h & \\ i & & k & l & \\ m & & o & p & \end{array} \right] + \left[\begin{array}{ccc|c} e & f & & x \\ \hline i & j & & h \\ m & n & & p \end{array} \right] - \left[\begin{array}{ccc|c} e & f & g & x \\ \hline i & j & k & \\ m & n & o & \end{array} \right]$$

As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the **+ - + -** pattern (**+**a... **-**b... **+**c... **-**d...). This is important to remember.

The pattern continues for 5×5 matrices and higher. Usually best to use a [Matrix Calculator](#) for those!

Not The Only Way

This method of calculation is called the "Laplace expansion" and I like it because the pattern is easy to remember. But there are other methods (just so you know).

Summary

- For a 2×2 matrix the determinant is **$ad - bc$**
- For a 3×3 matrix multiply **a** by the **determinant of the 2×2 matrix** that is **not** in **a** 's row or column, likewise for **b** and **c** , but remember that **b** has a negative sign!
- The pattern continues for larger matrices: multiply **a** by the **determinant of the matrix** that is **not** in **a** 's row or column, continue like this across the whole row, but remember the $+$ $-$ $+$ $-$ pattern.

What is the determinant of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$?

A 11

B 1

C -11

D -1

Generally, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant $|A| = ad - bc$.

Substitute $a=2, b=5, c=1$ and $d=-3$

So $|A| = 2 \times (-3) - 5 \times 1 = -11$

What is the determinant of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$?

A -13

B -7

C 7

D 13

Generally, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its determinant $|A| = ad - bc$.

In this case $a=3$, $b=-5$, $c=2$, and $d=1$.

So, $|A| = 3 \times 1 - (-5 \times 2) = 3 + 10 = 13$.

What is the determinant of the matrix $A = \begin{bmatrix} -2 & 4 \\ -6 & 2 \end{bmatrix}$?

A -28

B -20

C 20

D 28

Given, for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the Determinant $|A| = ad - bc$.

If $a=2, b=4, c=4$ and $d=1$

So $|A| = (2 \times 1) - (4 \times 4) = 4 - (16) = 4 - 16 = -12$

What is the determinant of the matrix $A = \begin{bmatrix} -3 & -1 \\ 4 & -5 \end{bmatrix}$?

A -19

B -11

C 11

D 19

Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Its determinant $|A| = ad - bc$.

Find a if $b = 3$, $d = 4$, $|A| = 10$.

$3(4) - 3(a) = 10$
 $12 - 3a = 10$
 $-3a = -2$
 $a = \frac{2}{3}$

What is the determinant of the matrix $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 4 \\ -3 & 1 & 3 \end{bmatrix}$?

A 70

B -44

C -60

D -74

```

def det(A):
    n = len(A)
    if n == 1:
        return A[0][0]
    if n == 2:
        return A[0][0]*A[1][1]-A[0][1]*A[1][0]
    det = 0
    for i in range(n):
        sign = (-1)**i
        subA = [row[:i]+row[i+1:] for row in A]
        det += sign*A[0][i]*det(subA)
    return det

```

What is the determinant of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -1 \\ -5 & 3 & -2 \end{bmatrix}$?

A -63

B -19

C -15

D 59

Generally, for the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,
 The determinant: $|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$
 In this case $a=2, b=-3, c=1, d=4, e=2, f=-1, g=-5, h=3, i=-2$
 So
 $= 2(2 \cdot (-2) - (-1) \cdot 3) - (-3)(4 \cdot (-2) - (-1) \cdot (-5)) + 1(4 \cdot 3 - 2 \cdot (-5))$
 $= 2(-4 + 3) - 3(-8 - 5) + 1(12 + 10)$
 $= 2(-1) - 3(-13) + 1(22)$
 $= -2 + 39 + 22$
 $= 59$

What is the determinant of the matrix $A = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -3 & 3 \end{bmatrix}$?

A -55

B 43

C 55

D 65

Similarly, for the matrix $B = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}$,
 The determinant $|B| = x(x^2 - yz) + y(z^2 - x^2) + z(y^2 - xz)$.
 With $x=y=5, z=1, x=2, y=4, z=1, x=1, y=4, z=2, x=2, y=1, z=3$
 $|B| = 4$
 $= 5(5^2 - 1 \cdot 1) + 1(1^2 - 5 \cdot 2) + 2(2^2 - 5 \cdot 4)$
 $= 5(25 - 1) + 1(1 - 10) + 2(4 - 20)$
 $= 5 \cdot 24 - 9 + (-16) = 120 - 9 - 16$
 $= 95$

What is the determinant of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \\ -4 & 1 & 4 \end{bmatrix}$?

A 13

B 21

C 53

D 59

Given \vec{u} , let the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

The determinant $|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$.

with $a=2, b=0, c=-1, d=3, e=5, f=2, g=-4, h=1, i=4$.

So A

$$= 2(5 \times 4 - 2 \times 1) - 0(3 \times 4 - 2 \times (-4)) + (-1)(3 \times 1 - 5 \times (-4))$$
$$= 2(20 - 2) + 0 - (-30)$$
$$= 2 \times 18 + 1 \times 30$$
$$= 36 + 30$$
$$= 66$$

What is the determinant of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$?

A -44

B -4

C 8

D 12

```

def determinant(a,b,c):
    return a*(b*c-c*a)+b*(c*a-a*b)+c*(a*b-b*a)

a,b,c=1,2,3
print(determinant(a,b,c))

# Output: 0
```

Inverse of a Matrix

Please read our [Introduction to Matrices](#) first.

What is the Inverse of a Matrix?

This is the [reciprocal](#) of a **number**:



Reciprocal of a Number

The **Inverse of a Matrix** is the **same idea** but we write it A^{-1}



Why not $\frac{1}{A}$? Because we don't divide by a matrix! And anyway $\frac{1}{8}$ can also be written 8^{-1}

And there are other similarities:

- When we **multiply a number** by its **reciprocal** we get **1**

$$8 \times (1/8) = \mathbf{1}$$

- When we **multiply a matrix** by its **inverse** we get the **Identity Matrix** (which is like "1" for matrices):

$$A \times A^{-1} = \mathbf{I}$$

- Same thing when the inverse comes first:

$$(1/8) \times 8 = \mathbf{1}$$

$$A^{-1} \times A = \mathbf{I}$$

Identity Matrix

We just mentioned the "Identity Matrix". It is the matrix equivalent of the number "1":

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1s** on the diagonal and **0s** everywhere else.
- Its symbol is the capital letter **I**.

The Identity Matrix can be 2×2 in size, or 3×3, 4×4, etc ...

Definition

Here is the definition:

The inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$


Sometimes there is no inverse at all.

2x2 Matrix

OK, how do we calculate the inverse?

Well, for a 2x2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$


determinant

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{aligned} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} &= \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \end{aligned}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{aligned} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} &= \begin{bmatrix} 4 \times 0.6 + 7 \times -0.2 & 4 \times -0.7 + 7 \times 0.4 \\ 2 \times 0.6 + 6 \times -0.2 & 2 \times -0.7 + 6 \times 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

And, hey!, we end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Why don't you have a go at multiplying these? See if you also get the Identity Matrix:

$$\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Why Do We Need an Inverse?

Because with matrices we **don't divide**! Seriously, there is no concept of dividing by a matrix.

But we can **multiply by an inverse**, which achieves the same thing.

Imagine we can't divide by numbers ...

... and someone asks "How do I share 10 apples with 2 people?"

But we can take the **reciprocal** of 2 (which is 0.5), so we answer:

$$10 \times 0.5 = 5$$

They get 5 apples each.

The same thing can be done with matrices:

Say we want to find matrix X , and we know matrix A and B :

$$XA = B$$

It would be nice to divide both sides by A (to get $X=B/A$), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from $1x = ab$ for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

In that example we were very careful to get the multiplications correct, because with matrices the order of multiplication matters. AB is almost never equal to BA .

A Real Life Example: Bus and Train



A group took a trip on a **bus**, at \$3 per child and \$3.20 per adult for a total of \$118.40.

They took the **train** back at \$3.50 per child and \$3.60 per adult for a total of \$135.20.

How many children, and how many adults?

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\begin{array}{cc} \text{Child} & \text{Adult} \\ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \end{array} \begin{array}{cc} \text{Bus} & \text{Train} \\ \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix} \end{array} = \begin{array}{cc} \text{Bus} & \text{Train} \\ \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \end{array}$$

This is just like the example above:

$$XA = B$$

So to solve it we need the inverse of "A":

$$\begin{aligned} \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} &= \frac{1}{3 \times 3.6 - 3.5 \times 3.2} \begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix} \end{aligned}$$

Now we have the inverse we can solve using:

$$X = BA^{-1}$$

$$\begin{aligned} \begin{bmatrix} x_1 & x_2 \end{bmatrix} &= \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix} \\ &= \begin{bmatrix} 118.4 \times -9 + 135.2 \times 8 & 118.4 \times 8.75 + 135.2 \times -7.5 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 22 \end{bmatrix} \end{aligned}$$

There were 16 children and 22 adults!

The answer almost appears like magic. But it is based on good mathematics.

Calculations like that (but using much larger matrices) help Engineers design buildings, are used in video games and computer animations to make things look 3-dimensional, and many other places.

It is also a way to solve [Systems of Linear Equations](#).

The calculations are done by computer, but the people must understand the formulas.

Order is Important

Say that we are trying to find "X" in this case:

$$AX = B$$

This is different to the example above! X is now **after** A.

With matrices the order of multiplication usually changes the answer. Do not assume that $AB = BA$, it is almost never true.

So how do we solve this one? Using the same method, but put A^{-1} in front:

$$A^{-1}AX = A^{-1}B$$

And we know that $A^{-1}A = I$, so:

$$IX = A^{-1}B$$

We can remove I:

$$X = A^{-1}B$$

And we have our answer (assuming we can calculate A^{-1})

Why don't we try our bus and train example, but with the data set up that way around.

It can be done that way, but we must be careful how we set it up.

This is what it looks like as $AX = B$:

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

It looks so neat! I think I prefer it like this.

Also note how the rows and columns are swapped over ("Transposed") compared to the previous example.

To solve it we need the inverse of "A":

$$\begin{aligned} \begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix}^{-1} &= \frac{1}{3 \times 3.6 - 3.2 \times 3.5} \begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix} \end{aligned}$$

It is like the inverse we got before, but Transposed (rows and columns swapped over).

Now we can solve using:

$$\begin{aligned} X &= A^{-1}B \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix} \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix} \\ &= \begin{bmatrix} -9 \times 118.4 + 8 \times 135.2 \\ 8.75 \times 118.4 - 7.5 \times 135.2 \end{bmatrix} \\ &= \begin{bmatrix} 16 \\ 22 \end{bmatrix} \end{aligned}$$

Same answer: 16 children and 22 adults.

So matrices are powerful things, but they do need to be set up correctly!

The Inverse May Not Exist

First of all, to have an inverse the matrix must be "square" (same number of rows and columns).

But also the **determinant cannot be zero** (or we end up dividing by zero). How about this:

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \frac{1}{3 \times 8 - 4 \times 6} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix} \\ = \frac{1}{24 - 24} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

24-24? That equals 0, and **1/0 is undefined**.

We cannot go any further! This Matrix has no Inverse.

Such a matrix is called "Singular", which only happens when the determinant is zero.

And it makes sense ... look at the numbers: the second row is just double the first row, and does **not add any new information**.

And the determinant lets us know this fact.

(Imagine in our bus and train example that the prices on the train were all exactly 50% higher than the bus: so now we can't figure out any differences between adults and children. There needs to be something to set them apart.)

Conclusion

- The inverse of A is A^{-1} only when $A \times A^{-1} = A^{-1} \times A = \mathbf{I}$
- To find the inverse of a 2x2 matrix: **swap** the positions of a and d , put **negatives** in front of b and c , and **divide** everything by the determinant ($ad-bc$).
- Sometimes there is no inverse at all

What is the inverse of the matrix

$$X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}?$$

A $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

B $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$

C $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

D $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$

$$X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Step 1: Find the determinant

$$\det(X) = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 3 \cdot 2 - 1 \cdot 5 = 6 - 5 = 1$$

Step 2: Find the adjugate matrix

$$\text{adj}(X) = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Step 3: Find the inverse

$$X^{-1} = \frac{1}{\det(X)} \text{adj}(X) = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

What is the inverse of the matrix

$$X = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix} ?$$

A $\begin{bmatrix} 1 & -4 \\ -2 & 7 \end{bmatrix}$

B $\begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$

C $\begin{bmatrix} -4 & 1 \\ 7 & -2 \end{bmatrix}$

$$X = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$$

So $a = 7$, $b = 4$, $c = 2$ and $d = 1$

First find the determinant of X:

$$\det(X) = ad - bc = 7 \times 1 - 4 \times 2 = 7 - 8 = -1$$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & -4 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$$

What is the inverse of the matrix

$$X = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}?$$

A $\begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$

B $\begin{bmatrix} -3.5 & 1.5 \\ 2 & -1 \end{bmatrix}$

C $\begin{bmatrix} 3.5 & -1.5 \\ -2 & 1 \end{bmatrix}$

D $\begin{bmatrix} -1.5 & 3.5 \\ 1 & -2 \end{bmatrix}$

$$X = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

$$\det(X) = 2 \cdot 7 - 3 \cdot 4 = 14 - 12 = 2$$

The inverse of the matrix X is:

$$X^{-1} = \frac{1}{\det(X)} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3.5 & -1.5 \\ -2 & 1 \end{bmatrix}$$

What is the inverse of the matrix

$$Y = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}?$$

A $\begin{bmatrix} 0.75 & 0.5 \\ 1 & 1 \end{bmatrix}$

B $\begin{bmatrix} -0.75 & -0.5 \\ -1 & -1 \end{bmatrix}$

C $\begin{bmatrix} 0.5 & 0.75 \\ 1 & 1 \end{bmatrix}$

D $\begin{bmatrix} -0.5 & -0.75 \\ -1 & -1 \end{bmatrix}$

$$\begin{aligned} Y &= \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix} \\ \text{So } y &= 4, b = -3, c = -4 \text{ and } d = 2 \\ \text{Find the determinant of } Y: \\ \det(Y) &= bc - da = 2 - (-4)(-3) = -10 \\ \text{The inverse of the matrix } \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \text{Therefore the inverse of } Y &= \frac{1}{-10} \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.3 \\ -0.4 & -0.4 \end{bmatrix} \end{aligned}$$

What is the inverse of the matrix

$$Z = \begin{bmatrix} -8 & 4 \\ 6 & -5 \end{bmatrix}?$$

A $\begin{bmatrix} \frac{5}{16} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{2} \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{4} & -\frac{5}{16} \\ -\frac{1}{2} & -\frac{3}{8} \end{bmatrix}$

C $\begin{bmatrix} -\frac{5}{16} & -\frac{1}{4} \\ -\frac{3}{8} & -\frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} \frac{1}{4} & \frac{5}{16} \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}$

$$Z = \begin{bmatrix} 4 & 5 \\ 6 & -3 \end{bmatrix}$$

So $a=4, b=5, c=6$ and $d=-3$
For $\det(A)$ we have $a \cdot d - bc$
 $\det(Z) = 4 \cdot (-3) - (5 \cdot 6) = -12 - 30 = -42$
The inverse of the matrix Z is $\frac{1}{\det(Z)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-42} \begin{bmatrix} -3 & -5 \\ -6 & 4 \end{bmatrix}$
The inverse of Z is $\frac{1}{-42} \begin{bmatrix} -3 & -5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} & \frac{5}{42} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix}$

Which one of the following matrices is singular?

A $P = \begin{bmatrix} -6 & 9 \\ 2 & 3 \end{bmatrix}$

B $Q = \begin{bmatrix} -6 & 9 \\ -2 & -3 \end{bmatrix}$

C $R = \begin{bmatrix} 6 & 9 \\ 2 & -3 \end{bmatrix}$

D $S = \begin{bmatrix} -6 & 9 \\ 2 & -3 \end{bmatrix}$

A singular matrix will have a determinant of 0.

$$\det(P) = (-6)(3) - (9)(2) = -30$$

$$\det(Q) = (-6)(-3) - (9)(-2) = 30$$

$$\det(R) = (6)(-3) - (9)(2) = -30$$

$$\det(S) = (-6)(-3) - (9)(2) = -30$$

$$\det(P) = (-6)(3) - (9)(2) = -30$$

$\det(S) = 0$ is singular.

Inverse of a Matrix using Elementary Row Operations

Also called the Gauss-Jordan method.

This is a fun way to find the Inverse of a Matrix:

Play around with the rows
(adding, multiplying or
swapping) until we make
Matrix **A** into the Identity
Matrix **I**

$$\begin{array}{c} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array} \right] \\ \text{"Elementary Row Operations"} \\ \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array} \right] \end{array}$$

And by ALSO doing the
changes to an Identity Matrix
it magically turns into the
Inverse!

The "**Elementary Row Operations**" are simple things like adding rows, multiplying and swapping
... but let's see with an example:

Example: find the Inverse of "A":

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix **A**, and write it down with an Identity Matrix **I** next to it:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

(This is called the "Augmented Matrix")

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

Now we do our best to turn "A" (the Matrix on the left) into an Identity Matrix. The goal is to make Matrix A have **1**s on the diagonal and **0**s elsewhere (an Identity Matrix) ... and the right hand side comes along for the ride, with every operation being done on it as well.

But we can only do these "**Elementary Row Operations**":

- **swap** rows
- **multiply** or divide each element in a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

$$\begin{array}{c} \swarrow A \quad \quad \swarrow I \\ \left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \end{array}$$

Start with **A** next to **I**

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \quad \rightarrow \text{Add}$$

Add row 2 to row 1,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \quad \text{Divide by 5}$$

then divide row 1 by 5,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \quad \rightarrow \text{Subtract } \times 2$$

Then take 2 times the first row, and subtract it from the second row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \quad \text{Multiply by } -\frac{1}{2}$$

Multiply second row by $-1/2$,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array} \right] \quad \rightarrow \text{Swap}$$

Now swap the second and third row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array} \right] \quad \rightarrow \text{Subtract}$$

Last, subtract the third row from the second row,

And we are done!

$$I \nearrow \quad A^{-1} \nearrow$$

And matrix **A** has been made into an Identity Matrix ...

... and at the same time an Identity Matrix got made into **A⁻¹**

$$A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

DONE! Like magic, and just as fun as solving any puzzle.

And note: there is no "right way" to do this, just keep playing around until we succeed!

Larger Matrices

We can do this with larger matrices, for example, try this 4x4 matrix:

$$B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Start Like this:

$$\left[\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

See if you can do it yourself (I would begin by dividing the first row by 4, but you do it your way).

You can check your answer using the [Matrix Calculator](#) (use the "inv(A)" button).

Why it Works

I like to think of it this way:

- when we turn "8" into "1" by dividing by 8,
- and do the same thing to "1", it turns into "1/8"

And "1/8" is the (multiplicative) **inverse of 8**

$$\begin{array}{c} \left[\begin{array}{c|c} 8 & 1 \end{array} \right] \\ \text{"Divide by 8"} \\ \left[\begin{array}{c|c} 1 & \frac{1}{8} \end{array} \right] \end{array}$$

Or, more technically:

The **total effect of all the row operations** is the same as **multiplying by \mathbf{A}^{-1}**

So **A** becomes **I** (because $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$)

And **I** becomes \mathbf{A}^{-1} (because $\mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$)

$$\begin{array}{c} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array} \right] \\ \left[\begin{array}{c|c} \mathbf{A}^{-1}\mathbf{A} & \mathbf{A}^{-1}\mathbf{I} \end{array} \right] \\ \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array} \right] \end{array}$$

Find the inverse of the matrix

$$A = \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} 2 & 1 \\ -3\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$

B $\begin{bmatrix} 2 & 1 \\ 3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$

C $\begin{bmatrix} -2 & -1 \\ 3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} 2 & -1 \\ -3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$

Find the inverse of the matrix

Step 1 of 4: $\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 2 & 7 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$

Subtract 2 times row 1 from row 2

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$$

Step 2 of 4: $\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$$

Divide row 2 by 3

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$$

Subtract 2 times row 2 from row 1

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & I_2 \end{array} \right]$$

Find the inverse of the matrix

$$B = \begin{bmatrix} 5 & -1 \\ 15 & -2 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -3 & 1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ 3 & -1 \end{bmatrix}$

C $\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ -3 & 1 \end{bmatrix}$

D $\begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ -3 & -1 \end{bmatrix}$

Algebra 2 answers

Chapter 3

Section 3.1

Exercise 3.1.1

Problem 1

Find the inverse of the matrix

$B = \begin{bmatrix} 5 & -1 \\ 15 & -2 \end{bmatrix}$

using Elementary Row Operations

Answer: $B^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -3 & 1 \end{bmatrix}$

Find the inverse of the matrix

$$D = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} -2 & -2\frac{1}{2} \\ 1 & 1\frac{1}{2} \end{bmatrix}$

B $\begin{bmatrix} 2 & -2\frac{1}{2} \\ 1 & -1\frac{1}{2} \end{bmatrix}$

C $\begin{bmatrix} -2 & 2\frac{1}{2} \\ -1 & 1\frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} 2 & 2\frac{1}{2} \\ -1 & -1\frac{1}{2} \end{bmatrix}$

This is just one of several possible methods:

Start with D and I and row reduce:

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

Add one row to four to force the second row to the first row:

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

Swap the first row to 2:

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

Subtract 3 times the first row from the second row:

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -2 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

Divide the second row by 2:

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

Finally: $D^{-1} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$

Find the inverse of the matrix

$$E = \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} 0.25 & 0.125 \\ -0.5 & -0.75 \end{bmatrix}$

B $\begin{bmatrix} -0.25 & -0.125 \\ 0.5 & 0.75 \end{bmatrix}$

C $\begin{bmatrix} 0.25 & 0.125 \\ 0.5 & 0.75 \end{bmatrix}$

D $\begin{bmatrix} 0.25 & -0.125 \\ -0.5 & 0.75 \end{bmatrix}$

Check part one of several possible methods.
Start with E^{-1} next to E .
 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$
Add one half times the second row to the first row.
 $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$
Add the first row to the second row.
 $\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$
Divide the first row by 2.
 $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$
Divide the second row by 2.
 $\begin{bmatrix} 1 & 0 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix}$
Therefore $E^{-1} = \begin{bmatrix} 0.25 & 0.125 \\ -0.5 & -0.75 \end{bmatrix}$

Find the inverse of the matrix

$$F = \begin{bmatrix} 4 & -3 \\ 2.5 & -2 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} -4 & 6 \\ 5 & -8 \end{bmatrix}$

B $\begin{bmatrix} 4 & -6 \\ 5 & -8 \end{bmatrix}$

C $\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$

D $\begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$

What is the inverse of the matrix?

Step 1: Write the matrix

$$A = \begin{bmatrix} 4 & -3 \\ 2.5 & -2 \end{bmatrix}$$

Step 2: Write the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 3: Write the augmented matrix

$$[A | I] = \left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 2.5 & -2 & 0 & 1 \end{array} \right]$$

Step 4: Use row operations to get the identity matrix

$$R_2 \rightarrow R_2 - \frac{5}{8}R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 0 & -\frac{1}{4} & -\frac{5}{8} & 1 \end{array} \right]$$

Step 5: Multiply row 2 by -4

$$R_2 \rightarrow -4R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 6: Add 3 times row 2 to row 1

$$R_1 \rightarrow R_1 + 3R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & \frac{11}{2} & -12 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 7: Divide row 1 by 4

$$R_1 \rightarrow \frac{1}{4}R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{11}{8} & -3 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 8: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & 1 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 9: Multiply row 1 by 4

$$R_1 \rightarrow 4R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -1 & 4 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 10: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -\frac{11}{2} & 14 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 11: Divide row 1 by 4

$$R_1 \rightarrow \frac{1}{4}R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{11}{8} & \frac{7}{2} \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 12: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & 1 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 13: Multiply row 1 by 4

$$R_1 \rightarrow 4R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -1 & 4 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 14: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -\frac{11}{2} & 14 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 15: Divide row 1 by 4

$$R_1 \rightarrow \frac{1}{4}R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{11}{8} & \frac{7}{2} \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 16: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & 1 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 17: Multiply row 1 by 4

$$R_1 \rightarrow 4R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -1 & 4 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 18: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -\frac{11}{2} & 14 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 19: Divide row 1 by 4

$$R_1 \rightarrow \frac{1}{4}R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{11}{8} & \frac{7}{2} \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 20: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & 1 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 21: Multiply row 1 by 4

$$R_1 \rightarrow 4R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -1 & 4 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 22: Subtract 5/2 times row 2 from row 1

$$R_1 \rightarrow R_1 - \frac{5}{2}R_2$$

$$[A | I] = \left[\begin{array}{cc|cc} 4 & 0 & -\frac{11}{2} & 14 \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Step 23: Divide row 1 by 4

$$R_1 \rightarrow \frac{1}{4}R_1$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 0 & -\frac{11}{8} & \frac{7}{2} \\ 0 & 1 & \frac{5}{2} & -4 \end{array} \right]$$

Find the inverse of the matrix

$$C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} -5 & -3 & -6 \\ -6 & -3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$

B $\begin{bmatrix} -5 & 3 & -6 \\ 6 & 3 & 7 \\ -2 & 1 & 2 \end{bmatrix}$

C $\begin{bmatrix} 5 & 3 & 6 \\ -6 & 3 & -7 \\ 2 & 1 & -2 \end{bmatrix}$

D $\begin{bmatrix} -5 & 3 & -6 \\ -6 & 3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$

Use your calculator to verify:

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 7 \\ 0 & -1 & 3 \end{bmatrix}$$

Reverse the last 2 rows and then the second row:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & 3 \\ 0 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & 3 \\ 0 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & 3 \\ 0 & 1 & -5 \end{bmatrix}$$

Add the third row to the first:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & 3 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & 3 \\ 0 & 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \\ 0 & -1 & 3 \\ 0 & 1 & -5 \end{bmatrix}$$

Swap the second and third rows:

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & -2 \end{bmatrix}$$

Swap the second and third rows:

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & -2 \end{bmatrix}$$

Multiply the third row by $-\frac{1}{2}$ and then the first row:

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtract 5 times the third row from the second row:

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtract 8 times the third row from the first row:

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the inverse of C is:

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the inverse of the matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ -3 & 0 & 1 \end{bmatrix}$$

using Elementary Row Operations

$$\mathbf{A} = \begin{bmatrix} -0.75 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ -2.25 & 0.75 & 1.75 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.75 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 2.25 & 0.75 & 1.75 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0.25 \\ 2.25 & 0.75 & 1.75 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0.75 & 0.25 & 0.25 \\ 2.25 & 0.75 & 1.75 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$$

Matrices and vector problems

Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$

Find $A+B$

$\begin{pmatrix} 1+9 & 2+8 & 3+7 \\ 4+6 & 5+5 & 6+4 \\ 7+3 & 8+2 & 9+1 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{pmatrix}$

Find $A-B$

$\begin{pmatrix} 1-9 & 2-8 & 3-7 \\ 4-6 & 5-5 & 6-4 \\ 7-3 & 8-2 & 9-1 \end{pmatrix} = \begin{pmatrix} -8 & -6 & -4 \\ -2 & 0 & 2 \\ 4 & 6 & 8 \end{pmatrix}$

Find $3A$

$\begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \\ 3 \times 7 & 3 \times 8 & 3 \times 9 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{pmatrix}$

Find $2B$

$\begin{pmatrix} 2 \times 9 & 2 \times 8 & 2 \times 7 \\ 2 \times 6 & 2 \times 5 & 2 \times 4 \\ 2 \times 3 & 2 \times 2 & 2 \times 1 \end{pmatrix} = \begin{pmatrix} 18 & 16 & 14 \\ 12 & 10 & 8 \\ 6 & 4 & 2 \end{pmatrix}$

Find $A+B$ and $A-B$

$\begin{pmatrix} 1+9 & 2+8 & 3+7 \\ 4+6 & 5+5 & 6+4 \\ 7+3 & 8+2 & 9+1 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{pmatrix}$

$\begin{pmatrix} 1-9 & 2-8 & 3-7 \\ 4-6 & 5-5 & 6-4 \\ 7-3 & 8-2 & 9-1 \end{pmatrix} = \begin{pmatrix} -8 & -6 & -4 \\ -2 & 0 & 2 \\ 4 & 6 & 8 \end{pmatrix}$

Find $3A$ and $2B$

$\begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \\ 3 \times 7 & 3 \times 8 & 3 \times 9 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{pmatrix}$

$\begin{pmatrix} 2 \times 9 & 2 \times 8 & 2 \times 7 \\ 2 \times 6 & 2 \times 5 & 2 \times 4 \\ 2 \times 3 & 2 \times 2 & 2 \times 1 \end{pmatrix} = \begin{pmatrix} 18 & 16 & 14 \\ 12 & 10 & 8 \\ 6 & 4 & 2 \end{pmatrix}$

Find $A+B$ and $3A$

$\begin{pmatrix} 1+9 & 2+8 & 3+7 \\ 4+6 & 5+5 & 6+4 \\ 7+3 & 8+2 & 9+1 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{pmatrix}$

$\begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \\ 3 \times 7 & 3 \times 8 & 3 \times 9 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{pmatrix}$

Find $A-B$ and $2B$

$\begin{pmatrix} 1-9 & 2-8 & 3-7 \\ 4-6 & 5-5 & 6-4 \\ 7-3 & 8-2 & 9-1 \end{pmatrix} = \begin{pmatrix} -8 & -6 & -4 \\ -2 & 0 & 2 \\ 4 & 6 & 8 \end{pmatrix}$

$\begin{pmatrix} 2 \times 9 & 2 \times 8 & 2 \times 7 \\ 2 \times 6 & 2 \times 5 & 2 \times 4 \\ 2 \times 3 & 2 \times 2 & 2 \times 1 \end{pmatrix} = \begin{pmatrix} 18 & 16 & 14 \\ 12 & 10 & 8 \\ 6 & 4 & 2 \end{pmatrix}$

$$G = \begin{bmatrix} 3 & 3 & 2 \\ -2 & 1 & 5 \\ 4 & -3 & -12 \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 3 & 30 & 13 \\ 4 & 44 & 19 \\ 2 & 21 & 9 \end{bmatrix}$$
$$C = \begin{bmatrix} -3 & -30 & -13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$$
$$\mathbf{D} = \begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ -2 & -21 & -9 \end{bmatrix}$$

[illegible]

[illegible]

Find the inverse of the matrix

$$H = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

using Elementary Row Operations

A $\begin{bmatrix} -2.2 & 1.6 & -1.4 \\ 4.8 & -3.4 & 3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

B $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ -4.8 & 3.4 & -3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

C $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ 4.8 & -3.4 & 3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

D $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ 4.8 & -3.4 & 3.6 \\ -0.8 & 0.4 & -0.6 \end{bmatrix}$

Before we start, let's write down the augmented matrix for the system $Ax = b$. The augmented matrix is $[A | b]$. In this case, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 3 \end{array} \right]$. Now, let's perform row operations to get the matrix into row echelon form. First, we'll subtract 2 times the first row from the second row, and 3 times the first row from the third row. This gives us $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array} \right]$. Next, we'll multiply the second row by -1 to get $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array} \right]$. Finally, we'll add 2 times the second row to the third row to get $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. So the row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. This means that the system has infinitely many solutions. We can express the solutions in terms of a free variable t . Let $x_3 = t$. Then $x_2 = -2t$ and $x_1 = 1 - 2t - 3t = 1 - 5t$. So the general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 5t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - t \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. This means that the solution set is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$.

Before we start, let's write down the augmented matrix for the system $Ax = b$. The augmented matrix is $[A | b]$. In this case, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 3 \end{array} \right]$. Now, let's perform row operations to get the matrix into row echelon form. First, we'll subtract 2 times the first row from the second row, and 3 times the first row from the third row. This gives us $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array} \right]$. Next, we'll multiply the second row by -1 to get $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array} \right]$. Finally, we'll add 2 times the second row to the third row to get $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. So the row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. This means that the system has infinitely many solutions. We can express the solutions in terms of a free variable t . Let $x_3 = t$. Then $x_2 = -2t$ and $x_1 = 1 - 2t - 3t = 1 - 5t$. So the general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 5t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - t \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. This means that the solution set is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$.

Inverse of a Matrix using Minors, Cofactors and Adjugate

(Note: also check out [Matrix Inverse by Row Operations](#) and the [Matrix Calculator](#).)

We can calculate the [Inverse of a Matrix](#) by:

- Step 1: calculating the Matrix of Minors,
- Step 2: then turn that into the Matrix of Cofactors,
- Step 3: then the Adjugate, and
- Step 4: multiply that by 1/Determinant.

But it is best explained by working through an example!

Example: find the Inverse of A:

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

It needs 4 steps. It is all simple arithmetic but there is a lot of it, so try not to make a mistake!

Step 1: Matrix of Minors

The first step is to create a "Matrix of Minors". This step has the most calculations.

For each element of the matrix:

- ignore the values on the current row and column
- calculate the determinant of the remaining values

Put those determinants into a matrix (the "Matrix of Minors")

Determinant

For a 2×2 matrix (2 rows and 2 columns) the determinant is easy: **ad-bc**

Think of a cross:

- Blue means positive (+ad),
- Red means negative (-bc)



(It gets harder for a 3×3 matrix, etc)

The Calculations

Here are the first two, and last two, calculations of the "**Matrix of Minors**" (notice how I ignore the values in the current row and columns, and calculate the determinant using the remaining values):

$$\begin{array}{c} \begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0 \times 1 - (-2) \times 1 = 2 \\ \\ \begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2 \times 1 - (-2) \times 0 = 2 \\ \\ \dots \\ \\ \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3 \times -2 - 2 \times 2 = -10 \\ \\ \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3 \times 0 - 0 \times 2 = 0 \end{array}$$

And here is the calculation for the whole matrix:

$$\begin{bmatrix} 0 \times 1 - (-2) \times 1 & 2 \times 1 - (-2) \times 0 & 2 \times 1 - 0 \times 0 \\ 0 \times 1 - 2 \times 1 & 3 \times 1 - 2 \times 0 & 3 \times 1 - 0 \times 0 \\ 0 \times (-2) - 2 \times 0 & 3 \times (-2) - 2 \times 2 & 3 \times 0 - 0 \times 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

Matrix of Minors

Step 2: Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, we need to change the sign of alternate cells, like this:

$$\begin{array}{c}
 \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix} \\
 \text{Matrix of Minors} \qquad \qquad \qquad \text{Matrix of CoFactors}
 \end{array}$$

+	-	+	-
-	+	-	+
+	-	+	-
-	+	-	+

Step 3: Adjugate (also called Adjoint)

Now "Transpose" all elements of the previous matrix... in other words swap their positions over the diagonal (the diagonal stays the same):

$$\begin{bmatrix} 2 & -2 & 0 \\ 2 & 3 & -10 \\ 0 & -3 & 0 \end{bmatrix}$$

Step 4: Multiply by 1/Determinant

Now find the determinant of the original matrix. This isn't too hard, because we already calculated the determinants of the smaller parts when we did "Matrix of Minors".

$$\left[\begin{array}{c|cc} a & f & i \\ \hline e & h & j \\ g & & \end{array} \right] - \left[\begin{array}{c|cc} b & f & i \\ \hline d & h & j \\ g & & \end{array} \right] + \left[\begin{array}{c|cc} c & f & i \\ \hline d & h & j \\ e & & \end{array} \right]$$

In practice we can just multiply each of the top row elements by the cofactor for the same location:

Elements of top row: 3, 0, 2

Cofactors for top row: 2, -2, 2

$$\text{Determinant} = 3 \times 2 + 0 \times (-2) + 2 \times 2 = \mathbf{10}$$

(Just for fun: try this for any other row or column, they should also get 10.)

And now multiply the Adjugate by 1/Determinant:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

Adjugate *Inverse*

And we are done!

Larger Matrices

It is exactly the same steps for larger matrices (such as a 4×4 , 5×5 , etc), but wow! there is a lot of calculation involved.

For a 4×4 Matrix we have to calculate 16 3×3 determinants. So it is often easier to use computers (such as the [Matrix Calculator](#).)

Conclusion

- For each element, calculate the **determinant of the values not on the row or column**, to make the Matrix of Minors
- Apply a **checkerboard** of minuses to make the Matrix of Cofactors
- **Transpose** to make the Adjugate
- Multiply by **$1/\text{Determinant}$** to make the Inverse

Find the inverse of the matrix

$$C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} -5 & -3 & -6 \\ 6 & 3 & 7 \\ -2 & -1 & -2 \end{bmatrix}$

B $\begin{bmatrix} 5 & 3 & 6 \\ -6 & -3 & -7 \\ 2 & 1 & 2 \end{bmatrix}$

C $\begin{bmatrix} -5 & 3 & -6 \\ -6 & 3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$

D $\begin{bmatrix} -5 & -6 & -2 \\ 3 & 3 & 1 \\ -6 & -7 & -2 \end{bmatrix}$

Step 3 Find the Minors of Matrix

Matrix of Minors

$$M = \begin{bmatrix} \det \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} & \det \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} & \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \\ \det \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} & \det \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} & \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} & \det \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} & \det \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \end{bmatrix}$$

Step 4 Find the Minors of Columns for finding the sign of the matrix

Matrix of Minors

$$M = \begin{bmatrix} \det \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} & \det \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} & \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \\ \det \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} & \det \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} & \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} & \det \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} & \det \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \end{bmatrix}$$

Step 5 Find the Cofactors of the Matrix

Matrix of Cofactors

$$C = \begin{bmatrix} 6 & 2 & -1 \\ -2 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

Step 6 Find the Adjugate of the Matrix

Adjugate

$$A = \begin{bmatrix} 6 & -2 & -1 \\ 2 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

Step 7 Find the Inverse of the Matrix

Inverse

$$C^{-1} = \frac{1}{\det C} A = \frac{1}{-1} A = -A = \begin{bmatrix} -6 & 2 & 1 \\ -2 & -2 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

Find the inverse of the matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ -3 & 0 & 1 \end{bmatrix}$$

Step 1 Find the Matrix of Minors:

Matrix of Minors

$$\begin{aligned} &= \begin{bmatrix} 3 \times 1 - (-1) \times 0 & 1 \times 1 - (-1) \times (-3) & 1 \times 0 - 3 \times (-3) \\ -1 \times 1 - 0 \times 0 & 2 \times 1 - 0 \times (-3) & 2 \times 0 - (-1) \times (-3) \\ (-1) \times (-1) - 0 \times 3 & 2 \times (-1) - 0 \times 1 & 2 \times 3 - (-1) \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 & 9 \\ -1 & 2 & -3 \\ 1 & -2 & 7 \end{bmatrix} \end{aligned}$$

Step 2 Find the Matrix of Cofactors by changing the signs of alternate cells:

$$\text{Matrix of cofactors} = \begin{bmatrix} 3 & 2 & 9 \\ 1 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

Step 3 Find the Adjugate by transposing all elements in the Matrix of Cofactors:

$$\text{Adjugate} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 9 & 3 & 7 \end{bmatrix}$$

Step 4 Multiply by $\frac{1}{\text{determinant}}$:

$$\det (M) = 2 \times 3 - (-1) \times (-2) + 0 \times 9 = 6 - 2 + 0 = 4$$

$$\text{So } \frac{1}{\text{determinant}} = \frac{1}{4}$$

$$\text{Therefore } M^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 9 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 2.25 & 0.75 & 1.75 \end{bmatrix}$$

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -2 & 3 & -4 \\ -3 & 1 & -4 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} -1.6 & 0.6 & -1.8 \\ 0.8 & 0.2 & 0.4 \\ -1.4 & 0.4 & -1.2 \end{bmatrix}$

B $\begin{bmatrix} -1.6 & 0.6 & -1.8 \\ 0.8 & 0.2 & 0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

C $\begin{bmatrix} 1.6 & -0.6 & 1.8 \\ 0.8 & 0.2 & 0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

D $\begin{bmatrix} -1.6 & 0.6 & -1.8 \\ -0.8 & -0.2 & -0.4 \\ 1.4 & -0.4 & 1.2 \end{bmatrix}$

Step 1: Find the Matrix of Minors

Matrix of Minors

$$\begin{bmatrix} 3 \times (2-4) - (-1) \times 1 & -2 \times (3-4) - (-1) \times (-3) & -2 \times 1 - 3 \times (-3) \\ 0 \times (-3) - 3 \times 1 & 2 \times (-4) - 3 \times (-2) & 2 \times 3 - 0 \times (-4) \\ 0 \times (-4) - 3 \times 3 & 2 \times (-4) - 0 \times (-2) & 2 \times 3 - 0 \times (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & 11 \\ -3 & -2 & 6 \\ -9 & -4 & 6 \end{bmatrix}$$

Step 2: Find the Matrix of Cofactors (or changing the signs of alternate cells)

$$\text{Matrix of cofactors} = \begin{bmatrix} 8 & 1 & 11 \\ 3 & 2 & -6 \\ 9 & 4 & -6 \end{bmatrix}$$

Step 3: Use the Adjugate by transposing all elements in the Matrix of Cofactors

$$\text{Adjugate} = \begin{bmatrix} 8 & 3 & 9 \\ 1 & 2 & 4 \\ 11 & -6 & -6 \end{bmatrix}$$

Step 4: Multiply by $\frac{1}{\text{determinant}}$

$$\det(A) = 2 \times (3 \times 4 - 0 \times (-2)) - 0 \times (2 \times 2 - (-6) \times (-6)) + 3 \times (2 \times (-9) - 0 \times 2) = 12$$

$$\text{So, } \frac{1}{\text{determinant}} = \frac{1}{12} = 0.08\bar{3}$$

$$\text{Therefore, } A^{-1} = \begin{bmatrix} 0.67 & 0.25 & 0.75 \\ 0.08 & 0.17 & 0.33 \\ 0.92 & -0.5 & -0.5 \end{bmatrix}$$

Find the inverse of the matrix

$$B = \begin{bmatrix} -3 & 1 & -6 \\ 2 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 4 & -4.5 & 2 \\ -1 & 1.5 & 0 \\ -2 & 2.5 & -1 \end{bmatrix}$

B $\begin{bmatrix} -4 & -4.5 & -2 \\ 1 & 1.5 & 0 \\ 2 & 2.5 & 1 \end{bmatrix}$

C $\begin{bmatrix} -4 & -4.5 & 2 \\ 1 & 1.5 & 0 \\ 2 & 2.5 & -1 \end{bmatrix}$

D $\begin{bmatrix} -4 & 4.5 & 2 \\ 1 & -1.5 & 0 \\ 2 & -2.5 & -1 \end{bmatrix}$

Step 1: Find the Determinant of Matrix

Determinant of Matrix

$$= \begin{vmatrix} 0 \times 0 \times 0 - 0 \times 0 & 2 \times 0 \times 0 - 0 \times 0 & 0 \times 0 - 0 \times 0 \\ 0 \times 0 \times 0 - 0 \times 0 & 0 \times 0 \times 0 - 0 \times 0 & 0 \times 0 - 0 \times 0 \\ 0 \times 0 - 0 \times 0 & 0 \times 0 - 0 \times 0 & 0 \times 0 - 0 \times 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Step 2: Find the Matrix of Minors by changing the signs of alternate cells

Matrix of Minors

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: Find the Adjugate by transposing all elements in the Matrix of Minors

Adjugate

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 4: Substitute for $\frac{1}{\text{determinant}}$

Final Answer

$$\frac{1}{0} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the inverse of the matrix

$$D = \begin{bmatrix} 6 & -3 & -2 \\ -5 & -1 & 3 \\ -2 & 4 & 0 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 1.2 & 0.8 & 1.1 \\ 0.6 & 0.4 & 0.8 \\ 2.2 & 1.8 & 2.1 \end{bmatrix}$

B $\begin{bmatrix} 1.2 & 0.8 & 1.1 \\ -0.6 & -0.4 & -0.8 \\ 2.2 & 1.8 & 2.1 \end{bmatrix}$

C $\begin{bmatrix} 1.2 & 0.8 & 1.1 \\ 0.6 & 0.4 & 0.8 \\ -2.2 & -1.8 & -2.1 \end{bmatrix}$

D $\begin{bmatrix} -1.2 & -0.8 & -1.1 \\ 0.6 & 0.4 & 0.8 \\ 2.2 & 1.8 & 2.1 \end{bmatrix}$

Step 4: Find the Minors of Matrix

Minors of Matrix:

$$M = \begin{bmatrix} 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \end{bmatrix}$$

Step 5: Find the Cofactors and Adjugate (the transpose of the Minors matrix)

Minors of cofactors:

$$C = \begin{bmatrix} 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 4 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 \end{bmatrix}$$

Step 6: Find the Inverse of the Matrix

The inverse of the matrix is:

$$D^{-1} = \frac{1}{\det(D)} \cdot C^T$$

Therefore, $D^{-1} = \begin{bmatrix} 1.2 & 0.8 & 1.1 \\ 0.6 & 0.4 & 0.8 \\ -2.2 & -1.8 & -2.1 \end{bmatrix}$

Find the inverse of the matrix

$$E = \begin{bmatrix} 5 & -2 & 3 \\ -2 & 6 & -1 \\ 3 & -4 & 2 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 1 & -1 & 2 \\ 0.125 & 0.125 & 0.125 \\ -1.25 & 1.75 & -3.25 \end{bmatrix}$

B $\begin{bmatrix} 1 & -1 & -2 \\ 0.125 & 0.125 & -0.125 \\ -1.25 & 1.75 & 3.25 \end{bmatrix}$

C $\begin{bmatrix} -1 & -1 & -2 \\ -0.125 & 0.125 & -0.125 \\ 1.25 & 1.75 & 3.25 \end{bmatrix}$

D $\begin{bmatrix} 1 & 1 & -2 \\ 0.125 & -0.125 & -0.125 \\ -1.25 & -1.75 & 3.25 \end{bmatrix}$

ANSWER:

$$E^{-1} = \frac{1}{\det(E)} \text{adj}(E)$$

$$\det(E) = \begin{vmatrix} 5 & -2 & 3 \\ -2 & 6 & -1 \\ 3 & -4 & 2 \end{vmatrix} = 5(12 - (-4)) - 2(-4 - 18) + 3(-12 - 18) = 5(16) - 2(-22) + 3(-30) = 80 + 44 - 90 = 34$$

$$\text{adj}(E) = \begin{bmatrix} 1 & 1 & -2 \\ 0.125 & -0.125 & -0.125 \\ -1.25 & -1.75 & 3.25 \end{bmatrix}$$

$$E^{-1} = \frac{1}{34} \begin{bmatrix} 1 & 1 & -2 \\ 0.125 & -0.125 & -0.125 \\ -1.25 & -1.75 & 3.25 \end{bmatrix}$$

Find the inverse of the matrix

$$F = \begin{bmatrix} -3 & 7 & -5 \\ 4 & -1 & 12 \\ -2 & 9 & -1 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\frac{1}{11} \begin{bmatrix} 107 & -38 & 79 \\ -20 & 7 & 16 \\ -34 & 13 & 25 \end{bmatrix}$

B $\frac{1}{11} \begin{bmatrix} -107 & 38 & 79 \\ 20 & -7 & 16 \\ -34 & 13 & 25 \end{bmatrix}$

C $\frac{1}{11} \begin{bmatrix} -107 & -38 & -79 \\ 20 & -7 & 16 \\ -34 & -13 & -25 \end{bmatrix}$

D $\frac{1}{11} \begin{bmatrix} -107 & -38 & 79 \\ -20 & -7 & 16 \\ 34 & 13 & -25 \end{bmatrix}$

Step 1: Find the Matrix of Minors

Matrix of Minors:

$$M = \begin{bmatrix} \begin{vmatrix} 1 & 4 & 1 \end{vmatrix} & \begin{vmatrix} -3 & 12 \end{vmatrix} & \begin{vmatrix} -3 & -5 \end{vmatrix} \\ \begin{vmatrix} 7 & -5 & -1 \end{vmatrix} & \begin{vmatrix} -3 & -1 \end{vmatrix} & \begin{vmatrix} -3 & 7 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 & 12 \end{vmatrix} & \begin{vmatrix} -2 & 12 \end{vmatrix} & \begin{vmatrix} -2 & 9 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -15 & 11 \\ 12 & 2 & -20 \\ 11 & -14 & -5 \end{bmatrix}$$

Step 2: Find the Matrix of Cofactors by changing the signs of alternate cells:

Matrix of cofactors = $\begin{bmatrix} 0 & 15 & -11 \\ 12 & -2 & 20 \\ 11 & 14 & -5 \end{bmatrix}$

Step 3: Find the Adjugate by transposing all cofactors in the Matrix of Cofactors

Adjugate = $\begin{bmatrix} 0 & 12 & 11 \\ 15 & -2 & 14 \\ -11 & 20 & -5 \end{bmatrix}$

Step 4: Multiply by $\frac{1}{\text{determinant}}$

$\det(F) = 0 \times 1 \times 107 - 7 \times 20 \times 1 \times 9 + 34 \times 107 - 340 - 378 = 11$

For $\frac{1}{\text{determinant}} = \frac{1}{11}$

Therefore $F^{-1} = \frac{1}{11} \begin{bmatrix} 0 & 12 & 11 \\ 15 & -2 & 14 \\ -11 & 20 & -5 \end{bmatrix}$

Find the inverse of the matrix

Step 1: Find the Matrix of Minors:

Matrix of Minors

$$\begin{aligned} &= \begin{bmatrix} 1 \times (-12) - 5 \times (-3) & -2 \times (-12) - 5 \times 4 & -2 \times (-3) - 1 \times 4 \\ 3 \times (-12) - 2 \times (-3) & 3 \times (-12) - 2 \times 4 & 3 \times (-3) - 3 \times 4 \\ 3 \times 5 - 2 \times 1 & 3 \times 5 - 2 \times (-2) & 3 \times 1 - 3 \times (-2) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 & 2 \\ -30 & -44 & -21 \\ 13 & 19 & 9 \end{bmatrix} \end{aligned}$$

Step 2: Find the Matrix of Cofactors by changing the signs of alternate cells:

$$\text{Matrix of cofactors} = \begin{bmatrix} 3 & -4 & 2 \\ 30 & -44 & 21 \\ 13 & -19 & 9 \end{bmatrix}$$

Step 3: Find the Adjugate by transposing all elements in the Matrix of Cofactors:

$$\text{Adjugate} = \begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$$

Step 4: Multiply by $\frac{1}{\text{determinant}}$:

$$\det(G) = 3 \times 3 - 3 \times 4 + 2 \times 2 = 9 - 12 + 4 = 1$$

$$\text{So } \frac{1}{\text{determinant}} = \frac{1}{1} = 1$$

$$\text{Therefore } G^{-1} = \begin{bmatrix} 3 & 30 & 13 \\ -4 & -44 & -19 \\ 2 & 21 & 9 \end{bmatrix}$$

Find the inverse of the matrix

$$H = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

using Minors, Cofactors and Adjugate

A $\begin{bmatrix} 2.2 & 1.6 & 1.4 \\ 4.8 & 3.4 & 3.6 \\ 0.8 & 0.4 & 0.6 \end{bmatrix}$

B $\begin{bmatrix} 2.2 & -1.6 & -1.4 \\ 4.8 & -3.4 & -3.6 \\ 0.8 & -0.4 & -0.6 \end{bmatrix}$

C $\begin{bmatrix} 2.2 & -1.6 & 1.4 \\ 4.8 & -3.4 & 3.6 \\ 0.8 & -0.4 & 0.6 \end{bmatrix}$

D $\begin{bmatrix} -2.2 & -1.6 & 1.4 \\ -4.8 & -3.4 & 3.6 \\ -0.8 & -0.4 & 0.6 \end{bmatrix}$

Step 1: Verify that the matrix is invertible.

Step 2: Find the Minors of the matrix.

Step 3: Find the Cofactors of the matrix.

Step 4: Find the Adjugate of the matrix.

Step 5: Find the Inverse of the matrix.

Step 6: Verify the result.

$$J = \begin{bmatrix} -2 & -2 & 1 \\ -4 & -8 & 4 \\ -1 & 5 & 0 \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} 1 & 0.25 & 0 \\ -0.2 & -0.05 & 0.2 \\ -1.4 & 0.6 & -0.4 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} -1 & -0.25 & 0 \\ -0.2 & 0.05 & -0.2 \\ 1.4 & 0.6 & 0.4 \end{bmatrix}$$
$$C = \begin{bmatrix} -1 & 0.25 & 0 \\ 0.2 & 0.05 & 0.2 \\ -1.4 & -0.6 & 0.4 \end{bmatrix}$$
$$\mathbf{D} = \begin{bmatrix} -1 & 0.25 & 0 \\ -0.2 & 0.05 & 0.2 \\ -1.4 & 0.6 & 0.4 \end{bmatrix}$$
[illegible]

Solving Systems of Linear Equations Using Matrices

Hi there! This page is only going to make sense when you know a little about [Systems of Linear Equations](#) and [Matrices](#), so please go and learn about those if you don't know them already!

The Example

One of the last examples on [Systems of Linear Equations](#) was this one:

Example: Solve

- $x + y + z = 6$
- $2y + 5z = -4$
- $2x + 5y - z = 27$

We then went on to solve it using "elimination" ... but we can solve it using Matrices!

Using Matrices makes life easier because we can use a computer program (such as the [Matrix Calculator](#)) to do all the "number crunching".

But first we need to write the question in Matrix form.

In Matrix Form?

OK. A Matrix is an array of numbers, right?

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

Well, think about the equations:

$$\begin{array}{rclcl} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}$$

They could be turned into a table of numbers like this:

$$\begin{array}{rclcl} 1 & & 1 & & 1 & = & 6 \\ 0 & & 2 & & 5 & = & -4 \\ 2 & & 5 & & -1 & = & 27 \end{array}$$

We could even separate the numbers before and after the "=" into:

$$\begin{array}{rcl} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{array} \quad \text{and} \quad \begin{array}{r} 6 \\ -4 \\ 27 \end{array}$$

Now it looks like we have 2 Matrices.

Now it looks like we have 2 Matrices.

In fact we have a third one, which is $[x \ y \ z]$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

Why does $[x \ y \ z]$ go there? Because when we [Multiply Matrices](#) the left side becomes:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y + z \\ 2y + 5z \\ 2x + 5y - z \end{bmatrix}$$

Which is the original left side of our equations above (you might like to check that).

The Matrix Solution

We can write this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

like this:

$$AX = B$$

where

- A is the 3x3 matrix of x, y and z **coefficients**
- X is **x, y and z**, and
- B is **6, -4 and 27**

Then (as shown on the [Inverse of a Matrix](#) page) the solution is this:

$$X = A^{-1}B$$

What does that mean?

It means that we can find the values of x, y and z (the X matrix) by multiplying the **inverse of the A matrix** by the **B matrix**.

So let's go ahead and do that.

First, we need to find the **inverse of the A matrix** (assuming it exists!)

Using the [Matrix Calculator](#) we get this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix}^{-1} = \frac{1}{-21} \begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}$$

(I left the 1/determinant outside the matrix to make the numbers simpler)

Then multiply A^{-1} by B (we can use the Matrix Calculator again):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

And we are done! The solution is:

$$\begin{aligned} x &= 5, \\ y &= 3, \\ z &= -2 \end{aligned}$$

Just like on the [Systems of Linear Equations](#) page.

Quite neat and elegant, and the human does the thinking while the computer does the calculating.

Just For Fun ... Do It Again!

For fun (and to help you learn), let us do this all again, but put matrix "X" first.

I want to show you this way, because many people think the solution above is so neat it must be the only way.

So we will solve it like this:

$$XA = B$$

And because of the way that matrices are multiplied we need to set up the matrices differently now. The rows and columns have to be switched over ("transposed"):

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix} = \begin{bmatrix} x + y + z & 2y + 5z & 2x + 5y - z \end{bmatrix}$$

And $XA = B$ looks like this:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 27 \end{bmatrix}$$

The Matrix Solution

Then (also shown on the [Inverse of a Matrix](#) page) the solution is this:

$$X = BA^{-1}$$

This is what we get for A^{-1} :

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix}^{-1} = \frac{1}{-21} \begin{bmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{bmatrix}$$

In fact it is just like the Inverse we got before, but Transposed (rows and columns swapped over).

Next we multiply B by A^{-1} :

$$\begin{aligned} \begin{bmatrix} x & y & z \end{bmatrix} &= \frac{1}{-21} \begin{bmatrix} 6 & -4 & 27 \end{bmatrix} \begin{bmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{bmatrix} \\ &= \frac{1}{-21} \begin{bmatrix} -105 & -63 & 42 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 3 & -2 \end{bmatrix} \end{aligned}$$

And the solution is the same:

$$x = 5, y = 3 \text{ and } z = -2$$

It didn't look as neat as the previous solution, but it does show us that there is more than one way to set up and solve matrix equations. Just be careful about the rows and columns!

Use matrices to solve the equations :

$$3x + y = 3$$

$$4x - 3y = 17$$

A $x = -2$ and $y = 3$

B $x = 2$ and $y = -3$

Write the pair of linear equations as a matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \end{bmatrix}$$

Find the inverse of the matrix:

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$a = 1, c = 4$ and $d = -3$

Determinant of X:

$$ad - bc = 3 \times (-3) - 1 \times 4 = -9 - 4 = -13$$

Inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{-13} \begin{bmatrix} -3 & -1 \\ -4 & 3 \end{bmatrix}$$

Multiply both sides of the matrix equation by X^{-1} :

$$\frac{1}{-13} \begin{bmatrix} -3 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-13} \begin{bmatrix} -3 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \end{bmatrix}$$

Therefore: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-13} \begin{bmatrix} -26 \\ 39 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow x = 2 \text{ and } y = -3$$

Use matrices to solve the equations:

$$2x + 3y = 11$$

$$-3x - 4y = -13$$

A $x = 1$ and $y = 3$

B $x = -2$ and $y = 5$

C $x = -5$ and $y = 7$

D $x = 5$ and $y = -7$

26. $xy = 12$
 $3x - 2y = 13$

Use the inverse of the coefficient matrix to solve.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

Find $\begin{bmatrix} x \\ y \end{bmatrix}$ by using the inverse.

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

Use the inverse of the coefficient matrix to solve.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

Use the inverse of the coefficient matrix to solve.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Find the inverse of A .

$$A^{-1} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}} \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

Multiply both sides of the matrix equation by A^{-1} .

$$\begin{bmatrix} -1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$\Rightarrow x = -1 \text{ and } y = 7$$

Use matrices to solve the equations:

$$2x - 5y = 31$$

$$-3x + 7y = -45$$

A $x = 1$ and $y = -6$

B $x = 3$ and $y = -5$

C $x = 8$ and $y = -3$

D $x = 13$ and $y = -1$

2x - 3y = 11
4x - 5y = -45

Use the inverse of the matrix to solve the system of equations

$$\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -45 \end{bmatrix}$$

Write the inverse of the matrix

$$M = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$$

Use the inverse of the matrix to solve the system of equations

Use the inverse of the matrix to solve the system of equations

Use the inverse of the matrix to solve the system of equations

The inverse of the matrix $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$ is $\frac{1}{2 \cdot 4 - 3 \cdot 5} \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix}$

Use the inverse of the matrix to solve the system of equations

$$M^{-1} = \frac{1}{2 \cdot 4 - 3 \cdot 5} \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix} = \frac{1}{-7} \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix}$$

Use the inverse of the matrix to solve the system of equations

$$\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -45 \end{bmatrix}$$

Use the inverse of the matrix to solve the system of equations

$$\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -45 \end{bmatrix}$$

Use the inverse of the matrix to solve the system of equations

Use the inverse of the matrix to solve the system of equations

Use matrices to solve the equations :

$$3x - y = -11$$

$$2x + 3y = 11$$

A $x = -2$ and $y = -5$

B $x = -2$ and $y = 5$

C $x = 2$ and $y = 5$

D $x = 3$ and $y = 20$

Ques
 20/04/2021
 Example 1: Use matrix method to solve the

$$\begin{cases} x + 2y = 7 \\ 3x - y = 11 \end{cases}$$

 Sol: Let $x = x$ and $y = y$

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$
 Then $AX = B$
 Multiplying both sides by A^{-1}
 $A^{-1}AX = A^{-1}B$
 $I X = A^{-1}B$
 $X = A^{-1}B$
 Therefore the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

Example 2: Solve the system $\begin{cases} x + 2y = 7 \\ 3x - y = 11 \end{cases}$
 Sol: Let $x = x$ and $y = y$

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$
 Then $AX = B$
 Multiplying both sides by A^{-1}
 $A^{-1}AX = A^{-1}B$
 $I X = A^{-1}B$
 $X = A^{-1}B$
 Therefore the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$
 where $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

Use matrices to solve the equations :

$$5x + 2y = 1$$

$$2x - y = 13$$

A $x = 5$ and $y = -12$

B $x = -3$ and $y = -7$

C $x = 3$ and $y = 7$

D $x = 3$ and $y = -7$

$5x + 2y = 1$
 $2x - y = 13$
 From matrix part two equations can be written as:

$$\begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

Now find the inverse of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\det A = 5 \times (-1) - 2 \times 2 = -9$$

For the inverse of A

$$\text{adj}(A) = \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix}$$

$$\text{The inverse of the matrix } A = \frac{1}{\det A} \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix} = \frac{1}{-9} \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix}$$

The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore the inverse of X is

$$X^{-1} = \frac{1}{-9} \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix}$$

Multiply both sides of the matrix equation by X^{-1} :

$$\frac{1}{-9} \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-9} \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

$$\text{Therefore: } \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-9} \begin{bmatrix} -27 \\ 63 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$$\text{so } x = 3 \text{ and } y = -7$$

Use matrices to solve the equations:

$$3x + 2y = -9$$

$$-5x - 7y = -7$$

A $x = -7$ and $y = 6$

B $x = 7$ and $y = -6$

C $x = 3$ and $y = -9$

D $x = 7$ and $y = 6$

```
def solve(A, b):
    """Solve the system Ax = b using Gaussian elimination.
    A is a 2x2 matrix, b is a 2x1 vector.
    Returns a tuple (x, y) where x and y are the solutions.
    """
    # Create an augmented matrix [A | b]
    aug = [
        [A[0][0], A[0][1], b[0]],
        [A[1][0], A[1][1], b[1]]
    ]

    # Perform row operations to get row echelon form
    # Swap rows if necessary
    if aug[0][0] == 0 and aug[1][0] != 0:
        aug[0], aug[1] = aug[1], aug[0]

    # Make the leading coefficient of the first row 1
    aug[0][0] = 1

    # Eliminate the x term from the second row
    aug[1][0] -= aug[0][0] * aug[1][0]

    # Make the leading coefficient of the second row 1
    aug[1][1] = 1

    # Back-substitute to find x
    x = -aug[1][2]

    # Find y using the first row
    y = -aug[0][2] + x * aug[0][1]

    return (x, y)
```

Claire and Dale shopped at the same store.

Claire bought 5 kg of apples and 2 kg of bananas and paid altogether \$22

Dale bought 4 kg of apples and 6 kg of bananas and paid altogether \$33

Use matrices to find the cost of 1 kg of bananas

A \$3

B \$3.50

C \$4

D \$4.50

```
11. Use the matrices A and B to find A + B.  
A =  $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$   
B =  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$   
Solution:  
A + B =  $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$   
=  $\begin{bmatrix} 2+1 & 3+2 \\ 4+3 & 5+4 \end{bmatrix}$   
=  $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$   
A + B =  $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$ 
```

Ann and Billy both entered a quiz. The quiz had twenty questions and points were allocated as follows:

- p points were added for each correctly answered question.
- q points were deducted for each incorrect (or unanswered) question.

Ann got 15 questions correct and scored 65 points.

Billy got 11 questions correct and scored 37 points.

Use matrices to find the value of q ?

A $q = 2$

B $q = 3$

C $q = 4$

D $q = 5$

Ann and Billy both entered a quiz. The quiz had twenty questions and points were allocated as follows:

• p points were added for each correctly answered question.

• q points were deducted for each incorrect (or unanswered) question.

Ann got 15 questions correct and scored 65 points.

Billy got 11 questions correct and scored 37 points.

Use matrices to find the value of q ?

Solution:

Let x be the number of questions correctly answered and y be the number of questions incorrectly answered (or unanswered).

For Ann:

$$15x - 5y = 65$$

For Billy:

$$11x - 5y = 37$$

Write the system of equations in matrix form:

$$\begin{bmatrix} 15 & -5 \\ 11 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 65 \\ 37 \end{bmatrix}$$

Let $A = \begin{bmatrix} 15 & -5 \\ 11 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 65 \\ 37 \end{bmatrix}$.

Then the system of equations can be written as:

$$AX = B$$

To solve for X , we need to find the inverse of A .

The determinant of A is:

$$\det(A) = (15)(-5) - (-5)(11) = -75 + 55 = -20$$

The inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -5 & 5 \\ 11 & -15 \end{bmatrix} = \frac{1}{-20} \begin{bmatrix} -5 & 5 \\ 11 & -15 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{11}{20} & \frac{3}{4} \end{bmatrix}$$

Now, multiply both sides of the equation $AX = B$ by A^{-1} :

$$A^{-1}AX = A^{-1}B$$

Since $A^{-1}A = I$, we have:

$$X = A^{-1}B$$

Substitute the values of A^{-1} and B :

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{11}{20} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 65 \\ 37 \end{bmatrix}$$

Calculate the matrix multiplication:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(65) - \frac{1}{4}(37) \\ -\frac{11}{20}(65) + \frac{3}{4}(37) \end{bmatrix} = \begin{bmatrix} \frac{65 - 37}{4} \\ -\frac{715}{20} + \frac{111}{4} \end{bmatrix} = \begin{bmatrix} \frac{28}{4} \\ -\frac{715}{20} + \frac{555}{20} \end{bmatrix} = \begin{bmatrix} 7 \\ -\frac{160}{20} \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

Therefore, $x = 7$ and $y = -8$.

Since y represents the number of questions incorrectly answered (or unanswered), it cannot be negative. This indicates that the system of equations is inconsistent, and there is no solution.

A store sells books and CD's. All the books have the same price and all the CD's have the same price, but different from that of the books.

Diane and Eric both shopped at the store.

Diane bought 5 books and 3 CD's and paid altogether \$90

Eric bought 2 books and 8 CD's and paid altogether \$138

Use matrices to find the cost of one CD.

A \$9

B \$11

C \$13

D \$15

```
def solve():
    # Create the coefficient matrix A
    A = [[5, 3], [2, 8]]
    # Create the constant vector b
    b = [90, 138]
    # Create the inverse of A
    A_inv = inverse(A)
    # Calculate the solution vector x
    x = multiply(A_inv, b)
    # Print the solution
    print(x)
```

Use matrices to solve the equations :

$$x + y - z = -3$$

$$2x - 3y + 4z = 23$$

$$-3x + y - 2z = -15$$

A $x = 2, y = -1$ and $z = 4$

B $x = 1, y = -2$ and $z = 2$

C $x = 2, y = -9$ and $z = -4$

D $x = -2, y = -1$ and $z = 0$

Step 1: Write the system of equations in matrix form $AX = B$, where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 23 \\ -15 \end{bmatrix}$$

Step 2: Find the inverse of matrix A , denoted as A^{-1} .

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Step 3: Multiply both sides of the equation $AX = B$ by A^{-1} to solve for X .

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Step 4: Substitute the values of A^{-1} and B to find the solution for X .

$$X = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Therefore, the solution is $x = 2, y = -1, z = 4$.

Step 1: Find the Matrix of Coefficients by changing the signs of alternate cells.

$$\text{Matrix } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$$

Step 2: Find the Adjoints by inverting all elements in the Matrix of Coefficients.

$$\text{Adjoints} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & -3 & 1 \\ 4 & 4 & -5 \end{bmatrix}$$

Step 3: Find the Determinant of the Matrix of Coefficients.

$$\det(A) = 1(-6 - 10) - 1(-4 - 12) - 1(-2 - 9) = -16 + 16 + 11 = 11$$

Step 4: Find the Inverse of the Matrix of Coefficients.

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{11} \begin{bmatrix} 1 & 2 & -3 \\ -1 & -3 & 1 \\ 4 & 4 & -5 \end{bmatrix}$$

Step 5: Multiply both sides of the matrix equation by A^{-1} .

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 23 \\ -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} \cdot \frac{1}{11} \begin{bmatrix} 1 & 2 & -3 \\ -1 & -3 & 1 \\ 4 & 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 1 & 2 & -3 \\ -1 & -3 & 1 \\ 4 & 4 & -5 \end{bmatrix} \begin{bmatrix} -3 \\ 23 \\ -15 \end{bmatrix}$$

Therefore, $x = 2, y = -1$ and $z = 4$.

