Tishk International University
Mechatronics Engineering Department
Finite Element Method ME 323
Lecture 5,6

Stress Analysis of Truss Element

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Stiffness Matrix for three dimensional Bar Element

- $K=\frac{AE}{L}\begin{bmatrix} C_x{}^2 & C_xC_y & C_xC_z & -C_x{}^2 & -C_xC_y & -C_xC_z \\ C_xC_y & C_y{}^2 & C_zC_y & -C_xC_y & -C_y{}^2 & -C_zC_y \\ C_xC_z & C_zC_y & C_z{}^2 & -C_xC_z & -C_zC_y & -C_z{}^2 \\ -C_x{}^2 & -C_xC_y & -C_xC_z & C_x{}^2 & C_xC_y & C_xC_z \\ -C_xC_y & -C_y{}^2 & -C_zC_y & C_xC_y & C_y{}^2 & C_zC_y \\ -C_xC_z & -C_zC_y & -C_z{}^2 & C_xC_z & C_zC_y & C_z{}^2 \end{bmatrix}$

Stiffness matrix for Three dimensional Bar element

- $\begin{bmatrix} C_x{}^2 & C_xC_y & C_xC_z \\ C_xC_y & C_y{}^2 & C_zC_y \\ C_xC_z & C_zC_y & C_z{}^2 \end{bmatrix}$

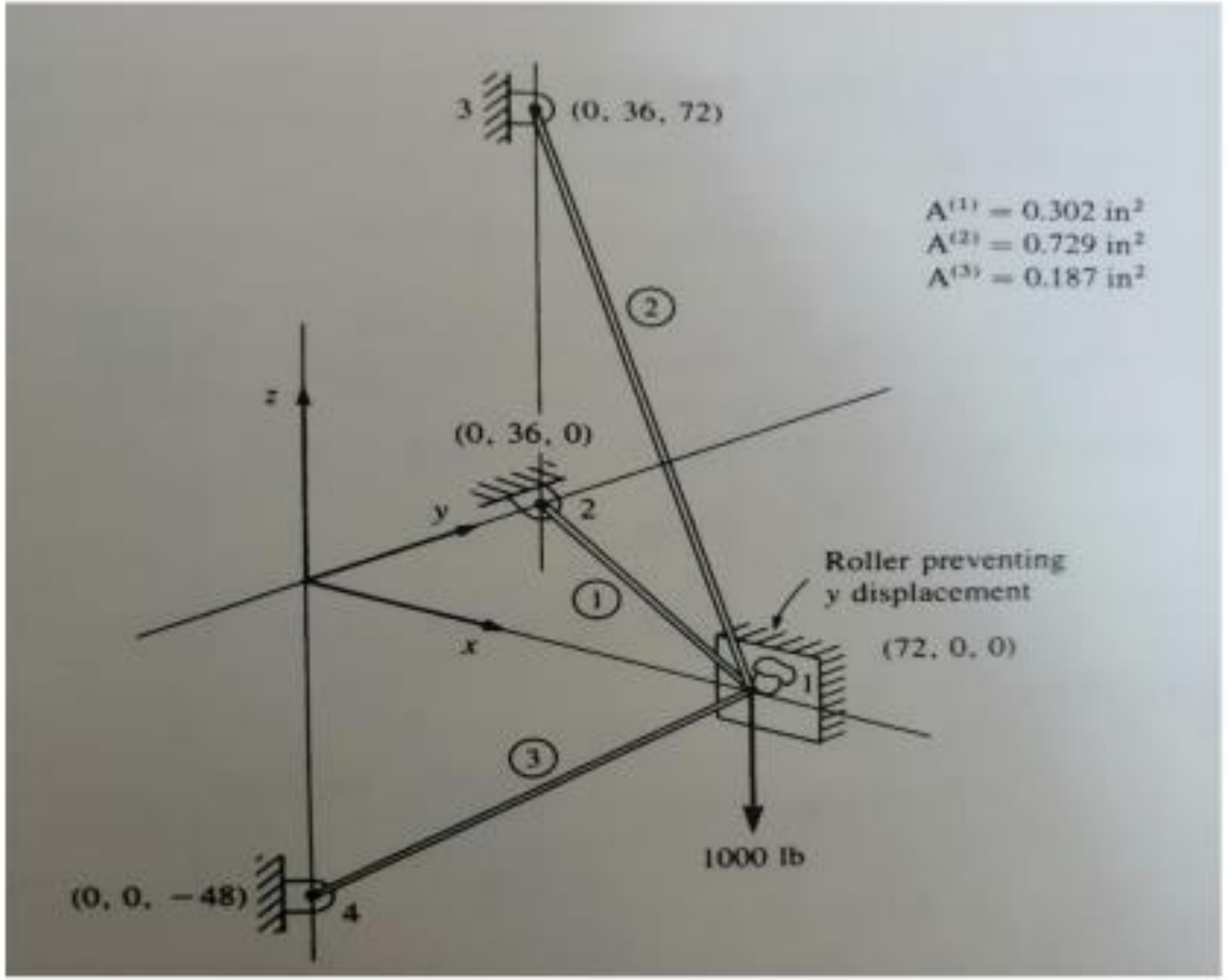
$$k = \frac{AE}{L}\left[\begin{array}{c|c} \lambda & -\lambda \\ \hline -\lambda & \lambda \end{array}\right]$$

- K=$\frac{AE}{L}\begin{bmatrix} C_x{}^2 & C_xC_y & C_xC_z & -C_x{}^2 & -C_xC_y & -C_xC_z \\ C_xC_y & C_y{}^2 & C_zC_y & -C_xC_y & -C_y{}^2 & -C_zC_y \\ C_xC_z & C_zC_y & C_z{}^2 & -C_xC_z & -C_zC_y & -C_z{}^2 \\ -C_x{}^2 & -C_xC_y & -C_xC_z & C_x{}^2 & C_xC_y & C_xC_z \\ -C_xC_y & -C_y{}^2 & -C_zC_y & C_xC_y & C_y{}^2 & C_zC_y \\ -C_xC_z & -C_zC_y & -C_z{}^2 & C_xC_z & C_zC_y & C_z{}^2 \end{bmatrix}$

Example 1

- Analyze the space truss shown in figure .The truss is composed of four nodes. Whose coordinates in inches are shown in the figure, and three elements whose cross sectional areas are given in the figure. The modulus of elasticity E=1.2 $*10^6$ psi for all elements. A load of 1000 Ib is applied at node 1in the negative z direction. Nodes 2-4 are supported by ball and socket joints and thus constrained from movements in x,y, and z directions. Node 1 is constrained from y direction by the roller shown in the figure

$$\underline{k} = \frac{AE}{L}\left[\begin{array}{c|c} \underline{\lambda} & -\underline{\lambda} \\ \hline -\underline{\lambda} & \underline{\lambda} \end{array}\right] \quad \text{...(1)}$$



3
(0, 36, 72)
$A^{(1)} = 0.302$ in^2
$A^{(2)} = 0.729$ in^2
$A^{(3)} = 0.187$ in^2
2
z
(0, 36, 0)
y
2
Roller preventing
y displacement
1
x
(72, 0, 0)
1
3
1000 lb
(0, 0, −48)
4

Figure1

Solution

- Therefore determing $\underset{\sim}{\lambda}$ will sufficiently describing k

$$\underset{\sim}{\lambda} = \begin{bmatrix} C_x^2 & C_xC_y & C_xC_z \\ C_yC_x & C_y^2 & C_yC_z \\ C_zC_x & C_zC_y & C_z^2 \end{bmatrix} \quad \text{...(2)}$$

Element 3

The direction cosines of element 3 are given, in general, by

$$C_x = \frac{x_4 - x_1}{L^{(3)}} \qquad C_y = \frac{y_4 - y_1}{L^{(3)}} \qquad C_z = \frac{z_4 - z_1}{L^{(3)}} \quad \text{...(3)}$$

Solution

- Where the notation of x_i, y_i, z_i is used to denote the coordinate of each node and L^e is used to denote element length from the coordinate information that given in the figure we obtain element length and the direction of cosine as By using the result of equation 4,5

$$L^{(3)} = [(-72.0)^2 + (-48.0)^2]^{1/2} = 86.5 \text{ in.}$$

....(4)

$$C_x = \frac{-72.0}{86.5} = -0.833 \qquad C_y = 0 \qquad C_z = \frac{-48.0}{86.5} = -0.550$$

....(5)

$$\underline{\lambda} = \begin{bmatrix} 0.69 & 0 & 0.46 \\ 0 & 0 & 0 \\ 0.46 & 0 & 0.30 \end{bmatrix}$$

....(6)

$$\underline{k}^{(3)} = \frac{(0.187)(1.2 \times 10^6)}{86.5} \begin{array}{c} \begin{array}{cc} d_{1x}d_{1y}d_{1z} & d_{4x}d_{4y}d_{4z} \end{array} \\ \left[\begin{array}{c:c} \underline{\lambda} & -\underline{\lambda} \\ \hdashline -\underline{\lambda} & \underline{\lambda} \end{array}\right] \end{array}$$

....(7)

$$k^3 = \frac{0.187 * 1.2 * 10^6}{86.5} \begin{bmatrix} 0.69 & 0 & 0.46 & -0.69 & 0 & -0.46 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.46 & 0 & 0.30 & -0.46 & 0 & -0.30 \\ -0.69 & 0 & -0.46 & 0.69 & 0 & 0.46 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.46 & 0 & -0.30 & 0.46 & 0 & 0.30 \end{bmatrix}$$

....(8)

Similarly for Element1

$$L^{(1)} = 80.5 \text{ in.}$$

$$C_x = -0.89 \qquad C_y = 0.45 \qquad C_z = 0$$

$$\underset{\sim}{\lambda} = \begin{bmatrix} 0.79 & -0.40 & 0 \\ -0.40 & 0.20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{...(9)}$$

$$\underset{\sim}{k}^{(1)} = \frac{(0.302)(1.2 \times 10^6)}{80.5} \begin{array}{c} \begin{array}{cc} d_{1x}\,d_{1y}\,d_{1z} & d_{2x}\,d_{2y}\,d_{2z} \end{array} \\ \left[\begin{array}{c|c} \underset{\sim}{\lambda} & -\underset{\sim}{\lambda} \\ \hline -\underset{\sim}{\lambda} & \underset{\sim}{\lambda} \end{array}\right] \end{array} \qquad \text{...(10)}$$

Stiffness matrix for Element 1

$$k^1 = \frac{0.302 * 1.2 * 10^6}{80.5} \begin{bmatrix} 0.79 & -0.40 & 0 & -0.79 & 0.40 & 0 \\ -0.40 & 0.20 & 0 & 0.40 & -0.20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.79 & 0.40 & 0 & 0.79 & -0.40 & 0 \\ 0.40 & -0.20 & 0 & -0.40 & 0.20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{....................(11)}$$

Element 2

$$L^{(2)} = 108 \text{ in.}$$

$$C_x = -0.667 \qquad C_y = 0.33 \qquad C_z = 0.667$$

$$\underset{\sim}{\lambda} = \begin{bmatrix} 0.45 & -0.22 & -0.45 \\ -0.22 & 0.11 & 0.22 \\ -0.45 & 0.22 & 0.45 \end{bmatrix}$$

....(12)

$$k^2 = \frac{0.729*1.2*10^6}{108}\begin{bmatrix} 0.45 & -0.22 & -0.45 & -0.45 & 0.22 & 0.45 \\ -0.22 & 0.11 & 0.22 & 0.22 & -0.11 & -0.22 \\ -0.45 & 0.22 & 0.45 & 0.45 & -0.22 & -0.45 \\ -0.45 & 0.22 & 0.45 & 0.45 & -0.22 & -0.45 \\ 0.22 & -0.11 & -0.22 & -0.22 & 0.11 & 0.22 \\ 0.45 & -0.22 & -0.45 & -0.45 & 0.22 & 0.45 \end{bmatrix}$$

...................(13)

Boundary Condition

- $d_{1y\ =0}$
- d_{2x}, d_{2y}, d_{2z} =0
- d_{3x}, d_{3y}, d_{3z}=0
- d_{4x}, d_{4y}, d_{4z} =0

Total stiffness matrix K

$$
=
\begin{array}{c}
\begin{array}{cccccccccccc}
dx1 & dy1 & dz1 & dx2 & dy2 & dz2 & dx3 & dy3 & dz3 & dx4 & dy4 & dz4
\end{array}
\\
\begin{bmatrix}
9000 & 3582 & -2450 & -3556 & 1800.4 & 0 & -3645 & 1782 & 3645 & -1790 & 0 & 1193.33 \\
-3582.4 & 1791.37 & -1782 & 1800.4 & -900.37 & 0 & 1782 & -891 & 1782 & 0 & 0 & 0 \\
-2450 & 1782 & 4450 & 0 & 0 & 0 & 3645 & -1782 & -3645 & -1193.33 & 0 & -778.2 \\
-3556 & 1800 & 0 & 3556 & -1800.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1800.4 & 900.37 & 0 & 1800 & 900.37 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3645 & 1782 & 3645 & 0 & 0 & 0 & 3645 & -1782 & -3645 & 0 & 0 & 0 \\
1782 & -891 & 1782 & 0 & 0 & 0 & -1782 & 891 & -1782 & 0 & 0 & 0 \\
3645 & -1782 & -3645 & 0 & 0 & 0 & -3645 & 1782 & 3645 & 0 & 0 & 0 \\
-1790 & 0 & 1193.33 & 0 & 0 & 0 & 0 & 0 & 0 & 1790 & 0 & 1119.33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1193.33 & 0 & -778.2 & 0 & 0 & 0 & 0 & 0 & 0 & 1193.33 & 0 & 778.2
\end{bmatrix}
\end{array}
$$

Solution

$$\underline{K} = \begin{array}{cc} d_{1x} & d_{1z} \\ \end{array}$$
$$\underline{K} = \begin{bmatrix} 9000 & -2450 \\ -2450 & 4450 \end{bmatrix}$$

$$\begin{Bmatrix} 0 \\ -1000 \end{Bmatrix} = \begin{bmatrix} 9000 & -2450 \\ -2450 & 4450 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1z} \end{Bmatrix}$$

$$d_{1x} = -0.072 \text{ in.}$$
$$d_{1z} = -0.264 \text{ in.}$$

Elemental stress

$$\underline{\sigma} = \frac{E}{L}[-C_x \quad -C_y \quad -C_z \quad C_x \quad C_y \quad C_z] \begin{Bmatrix} d_{ix} \\ d_{iy} \\ d_{iz} \\ d_{jx} \\ d_{jy} \\ d_{jz} \end{Bmatrix}$$

$$\underline{\sigma}^{(3)} = \frac{1.2 \times 10^6}{86.5}[0.83 \quad 0 \quad 0.55 \quad -0.83 \quad 0 \quad -0.55] \begin{Bmatrix} -0.072 \\ 0 \\ -0.264 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

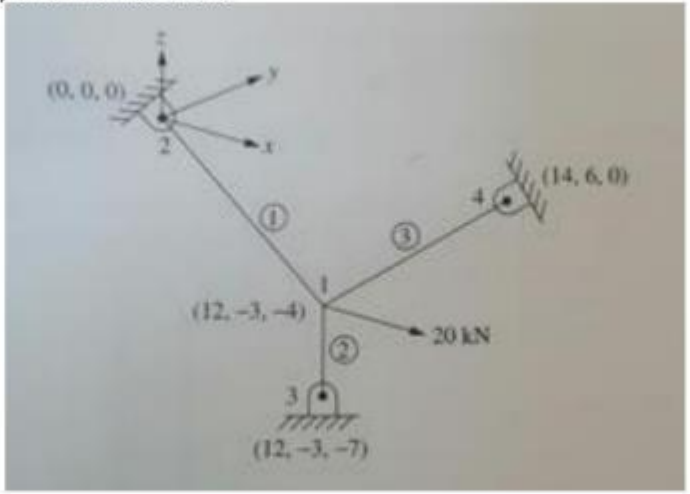
σ^1=-945 psi

σ^2=-1440 psi

σ^3= -2850 psi

Example 2

- Analyze the space truss shown in figure 2. The truss is composed of four nodes whose coordinates(in meters) are shown in figure, and the cross section area of three elements are $10*10^{-4}$. The modulus of elasticity E= 210 Gpa for all elements. A load of 20 KN is applied at node1 in the global x direction Nodes 2-4 are pin supported and thus constrained from movement in the x , y, and z directions.



Solution

$$L^{(1)} = [(0-12)^2 + (0-(-3))^2 + (0-(-4))^2]^{1/2} = 13\,\text{m}$$

$$L^{(2)} = [(12-12)^2 + (-3+3)^2 + (-7+4)^2]^{1/2} = 3\,\text{m}$$

$$L^{(3)} = [(14-12)^2 + (6+3)^2 + (0+4)^2]^{1/2} = 10.05\,\text{m}$$

Element Number	$C_x = \frac{x_j - x_i}{L^{(e)}}$	$C_y = \frac{y_j - y_i}{L^{(e)}}$	$C_z = \frac{z_j - z_i}{L^{(e)}}$
1	$-12/13$	$3/13$	$4/13$
2	0	0	-1
3	$2/10.05$	$9/10.05$	$4/10.05$

Element Number	C_x^2	C_xC_y	C_xC_z	C_y^2	C_yC_z	C_z^2
1	0.852	-0.213	-0.284	0.053	-0.071	0.095
2	0	0	0	0	0	1
3	0.040	0.178	0.079	0.802	0.356	0.158

Solution

$$\underline{\lambda}^{(1)} = \begin{bmatrix} 0.852 & -0.213 & -0.284 \\ -0.213 & 0.053 & 0.071 \\ -0.284 & 0.071 & 0.095 \end{bmatrix} \quad \underline{\lambda}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{\lambda}^{(3)} = \begin{bmatrix} 0.040 & 0.178 & 0.079 \\ 0.128 & 0.802 & 0.356 \\ 0.079 & 0.356 & 0.158 \end{bmatrix}$$

$$d_{2x} = d_{2y} = d_{2z} = 0, \quad d_{3x} = d_{3y} = d_{3z} = 0, \quad d_{4x} = d_{4y} = d_{4z} = 0$$

$$\underline{k}^{(1)} = \frac{AE}{13}\left[\begin{array}{c|c} \underline{\lambda}^{(1)} & -\underline{\lambda}^{(1)} \\ \hline -\underline{\lambda}^{(1)} & \underline{\lambda}^{(1)} \end{array}\right] \quad \underline{k}^{(2)} = \frac{AE}{3}\left[\begin{array}{c|c} \underline{\lambda}^{(2)} & -\underline{\lambda}^{(2)} \\ \hline -\underline{\lambda}^{(2)} & \underline{\lambda}^{(2)} \end{array}\right] \quad \underline{k}^{(3)} = \frac{AE}{10.05}\left[\begin{array}{c|c} \underline{\lambda}^{(3)} & -\underline{\lambda}^{(3)} \\ \hline -\underline{\lambda}^{(3)} & \underline{\lambda}^{(3)} \end{array}\right]$$

$$\begin{Bmatrix} 20\,\text{kN} \\ 0 \\ 0 \end{Bmatrix} = 210 \times 10^3 \begin{bmatrix} 69.519 & 1.327 & -13.985 \\ 1.327 & 83.879 & 40.885 \\ -13.985 & 40.885 & 356.363 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{1z} \end{Bmatrix}$$

$$d_{1x} = 1.383 \times 10^{-3}\,\text{m}$$

$$d_{1y} = -5.119 \times 10^{-5}\,\text{m}$$

$$d_{1z} = 6.015 \times 10^{-5}\,\text{m}$$

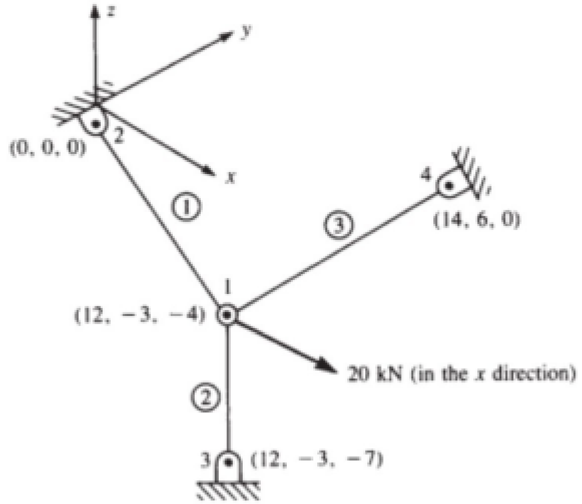
$$\sigma^{(1)} = \frac{210 \times 10^6}{13} [12/13 \quad -3/13 \quad -4/13 \quad -12/13 \quad 3/13 \quad 4/13] \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 20.51\,\text{MPa}$$

$$\sigma^{(3)} = -5.29\,\text{MPa}$$

Homework

- For the space trusses shown in figure determine the nodal displacements and the stress in each elements. Let E=210 Mpa and A= $10*10^{-4}m^2$ for all elements. Verify force equilibrium at node 1.The coordinates of each node in meters, are shown in figure.All supports are ball and socket joints.



Next Lecture

- Solving Problem on Three dimensional Bar Elements.