Tishk International University
Faculty of Administrative Sciences and Economics

MATHEMATICS
FOR ECONOMICS AND BUSINESS

BUS 143
Part 7

I Grade- Fall

Assist. Prof. Dr. Hamdi Serin

Interest *(An Introduction)*

Interest: how much is paid for the use of money (as a percent, or an amount)

Money is Not Free to Borrow

People can always find a use for money, so it **costs to borrow money**.

How Much does it Cost to Borrow Money?

Different places charge different amounts at different times!

But they usually charge this way:

%	As a percent (per year) of the amount borrowed	%
	It is called **Interest**	

Example: Borrow $1,000 from the Bank

Alex wants to borrow $1,000. The local bank says "***10% Interest***". So to borrow the $1,000 for 1 year will cost:

$1,000 × 10% = $100

In this case the "Interest" is $100, and the "Interest Rate" is 10% (but people often say "10% Interest" without saying "Rate")

Of course, Alex will have to pay back the original $1,000 after one year, so this is what happens:

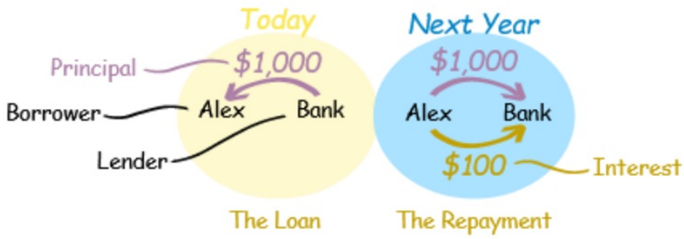
Alex Borrows $1,000, but has to pay back $1,100

This is the idea of Interest ... paying for the use of the money.

Note: This example is a simple full year loan, but banks often want the loan paid back in monthly amounts, and they also charge extra fees too!

Words

There are special words used when borrowing money, as shown here:



Alex is the **Borrower**, the Bank is the **Lender**

The **Principal** of the Loan is $1,000

The **Interest** is $100

The important part of the word "Interest" is *Inter-* meaning *between* (we see *inter-* in words like *interior* and *interval*), because the interest happens between the start and end of the loan.

More Than One Year ...

What if Alex wanted to borrow the money for 2 Years?

Simple Interest

If the bank charges "Simple Interest" then Alex just pays another 10% for the extra year.

Alex pays Interest of ($1,000 × 10%) x 2 Years = $200

That is how simple interest works ... pay the same amount of interest every year.

> Example: Alex borrows $1,000 for 5 Years, at 10% simple interest:
>
> - Interest = $1,000 × 10% x 5 Years = $500
> - Plus the Principal of $1,000 means Alex needs to pay $1,500 after 5 Years

> Example: Alex borrows $1,000 for 7 Years, at 6% simple interest:
>
> - Interest = $1,000 × 6% x 7 Years = $420
> - Plus the Principal of $1,000 means Alex needs to pay $1,420 after 7 Years

There is a formula for simple interest

$$I = Prt$$

where

- I = interest
- P = amount borrowed (called "Principal")
- r = interest rate
- t = time

Like this:

Example: Jan borrowed $3,000 for 4 Years at 5% interest rate, how much interest is that?

$I = Prt$

$I = \$3,000 \times 5\% \times 4$ years

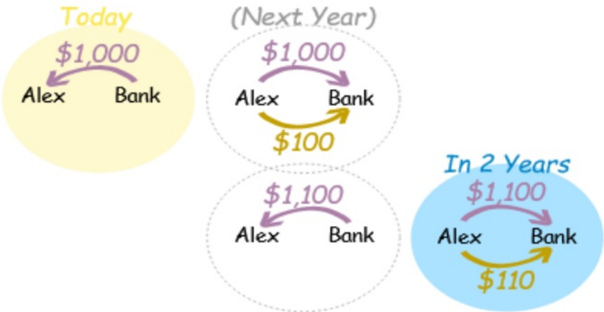
$I = \$3000 \times 0.05 \times 4$

$I = \$600$

But banks almost NEVER charge simple interest, they prefer Compound Interest:

Compound Interest

But the bank says "If you paid me everything back after one year, and then I loaned it to you again, I would be loaning you **$1,100 for the second year**!" so I want more interest:



And Alex pays **$110** interest in the second year, not just $100.

Because Alex is paying 10% on $1,100 not just $1,000

This may seem unfair ... but imagine YOU lend the money to Alex. After a year you think *"Alex owes me $1,100 now, and is still using my money, I should get more interest!"*

And so this is the normal way of calculating interest. It is called **compounding**.

With **compounding** we work out the interest for the first period, add it the total, and **then** calculate the interest for the next period, and so on ..., like this:

It is like paying interest on interest: after a year Alex owed $100 interest, the Bank thinks of that as another loan and charges interest on it, too.

After a few years it can get really large. This is what happens on a 5 Year Loan:

Year	Loan at Start	Interest	Loan at End
0 (Now)	$1,000.00	($1,000.00 × 10% =) **$100.00**	$1,100.00
1	$1,100.00	($1,100.00 × 10% =) **$110.00**	$1,210.00
2	$1,210.00	($1,210.00 × 10% =) **$121.00**	$1,331.00
3	$1,331.00	($1,331.00 × 10% =) **$133.10**	$1,464.10
4	$1,464.10	($1,464.10 × 10% =) **$146.41**	$1,610.51
5	$1,610.51		

So, after 5 Years Alex has to pay back **$1,610.51**

And the Interest for the last year was $146.41 ... it sure grew quickly!

(Compare that to the Simple Interest of only $100 each year)

What is Year 0?

Year 0 is the year that starts with the "Birth" of the Loan, and ends just before the 1st Birthday.

> Just like when a baby is born its age is **zero**, and will not be 1 year old until the first birthday.

So the start of **Year 1** is the "1st Birthday". And the **start of Year 5** is exactly when the loan is 5 years old.

> **In Summary:**
>
> To calculate compound interest, work out the interest for the first period, add it on, and then calculate the interest for the next period, etc.
>
> (There are quicker methods, see Compound Interest)

Why Borrow?

Well ... you may want to buy something you like. Paying it back will end up costing you more though.

But a business may be able to use the money to make **even more** money.

Example: Chicken Business

You borrow $1,000 to start a chicken business (to buy chicks, chicken food and so on).

A year later you sell all the grown chickens for $1,200.

You pay back the bank $1,100 (the original $1,000 plus 10% interest) and you are left with **$100 profit**.

And you used someone else's money to do it!

But be careful! What if you only sold the chickens for $800? ... the bank still wants $1,100 and you end up with a **$300 loss.**

Investment

Compound Interest can **work for you**!

Investment is when you put money where it **can grow**, such as a bank, or a business.

If you invest your money at a good interest rate it can grow very nicely.

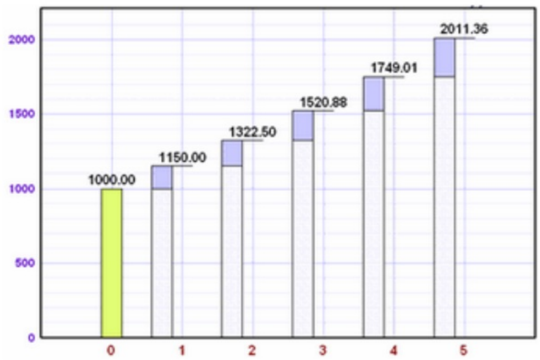
This is what 15% interest on $1,000 can do:

Year	Loan at Start	Interest	Loan at End
0 (Now)	$1,000.00	($1,000.00 × 15% =) **$150.00**	$1,150.00
1	$1,150.00	($1,150.00 × 15% =) **$172.50**	$1,322.50
2	$1,322.50	($1,322.50 × 15% =) **$198.38**	$1,520.88
3	$1,520.88	($1,520.88 × 15% =) **$228.13**	$1,749.01
4	$1,749.01	($1,749.01 × 15% =) **$262.35**	$2,011.36
5	$2,011.36		

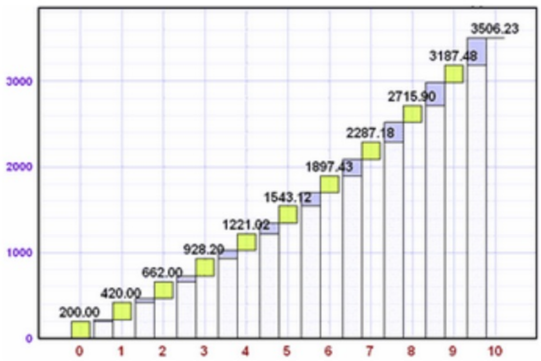
It more than doubles in 5 Years!

An investment at 15% is not likely to be safe (see Investing introduction) ... but it does show us the power of compounding.

The graph of that investment looks like this:



Maybe you don't have $1,000? Here is what saving $200 every year for 10 Years at 10% interest can do:



$3,506.23 after 10 Years!
For 10 Years of $200 each year.

Jerry borrowed $4,000 for 5 years at 6% simple interest rate. How much interest is that?

A $800	B $1,000
C $1,200	D $1,500

Use I = Prt

where

I = Interest

P = Principal = $4,000

r = interest rate = 6% = 0.06

t = time = 5 years

I = $4,000 × 0.06 × 5 = $1,200

Julie borrowed $3,500 for 3 years at 7½% simple interest rate.

How much interest is that?

A $735

B $787.50

C $810

D $812.50

Use I = Prt

where

I = Interest

P = Principal = $3,500

r = interest rate = 7½% = 0.075

t = time = 3 years

Therefore, I = $3,500 × 0.075 × 3 = $787.50

Dan borrowed $2,000 for 6 months at 12% annual simple interest rate. How much interest is that?

A $120

B $144

C $1,200

D $1,440

Jenna borrowed $5,000 for 3 years and had to pay $1,350 simple interest at the end of that time. What rate of interest did she pay?

A 6%	B 7%
C 8%	D 9%

Sam borrowed $4,500 for 2 years and had to pay $630 simple interest at the end of that time. What rate of interest did he pay?

A 6%

B 7%

C 8%

D 9%

Use I = Prt

where

I = Interest = $630

P = Principal = $4,500

r = interest rate

t = time = 2 years

So; 630 = 4,500 × r × 2

Simplify: 630 = 9,000 × r

Rearrange: r = 630 ÷ 9,000

Solve: r = 0.07 = 7%

Sanjay borrowed $7,000 at a simple interest rate of 3% per year.
After a certain number of years he had paid $840 in interest altogether.
How many years was that?

A 2 years

B 3 years

C 3½ years

D 4 years

[illegible]

where

I = Interest = $840

P = Principal = $7000

r = interest rate = 3% = 0.03

t = time

So, 840 = 7000 × 0.03 × t

Simplify, 840 = 210 × t

Rearrange t = 840 ÷ 210

Solve: t = 4

The time was 4 years

Sarina borrowed $5,800 at a simple interest rate of 7½% per year.
After a certain number of years she had paid $1,305 in interest altogether.
How many years was that?

A 3 years	B 3½ years
C 4 years	D 5 years

Alex borrowed $2,000 for 2 years at 5% compound interest rate. How much interest is that?

A $200

B $205

C $2,100

D $2,205

Interest for the first year = 5% of $2,000 = 0.05 × $2,000 = $100
So at the end of the first year he owes $2,000 + $100 = $2,100

Interest for the second year = 5% of $2,100 = 0.05 × $2,100 = $105

So the total amount of compound interest = $100 + $105 = $205

Alice borrowed $4,000 for 3 years at 10% compound interest rate. How much interest is that?

A $1,200

B $1,240

C $1,280

D $1,324

Simon borrowed $1,000 for 3 years at 5% compound interest rate. How much did he owe after 3 years?

A $1,150	B $1,155
C $1,157.63	D $1,175.13

Compound Interest

You may wish to read [Introduction to Interest](#) first

With Compound Interest, you work out the interest for the first period, add it to the total, and **then** calculate the interest for the next period, and so on ..., like this:

It grows faster and faster like this:

Here are the calculations for 5 Years at 10%:

Year	Loan at Start	Interest	Loan at End
0 (Now)	$1,000.00	($1,000.00 × 10% =) **$100.00**	$1,100.00
1	$1,100.00	($1,100.00 × 10% =) **$110.00**	$1,210.00
2	$1,210.00	($1,210.00 × 10% =) **$121.00**	$1,331.00
3	$1,331.00	($1,331.00 × 10% =) **$133.10**	$1,464.10
4	$1,464.10	($1,464.10 × 10% =) **$146.41**	$1,610.51
5	$1,610.51		

Those calculations are done one step at a time:

- Calculate the Interest (= "Loan at Start" × Interest Rate)
- Add the Interest to the "Loan at Start" to get the "Loan at End" of the year
- The "Loan at End" of the year is the "Loan at Start" of the **next** year

A simple job, with lots of calculations.

But there are quicker ways, using some clever mathematics.

Make A Formula

Let us make a formula for the above ... just looking at the first year to begin with:

$1,000.00 + ($1,000.00 × 10%) = **$1,100.00**

We can rearrange it like this:

Loan at Start		Interest ($100)
$1,000	+	$1,000 × 10%
→ $1,000 × 1	+	$1,000 × 0.10
→ $1,000 × (1	+	0.10)
→ $1,000 × 1.10		

So, adding 10% interest is the same as multiplying by 1.10

+10%

×1.10

so this: $1,000 + ($1,000 x 10%) = $1,000 + $100 = **$1,100**

is the same as: $1,000 × 1.10 = **$1,100**

Note: the Interest Rate was turned into a decimal by dividing by 100:

10% = 10/100 = 0.10

Read Percentages to learn more, but in practice just move the decimal point 2 places, like this:

10% → 1.0 → **0.10**

Or this:

6% → 0.6 → **0.06**

The result is that we can do a year in one step:

- **Multiply the "Loan at Start" by (1 + Interest Rate) to get "Loan at End"**

Now, here is the magic ...

... the same formula works for any year!

- We could do the next year like this: **$1,100 × 1.10 = $1,210**
- And then continue to the following year: **$1,210 × 1.10 = $1,331**
- etc...

So it works like this:

$1,000 →(×1.10) $1,100 →(×1.10) $1,210 →(×1.10) $1,331

In fact we could go from the start straight to Year 5, if we **multiply 5 times**:

$1,000 × 1.10 × 1.10 × 1.10 × 1.10 × 1.10 = **$1,610.51**

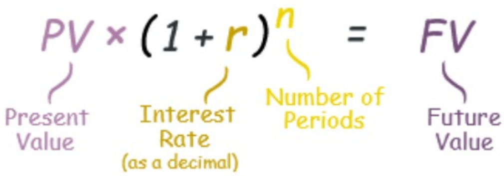
But it is easier to write down a series of multiplies using Exponents (or Powers) like this:

$$\$1{,}000 \times 1.10^{5} = \$1{,}610.51$$

This does all the calculations in the top table in one go.

The Formula

We have been using a real example, but let's be more general by **using letters instead of numbers**, like this:



(This is the same as above, but with PV = $1,000, r = 0.10, n = 5, and FV = $1,610.51)

Here is is written with "FV" first:

$$FV = PV \times (1+r)^n$$

where **FV** = Future Value
PV = Present Value
r = annual interest rate
n = number of periods

This is the basic formula for Compound Interest.

Remember it, because it is very useful.

Examples

How about some examples ...
... what if the loan went for **15 Years**? ... just change the "n" value:

$$\$1{,}000 \times 1.10^{15} = \$4{,}177.25$$

... and what if the loan was for 5 years, but the interest rate was only 6%? Here:

$$\$1{,}000 \times 1.06^{5} = \$1{,}338.23$$

Did you see how we just put the 6% into its place like this: 6% → 1.06

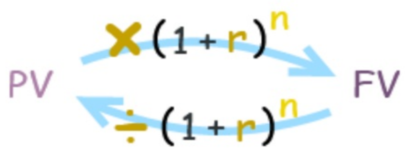
... and what if the loan was for 20 years at 8%? ... you work it out!

Going "Backwards" to Work Out the Present Value

Let's say your goal is to have $2,000 in 5 Years. You can get 10%, so **how much should you start with**?

In other words, you know a Future Value, and **want to know a Present Value**.

We know that **multiplying** a Present Value (PV) by $(1+r)^n$ gives us the Future Value (FV), so we can go backwards by **dividing**, like this:



So the Formula is:

$$PV = \frac{FV}{(1+r)^n}$$

And now we can calculate the answer:

$$PV = \frac{\$2{,}000}{(1+0.10)^5}$$

$$= \frac{\$2{,}000}{1.61051}$$

$$= \$1{,}241.84$$

In other words, $1,241.84 will grow to $2,000 if you invest it at 10% for 5 years.

Another Example: How much do you need to invest now, to get $10,000 in 10 years at 8% interest rate?

$$PV = \frac{\$10,000}{(1+0.08)^{10}}$$

$$= \frac{\$10,000}{2.1589}$$

$$= \$4,631.93$$

So, **$4,631.93** invested at 8% for 10 Years grows to $10,000

Compounding Periods

Compound Interest is not always calculated per year, it could be per month, per day, etc. **But if it is not per year it should say so!**

Example: you take out a $1,000 loan for 12 months and it says "**1% per month**", how much do you pay back?

Just use the Future Value formula with "n" being the number of months:

$$\begin{aligned} FV &= PV \times (1+r)^n \\ &= \$1{,}000 \times (1.01)^{12} \\ &= \$1{,}000 \times 1.12683 \\ &= \mathbf{\$1{,}126.83} \text{ to pay back} \end{aligned}$$

And it is also possible to have yearly interest *but with several compoundings **within** the year*, which is called Periodic Compounding.

Example, 6% interest with "**monthly compounding**" does not mean 6% per month, it means 0.5% per month (6% divided by 12 months), and is worked out like this:

$$\begin{aligned} FV &= PV \times (1+r/n)^n \\ &= \$1{,}000 \times (1 + 6\%/12)^{12} \\ &= \$1{,}000 \times (1 + 0.5\%)^{12} \\ &= \$1{,}000 \times (1.005)^{12} \\ &= \$1{,}000 \times 1.06168... \\ &= \mathbf{\$1{,}061.68} \text{ to pay back} \end{aligned}$$

This is equal to a **6.168%** *($1,000 grew to $1,061.68)* for the whole year.

So be careful to understand what is meant!

APR

Because it is easy for loan ads to be confusing (sometimes on purpose!), the "**APR**" is often used.

APR means "***Annual Percentage Rate***": it shows how much you will actually be paying for the year (including compounding, fees, etc).

This ad looks like 6.25%, but ***is really 6.335%***

Here are some examples:

> Example 1: "**1% per month**" actually works out to be **12.683% APR** (if no fees)**.**

> Example 2: "**6% interest with monthly compounding**" works out to be **6.168% APR** (if no fees)**.**

If you are shopping around, ask for the APR.

Working Out The Interest Rate

You can calculate the Interest Rate if you know a Present Value, a Future Value and how many Periods.

Example: you have $1,000, and want it to grow to $2,000 in 5 Years, what **interest rate** do you need?

The formula is:

$$r = (FV / PV)^{1/n} - 1$$

Note: the little "1/n" is a Fractional Exponent, first calculate 1/n, then use that as the exponent on your calculator.

For example $2^{0.2}$ *is entered as* ***2, "x^y", 0, ., 2,*** *=*

Now we can "plug in" the values to get the result:

$$\begin{aligned} r &= (\$2,000 / \$1,000)^{1/5} - 1 \\ &= (2)^{0.2} - 1 \\ &= 1.1487 - 1 \\ &= 0.1487 \end{aligned}$$

And 0.1487 as a percentage is **14.87%**, So you need **14.87%** interest rate to turn $1,000 into $2,000 in 5 years.

Another Example: What interest rate do you need to turn $1,000 into $5,000 in 20 Years?

$$r = (\$5{,}000 / \$1{,}000)^{1/20} - 1$$

$$= (5)^{0.05} - 1$$

$$= 1.0838 - 1$$

$$= 0.0838$$

And 0.0838 as a percentage is **8.38%**.

So **8.38%** will turn $1,000 into $5,000 in 20 Years.

Working Out How Many Periods

You can calculate how many Periods if you know a Future Value, a Present Value and the Interest Rate.

Example: you want to know how many periods it will take to turn \$1,000 into \$2,000 at 10% interest.

This is the formula (note: it uses the natural logarithm function ***ln***):

$$n = \ln(FV / PV) / \ln(1 + r)$$

*The "**ln**" function should be on a good calculator.*

*You could also use **log**, just don't mix the two.*

Anyway, let's "plug in" the values:

$$\begin{aligned} n &= \ln(\$2{,}000/\$1{,}000) / \ln(1 + 0.10) \\ &= \ln(2)/\ln(1.10) \\ &= 0.69315/0.09531 \\ &= 7.27 \end{aligned}$$

Magic! It will need **7.27 years** to turn \$1,000 into \$2,000 at 10% interest.

Example: How many years to turn $1,000 into $10,000 at 5% interest?

$$n = \ln(\$10{,}000/\$1{,}000) / \ln(1 + 0.05)$$
$$= \ln(10)/\ln(1.05)$$
$$= 2.3026/0.04879$$
$$= 47.19$$

47 Years! But we are talking about a 10-fold increase, at only 5% interest.

Summary

The basic formula for Compound Interest is:

$$FV = PV\ (1+r)^n$$

Finds the **Future Value**, where:

- FV = Future Value,
- PV = Present Value,
- r = Interest Rate (as a decimal value), and
- n = Number of Periods

And by rearranging that formula *(see Compound Interest Formula Derivation)* we can find any value when we know the other three:

$$PV = \frac{FV}{(1+r)^n}$$

Finds the **Present Value** when you know a Future Value, the Interest Rate and number of Periods.

$$r = (FV/PV)^{(1/n)} - 1$$

Finds the **Interest Rate** when you know the Present Value, Future Value and number of Periods.

$$n = \frac{ln(FV\ /\ PV)}{ln(1\ +\ r)}$$

Finds the number of **Periods** when you know the Present Value, Future Value and Interest Rate (note: ***ln*** is the logarithm function)

If the present value of my investment is $1,000 and the rate of interest is 10% compounded annually, what will the value be after 6 years?

A $1,600	B $1,771.56
C $1,790.85	D $1,948.72

Use the formula:

$FV = PV \times (1 + r)^n$

Substitute PV = $1,000, r = 10% = 0.10 and n = 6

$FV = \$1,000 \times (1 + 0.10)^6 = \$1,000 \times (1.10)^6 = \$1,000 \times 1.771561 = \$1,771.561$

So the value after 6 years = $1,771.56

If the present value of my investment is $1,000 and the rate of interest is 6% compounded annually, what will the value be after 10 years?

A $1,600

B $1,771.56

C $1,790.85

D $1,898.30

Use the formula:

$FV = PV \times (1 + r)^n$

Substitute: PV = $1,000, r = 6% = 0.06, and n = 10

$FV = \$1,000 \times (1 + 0.06)^{10} = \$1,000 \times (1.06)^{10} = \$1,000 \times 1.790847... = \$1,790.847...$

The value after 10 years = $1,790.85

If the present value of my investment is $2,500 and the rate of interest is 2% compounded annually, what will the value be after 15 years?

A $3,364.67

B $3,306.25

C $3,250

D $3,047.49

If the present value of my investment is $9,000 and the rate of interest is 3½% compounded annually, what will the value be after 4 years?

A $10,176.87

B $10,260

C $10,324.27

D $10,327.71

Your goal is to have $2,000 in 6 years. The rate of interest is 10% compounded annually, so how much should you start with?

A $800

B $1,026.32

C $1,116.79

D $1,128.95

Your goal is to have $3,500 in 10 years. The rate of interest is 3% compounded annually, so how much should you start with?

A $2,126.16

B $2,450

C $2,604.33

D $2,629.60

Use the formula $PV = \frac{FV}{(1+r)^n}$

Substitute $FV = \$3{,}500$, $r = 3\% = 0.03$ and $n = 10$

Therefore $PV = \frac{\$3{,}500}{(1.03)^{10}} = \frac{\$3{,}500}{1.3439...} = \$2{,}604.328...$

So you should start with $2,604.33

You have $1,000, and want it to grow to $2,000 in 4 years, what compound interest rate do you need?

A 11.89%

B 18.92%

C 25%

D 41.42%

You have $2,500, and want it to grow to $4,000 in 10 years, what annual compound interest rate do you need?

A 4.81%

B 6%

C 6.05%

D 7.18%

Use the formula $r = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1$

Substitute FV = $4,000, PV = $2,500 and n = 10

Therefore $r = \left(\frac{4,000}{2,500}\right)^{\frac{1}{10}} - 1$

$= (1.6)^{0.1} - 1$

$= 1.0481 - 1$

$= 0.0481$

$= 4.81\%$

How many years will it take to turn $1,000 into $1,500 at 8% compound interest?

A 4.254 years

B 5 years

C 5.268 years

D 6.25 years

Approximately how many years will it take to turn $2,500 into $3,000 at 3.7% interest, compounded annually?

A 4 years

B 5 years

C 6 years

D 7 years

Compound Interest Formula Derivations

Showing how the formulas are worked out, with Examples!

With Compound Interest we work out the interest for the first period, add it to the total, and **then** calculate the interest for the next period, and so on ..., like this:

Make A Formula

Let's look at the first year to begin with:

$1,000.00 + ($1,000.00 × 10%) = **$1,100.00**

We can rearrange it like this:

Loan at Start   Interest ($100)

$$\$1{,}000 + \$1{,}000 \times 10\%$$

$$\rightarrow \$1{,}000 \times 1 + \$1{,}000 \times 0.10$$

$$\rightarrow \$1{,}000 \times (1 + 0.10)$$

$$\rightarrow \$1{,}000 \times 1.10$$

So, adding 10% interest is the same as multiplying by 1.10

*(Note: the Interest Rate was turned into a decimal by dividing by 100: **10% = 10/100 = 0.10**, read Percentages to learn more.)*

And that formula works for any year:

- We could do the next year like this: **\$1,100 × 1.10 = \$1,210**
- And then continue to the following year: **\$1,210 × 1.10 = \$1,331**
- etc...

So it works like this:

$$\$1{,}000 \xrightarrow{\times 1.10} \$1{,}100 \xrightarrow{\times 1.10} \$1{,}210 \xrightarrow{\times 1.10} \$1{,}331$$

In fact we could go straight from the start to Year 5 if we **multiply 5 times**:

$$\$1{,}000 \times 1.10 \times 1.10 \times 1.10 \times 1.10 \times 1.10 = \mathbf{\$1{,}610.51}$$

But it is easier to write down a series of multiplies using Exponents (or Powers) like this:

$$\$1{,}000 \times 1.10^5 = \$1{,}610.51$$

The Formula

We have been using a real example, but let us make it more general by **using letters instead of numbers**, like this:

$$PV \times (1 + r)^n = FV$$

PV: Present Value; r: Interest Rate (as a decimal); n: Number of Periods; FV: Future Value

(Compare this to the calculation above it: PV = $1,000, r = 0.10, n = 5, and FV = $1,610.51)

- When the interest rate is annual, then **n** is the number of years
- When the interest rate is monthly, then **n** is the number of months
- and so on

Examples

How about some examples ...
... what if the loan went for **15 Years**? ... just change the "n" value:

$$\$1,000 \times 1.10^{15} = \$4,177.25$$

... and what if the loan was for 5 years, but the interest rate was only 6%? Here:

$$\$1,000 \times 1.06^{5} = \$1,338.23$$

(Note that it is **1.06**, not 1.6)

The Four Formulas

So, the basic formula for Compound Interest is:

$$FV = PV\ (1+r)^n$$

- FV = Future Value,
- PV = Present Value,
- r = Interest Rate (as a decimal value), and
- n = Number of Periods

With that we can work out the Future Value **FV** when we know the Present Value **PV**, the Interest Rate **r** and Number of Periods **n**

And we can **rearrange** that formula to find FV, the Interest Rate or the Number of Periods when we know the other three.

Here are all four furmulas:

$FV = PV\ (1+r)^n$	Find the **Future Value** when we know a Present Value, the Interest Rate and number of Periods.
$PV = FV\ /\ (1+r)^n$	Find the **Present Value** when we know a Future Value, the Interest Rate and number of Periods.
$r = (\ FV\ /\ PV\ )^{1/n} - 1$	Find the **Interest Rate** when we know the Present Value, Future Value and number of Periods.
$n = \dfrac{\ln(FV\ /\ PV)}{\ln(1 + r)}$	Find the number of **Periods** when we know the Present Value, Future Value and Interest Rate

Working Out the Present Value

Example: Sam wants to reach $2,000 in 5 Years at 10% annual interest. How much should Sam start with?

In other words, we know a Future Value, and **want to know a Present Value**.

We can just rearrange the formula to suit ... dividing both sides by $(1+r)^n$ to give us:

Start with: $FV = PV\ (1+r)^n$

Swap sides: $PV\ (1+r)^n = FV$

Divide both sides by $(1+r)^n$: $PV = \frac{FV}{(1+r)^n}$

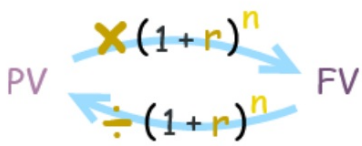
So now we can calculate the answer:

Example (continued):

PV = \$2,000 / $(1+0.10)^5$ = \$2,000 / 1.61051 = **\$1,241.84**

So Sam should start with **$1,241.84**

It works like this:



Another Example: How much do you need to invest now, to get $10,000 in 10 years at 8% interest rate?

$$PV = \$10{,}000 / (1+0.08)^{10} = \$10{,}000 / 2.1589 = \mathbf{\$4{,}631.93}$$

So, **$4,631.93** invested at 8% for 10 Years grows to $10,000

Working Out The Interest Rate

Example: Sam has only $1,000, and wants it to grow to $2,000 in 5 Years, what interest rate should Sam be looking for?

We need a rearrangement of the first formula to work it out:

Start with: $FV = PV\,(1+r)^n$

Swap sides: $PV\,(1+r)^n = FV$

Divide both sides by PV: $(1+r)^n = \frac{FV}{PV}$

Take nth root of both sides: $1+r = \left(\frac{FV}{PV} \right)^{1/n}$

Subtract 1 from both sides: $r = \left(\frac{FV}{PV} \right)^{1/n} - 1$

(Note: to understand the step "take nth root" please read Fractional Exponents)

The result is:

$$r = (FV / PV)^{1/n} - 1$$

Now we have the formula, it is just a matter of "plugging in" the values to get the result:

Example (continued):

$$r = (\ \$2{,}000\ /\ \$1{,}000\ )^{1/5} - 1$$
$$= (\ 2\ )^{0.2} - 1$$
$$= 1.1487 - 1$$
$$= \mathbf{0.1487}$$

And 0.1487 as a percentage is **14.87%**

So Sam needs **14.87%** to turn $1,000 into $2,000 in 5 years.

Another Example: What interest rate do you need to turn $1,000 into $5,000 in 20 Years?

$$r = (\ \$5{,}000\ /\ \$1{,}000\ )^{1/20} - 1 = (\ 5\ )^{0.05} - 1 = 1.0838 - 1 = \mathbf{0.0838}$$

And 0.0838 as a percentage is **8.38%**. So 8.38% will turn $1,000 into $5,000 in 20 Years.

Working Out How Many Periods

> Example: Sam can only get a 10% interest rate. How many years will it take Sam to get $2,000?

When we want to know how many periods it takes to turn $1,000 into $2,000 at 10% interest, we can rearrange the basic formula.

But we need to use the natural logarithm function ***ln()*** to do it.

Start with: $FV = PV\ (1+r)^n$

Swap sides: $PV\ (1+r)^n = FV$

Divide both sides by PV: $(1+r)^n = FV\ /\ PV$

Use logarithms: $\ln(1+r) \times n = \ln(\ FV\ /\ PV\ )$

Divide both sides by ln(1+r): $n = \dfrac{\ln(\ FV\ /\ PV\ )}{\ln(1+r)}$

Now let's "plug in" the values:

Example (continued):

n = ln($2,000 / $1,000) / ln(1 + 0.10) = ln(2)/ln(1.10) = 0.69315/0.09531 = **7.27**

Magic! It will need **7.27 years** to turn $1,000 into $2,000 at 10% interest.

Poor Sam will have to wait over 7 years.

Another Example: How many years to turn $1,000 into $10,000 at 5% interest?

n = ln($10,000 / $1,000) / ln(1 + 0.05) = ln(10)/ln(1.05) = 2.3026/0.04879 = **47.19**

47 Years! But we are talking about a 10-fold increase, at only 5% interest.

Conclusion

Knowing how the formulas are derived and used makes it easier for you to remember them, and to use them in different situations.

Compound Interest: Periodic Compounding

You may like to read about Compound Interest first.
You can skip straight down to Periodic Compounding.

Quick Explanation of Compound Interest

With Compound Interest, you work out the interest for the first period, add it to the total, and **then** calculate the interest for the next period, and so on ..., like this:

But adding 10% interest is the same as multiplying by 1.10 *(explained here)*

So it also works like this:

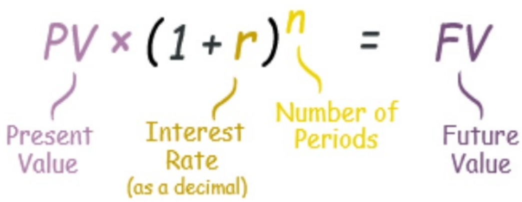
$$\$1{,}000 \xrightarrow{\times 1.10} \$1{,}100 \xrightarrow{\times 1.10} \$1{,}210 \xrightarrow{\times 1.10} \$1{,}331$$

In fact we can go from the Start to Year 5 if we **multiply 5 times** using Exponents (or Powers):

$$\$1{,}000 \times 1.10^5 = \$1{,}610.51$$

The Formula

This is the formula for Compound Interest (like above but **using letters instead of numbers**):



Example: $1,000 invested at 10% for 5 Years:

Present Value **PV** = **$1,000**

Interest Rate is 10%, which as a decimal **r** = **0.10**

Number of Periods **n** = **5**

$PV \times (1 + r)^n = FV$

$\$1{,}000 \times (1 + 0.10)^5 = FV$

$\$1{,}000 \times 1.10^5 =$ **$1,610.51**

Now we can choose different values, such as an interest rate of 6%:

Example: $1,000 invested at 6% for 5 Years:

Present Value **PV** = **$1,000**

Interest Rate is 6%, which as a decimal **r** = **0.06**

Number of Periods **n** = **5**

$$PV \times (1 + r)^n = FV$$

$$\$1{,}000 \times (1 + 0.06)^5 = FV$$

$$\$1{,}000 \times 1.06^5 = \mathbf{\$1{,}338.23}$$

Periodic Compounding (Within The Year)

But sometimes interest is charged Yearly ...

... but it is calculated more than once within the year, with the interest added each time ...

... so there are compoundings **within** the Year.

Example: "10%, Compounded Semiannually"

Semiannual means twice a year. So the 10% is split into two:

- 5% halfway through the year,
- and another 5% at the end of the year,

but each time it is **compounded** (meaning the interest is added to the total):

10%, Compounded Semiannually

This results in $1,102.50, which is equal to **10.25%**, not 10%

Two Annual Interest Rates?

Yes, there are two annual interest rates:

Example		
10%		The **Nominal Rate** (the rate they mention)
10.25%		The **Effective Annual Rate** (the rate after compounding)

The **Effective Annual Rate** is what actually gets paid!

When interest is compounded **within** the year, the Effective Annual Rate is **higher** than the rate mentioned.

How much higher depends on the interest rate, and how many times it is compounded within the year.

Working It Out

Let's come up with a formula to work out the **Effective Annual Rate** if we know:

- the rate mentioned (the **Nominal Rate, "r"**)
- how many times it is compounded (**"n"**)

Our task is to take an interest rate (like 10%) and chop it up into "n" periods, compounding each time.

From the Compound Interest formula (shown above) we can compound "n" periods using

$$FV = PV\,(1+r)^n$$

But the interest rate won't be "r", because it has to be chopped into "n" periods like this:

$$r\,/\,n$$

So we change the compounding formula into:

This is the formula for Periodic Compounding:

$$FV = PV\ (1+(r/n))^n$$

where **FV** = Future Value
PV = Present Value
r = annual interest rate
n = number of periods within the year

Let's try it on our "10%, Compounded Semiannually" example:

$$FV = \$1{,}000\ (1+(0.10/2))^2 = \$1{,}000(1.05)^2 = \$1{,}000 \times 1.1025 = \$1{,}102.50$$

That worked! But we want to know what the new **interest rate** is, we don't want the dollar values in there, so let's remove them:

$$(1+(r/n))^n = (1.05)^2 = 1.\mathbf{1025}$$

That has the interest rate in there (0.1025 = 10.25%), but we should subtract the extra 1:

$$(1+(r/n))^n - 1 = 0.1025 = \mathbf{10.25\%}$$

And so the formula is:

$$\text{Effective Annual Rate} = (1+(r/n))^n - 1$$

Example: what rate do you get when the ad says "6% compounded monthly"?

r = 0.06 (which is 6% as a decimal)
n = 12

$$\text{Effective Annual Rate} = (1+(r/n))^n - 1$$

$$= (1+(0.06/12))^{12} - 1$$

$$= (1.005)^{12} - 1 = 0.06168 = \mathbf{6.168\%}$$

So you actually get 6.168%

Example: 7% interest, compounded 4 times a year.

r = 0.07 (which is 7% as a decimal)
n = 4

So:

$$FV = PV\ (1+(0.07/4))^4$$

$$FV = PV\ (1+(0.07/4))^4$$

$$FV = PV\ (1.0719...)$$

The effective annual rate is **7.19%**

Table of Values

Here are some example values. Notice that compounding has a very small effect when the interest rate is small, but a large effect for high interest rates.

Compounding	Periods		1.00%	5.00%	10.00%	20.00%	100.00%
Yearly	1		1.00%	5.00%	10.00%	20.00%	100.00%
Semiannually	2		1.00%	5.06%	10.25%	21.00%	125.00%
Quarterly	4		1.00%	5.09%	10.38%	21.55%	144.14%
Monthly	12		1.00%	5.12%	10.47%	21.94%	161.30%
Daily	365		1.01%	5.13%	10.52%	22.13%	171.46%
...	...						
Continuously	Infinite		1.01%	5.13%	10.52%	22.14%	171.83%

Continuously?

Yes, if you have smaller and smaller periods (hourly, minutely, etc) you eventually reach a limit, and we even have a formula for it:

$$e^r - 1$$

Continuous Compounding Formula
Note: ***e****=2.71828..., which is* *Euler's number*.

Example: Continuous Compounding for 20%

$$\mathbf{e}^{0.20} - 1 = 1.2214... - 1 = \mathbf{0.2214...}$$

Or about **22.14%**

Using It

Now that you can calculate the Effective Annual Rate (for specific periods, or continuous), we can use it in any normal compound interest calculations.

Example: Continuous Compounding of $10,000 for 2 years at 8%

Continuous Compounding for 8% is: $e^{0.08} - 1 = 1.08329... - 1 = \mathbf{0.08329...}$

That is about **8.329%**

Over 2 years (see *Compound Interest*) we get:

$$FV = PV \times (1+r)^n$$

$$FV = \$10,000 \times (1+0.08329)^2$$

$$FV = \$10,000 \times 1.173511... = \$11,735.11$$

Summary

$$\text{Effective Annual Rate} = (1+(r/n))^n - 1$$

Where:

- r = **Nominal Rate** (the rate they mention)
- n = number of periods that are compounded (example: monthly=12)

What rate would you get if the ad says "8% compounded four times a year"?

A 8.14%	B 8.21%
C 8.24%	D 32%

$r = 0.08$ (which is 8% as a decimal)

$n = 4$

Effective Annual Rate $= (1+(r/n))^n - 1 = (1+(0.08/4))^4 - 1 = (1.02)^4 - 1 = 0.0824 = 8.24\%$

So you would actually get 8.24%.

What rate would you get if the ad says "12% compounded monthly"?

A 12.5%

B 12.68%

C 12.86%

D 14.4%

r = 0.12 (which is 12% as a decimal)

n = 12

Effective Annual Rate = $(1 + r/n)^n - 1 = (1 + 0.12/12)^{12} - 1 = (1.01)^{12} - 1 = 0.1268 = 12.68\%$

So you would actually get 12.68%.

What rate would you get if the ad says "9% compounded every two months"?

A 9.20%

B 9.34%

C 9.43%

D 18%

r = 0.09 (which is 9% as a decimal)

n = 6

Effective Annual Rate = $(1 + r/n)^n - 1 = (1 + 0.09/6)^6 - 1 = (1.015)^6 - 1 = 0.0934 = 9.34\%$

So you would actually get 9.34%

What rate would you get if the ad says "13% compounded weekly"?

A 9.1%

B 13.68%

C 13.7%

D 13.86%

r = 0.13 (which is 13% as a decimal)

n = 52

Effective Annual Rate = [illegible]

So you would actually get 13.86%.

What rate would you get if the ad says "50% compounded daily"?

A 60%

B 64.28%

C 64.82%

D 64.87%

r = 0.5 (which is 50% as a decimal)

n = 365

Effective Annual Rate = $(1 + (r/n))^n - 1 = (1 + (0.5/365))^{365} - 1 = (1.00137)^{365} - 1 = 0.6482 = 64.82\%$

So you would actually get 64.82%

What rate would you get if the ad says "6.25% compounded continuously"?

A 6.25%

B 6.35%

C 6.43%

D 6.45%

Continuous compounding for 6.25% is $e^{0.0625} - 1 = 1.0644... - 1 = 0.0644... = 6.45\%$

So you would actually get 6.45%

What rate would you get if the ad says "15% compounded continuously"?

A 15%

B 16.08%

C 16.18%

D 16.58%

[illegible]

Dillon deposited $2,000 at an interest rate of 9% compounded monthly.
How much money did he have in the bank at the end of one year?

A $2,180

B $2187.61

C $4,160

D $4,763.56

[illegible]

Diane deposited $5,000 at an interest rate of 3.25% compounded semi-annually.
How much interest was added in the first year?

A $163.82	B $164.49
C $164.72	D $325

[illegible]

$4,000 is deposited at a continuous rate of 2.5%
What is its value after 3 years?

A $4,300

B $4,303.78

C $4,307.56

D $4,311.54

Annuities

An annuity is a **fixed income over a period of time**.

> Example: You get $200 a week for 10 years.

How do you get such an income? **You buy it!**

So:

- **you pay them** one large amount, then
- **they pay you** back a series of small payments over time

> Example: You buy an annuity
>
> It costs you **$20,000**
>
> And in return you get $400 a month for 5 years

Is that a good deal?

> Example (continued):
>
> $400 a month for 5 years = $400 × 12 × 5 = **$24,000**
>
> Seems like a good deal ... you get back more than you put in.

Why do you get **more** income *($24,000)* than the annuity originally cost *($20,000)*?

Because **money now** is more valuable than money later.

The people who got your $20,000 can invest it and earn interest, or do other clever things to make more money.

So how much *should* an annuity cost?

Value of an Annuity

First: let's see the effect of an **interest rate of 10%** (imagine a bank account that earns 10% interest):

Example: 10% interest on $1,000

$1,000 now could earn $1,000 x 10% = **$100** in a year.

$1,000 now becomes **$1,100 in a year's time**.

So $1,100 next year is the **same** as $1,000 now (at 10% interest).

The **Present Value** of $1,100 next year is **$1,000**

So, at 10% interest:

- to go from **now** to **next year**: multiply by 1.10
- to go from **next year** to **now**: divide by 1.10

Now let's imagine an annuity of **4 yearly payments of $500**.

Your first payment of $500 is next year ... how much is that worth **now**?

→ $500 ÷ 1.10 = $454.55 now (to nearest cent)

Your second payment is 2 years from now. How do we calculate that? Bring it back one year, then bring it back another year:

→ $500 ÷ 1.10 ÷ 1.10 = $413.22 now

The third and 4th payment can also be brought back to today's values:

→ $500 ÷ 1.10 ÷ 1.10 ÷ 1.10 = $375.66 now

→ $500 ÷ 1.10 ÷ 1.10 ÷ 1.10 ÷ 1.10 = $341.51 now

Finally we add up the 4 payments (in today's value):

→ Annuity Value = $454.55 + $413.22 + $375.66 + $341.51

→ Annuity Value = $1,584.94

We have done our first annuity calculation!

4 annual payments of $500 at 10% interest is worth **$1,584.94 now**

How about another example:

Example: An annuity of $400 a month for 5 years.

Use a Monthly interest rate of 1%.

12 months a year, 5 years, that is **60 payments** ... and a LOT of calculations.

We need an easier method. Luckily there is a neat formula:

$$\text{Present Value of Annuity: } PV = P \times \frac{1 - (1+r)^{-n}}{r}$$

- **P** is the value of each payment
- **r** is the interest rate per period, as a decimal, so 10% is 0.10
- **n** is the number of periods

First, let's try it on our $500 for 4 years example.

The interest rate per year is 10%, so **r = 0.10**

There are 4 payments, so **n=4**, and each payment is $500, so **P = $500**

$$PV = \$500 \times \frac{1 - (1.10)^{-4}}{0.10}$$

$$PV = \$500 \times \frac{1 - 0.68301...}{0.10}$$

$$PV = \$500 \times 3.169865...$$

$$PV = \$1584.93$$

It matches our answer above (and is 1 cent more accurate)

Now let's try it on our $400 for 60 months example:

The interest rate is 1% per month, so **r = 0.01**

There are 60 monthly payments, so **n=60**, and each payment is $400, so **P = $400**

$$PV = \$400 \times \frac{1 - (1.01)^{-60}}{0.01}$$

$$PV = \$400 \times \frac{1 - 0.55045...}{0.01}$$

$$PV = \$400 \times 44.95504...$$

$$PV = \$17,982.02$$

Certainly easier than 60 separate calculations.

Going the Other Way

What if you know the annuity value and want to work out the payments?

> Say you have $10,000 and want to get a monthly income for 6 years, how much do you get each month (assume a monthly interest rate of 0.5%)

We need to change the subject of the formula above

Start with: $PV = P \times \frac{1 - (1+r)^{-n}}{r}$

Swap sides: $P \times \frac{1 - (1+r)^{-n}}{r} = PV$

Multiply both sides by **r**: $P \times (1 - (1+r)^{-n}) = PV \times r$

Divide both sides by $\mathbf{1 - (1+r)^{-n}}$: $P = PV \times \frac{r}{1 - (1+r)^{-n}}$

And we get this:

$$P = PV \times \frac{r}{1 - (1+r)^{-n}}$$

- **P** is the value of each payment
- **PV** is the Present Value of Annuity
- **r** is the interest rate per period as a decimal, so 10% is 0.10
- **n** is the number of periods

Say you have $10,000 and want to get a monthly income for 6 years out of it, how much could you get each month (assume a monthly interest rate of 0.5%)

The monthly interest rate is 0.5%, so **r = 0.005**

There are 6x12=72 monthly payments, so **n=72**, and **PV = $10,000**

$$P = PV \times \frac{r}{1 - (1+r)^{-n}}$$

$$P = \$10,000 \times \frac{0.005}{1 - (1.005)^{-72}}$$

$$P = \$10,000 \times 0.016572888...$$

$$P = \$165.73$$

What do you prefer? **$10,000 now** or 6 years of **$165.73 a month**

Footnote:

You don't need to remember this, but you may be curious how the formula comes about:

With **n** payments of **P**, and an interest rate of **r** we add up like this:

$$P \times \frac{1}{1+r} + P \times \frac{1}{(1+r)\times(1+r)} + P \times \frac{1}{(1+r)\times(1+r)\times(1+r)} + \ldots \text{(n terms)}$$

We can use exponents to help. $\frac{1}{1+r}$ is actually $(1+r)^{-1}$ and $\frac{1}{(1+r)\times(1+r)}$ is $(1+r)^{-2}$ etc:

$$P \times (1+r)^{-1} + P \times (1+r)^{-2} + P \times (1+r)^{-3} + \ldots \text{(n terms)}$$

And we can bring the "P" to the front of all terms:

$$P \times [\ (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \ldots \text{(n terms)}\ ]$$

To simplify that further is a little harder! We need some clever work using Geometric Sequences and Sums but trust me, it can be done ... and we get this:

$$PV = P \times \frac{1 - (1+r)^{-n}}{r}$$

You wish to receive an annuity of $800 each year for 5 years.
The annual interest rate is 10%
What is the present value of the annuity (to the nearest cent)?

A $2535.89

B 2653.55

C $3032.63

D $3040.00

You wish to receive an annuity of $500 a month for 10 years.
The monthly interest rate is 1%
What is the present value of the annuity (to the nearest cent)?

A $4,735.65

B $22,477.52

C $27,880.21

D $34,850.26

[illegible]

You wish to receive an annuity of $300 a month for 6 years.
The monthly interest rate is 0.9%
What is the present value of the annuity (to the nearest cent)?

A $14,261.72

B $15,846.35

C $17,430.99

D $17,446.34

[illegible]

You wish to receive an annuity of $1,000 a quarter for 5 years.
The quarterly interest rate is 2.5%
What is the present value of the annuity (to the nearest cent)?

A $15,589.16	B $15,815.53
C $18,583.31	D $18,706.99

You wish to receive an annuity of $6,500 a year for 15 years.
The annual interest rate is 5.5%
What is the present value of the annuity (to the nearest cent)?

A $52,937.54

B $60,225.49

C $65,244.28

D $118,174.11

You have $10,000 and wish to receive an annual annuity for 10 years.
The annual interest rate is 8%
How much will you receive each year (to the nearest cent)?

A $800.08	B $1,481.15
C $1,490.29	D $1,727.14

[illegible]

You have $6,000 and wish to receive a monthly annuity for 5 years.
The monthly interest rate is 0.5%
How much will you receive each year (to the nearest cent)?

A $40.47

B $101.50

C $115.38

D $116.00

[illegible]

You have $15,000 and wish to receive a monthly annuity for 20 years.
The monthly interest rate is 0.25%
How much will you receive each year (to the nearest cent)?

A $84.00	B $83.19
C $68.28	D $64.15

You have $7,500 and wish to receive a quarterly annuity for 12 years.
The quarterly interest rate is 1.5%
How much will you receive each year (to the nearest cent)?

A $156.25	B $171.90
C $220.31	D $229.89

You have $20,000 and wish to receive a monthly annuity for 10 years.
The monthly interest rate is 0.1%
How much extra money on top of your investment will you receive over the 10-year period?

A $110	B $1,234
C $1,324	D $14,433

FINISH LINE