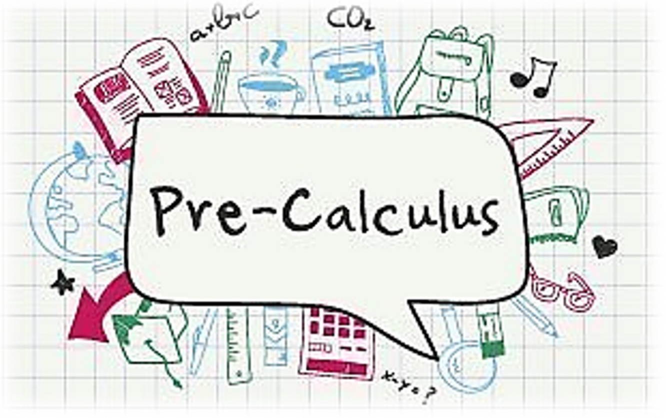
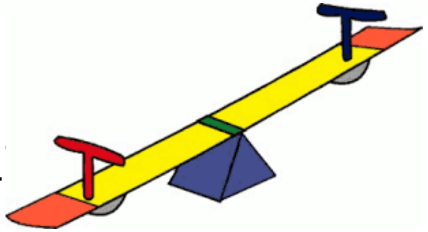


Lecture 3

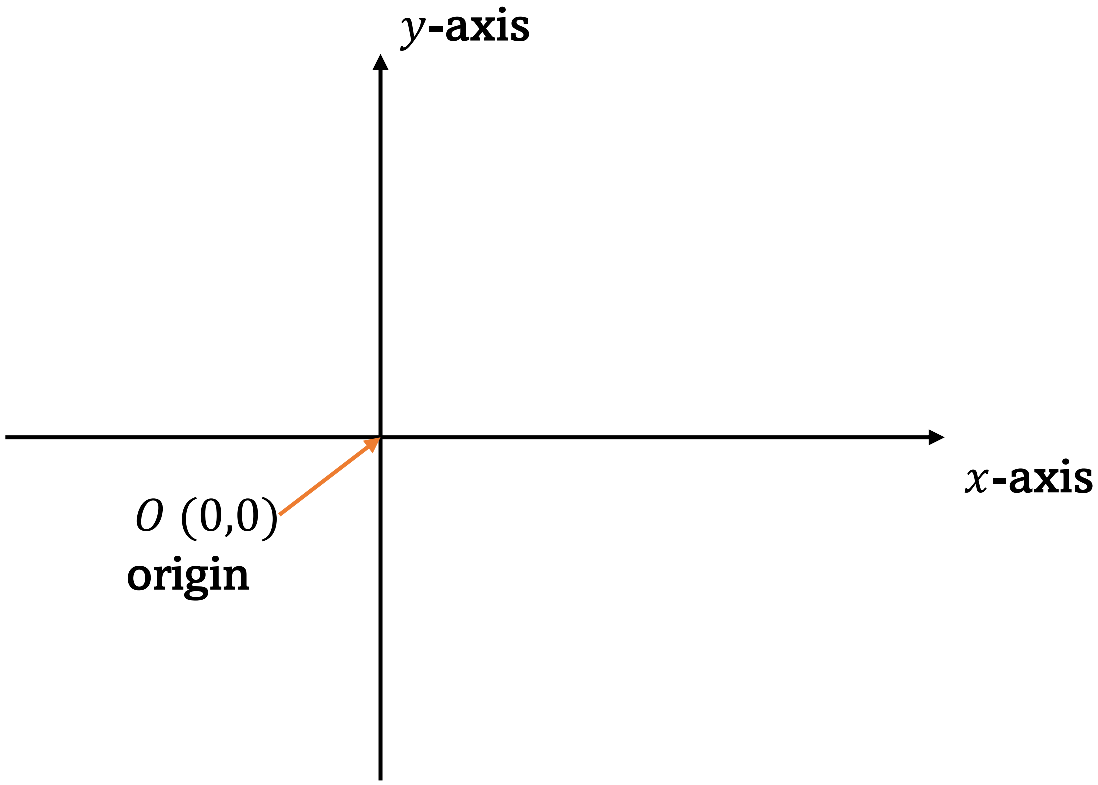
- Distance and Midpoint
- Absolute Value: equation & inequal
- Line Equation
- Systems of Equations



$$|x|$$

The Cartesian Plane

The **Cartesian plane** (coordinate plane, xy**-plane**) is a flat surface used to locate points using **ordered pairs** (x, y).

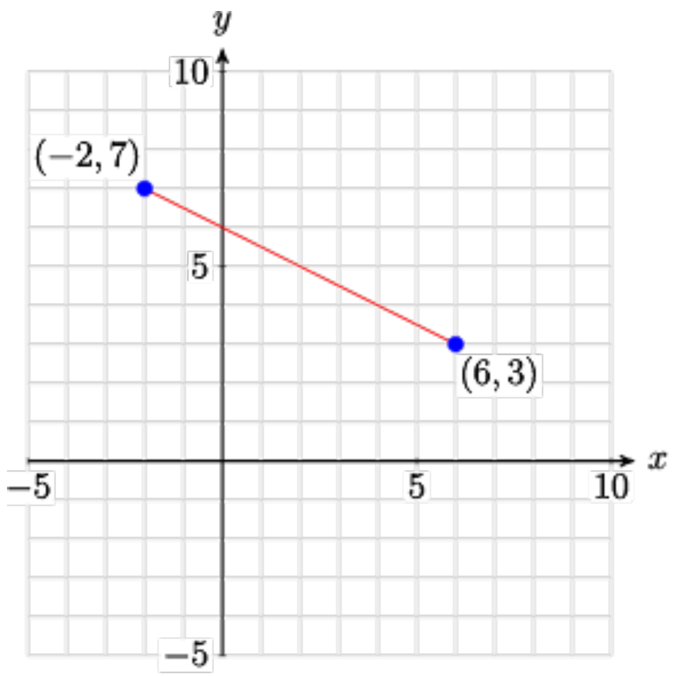


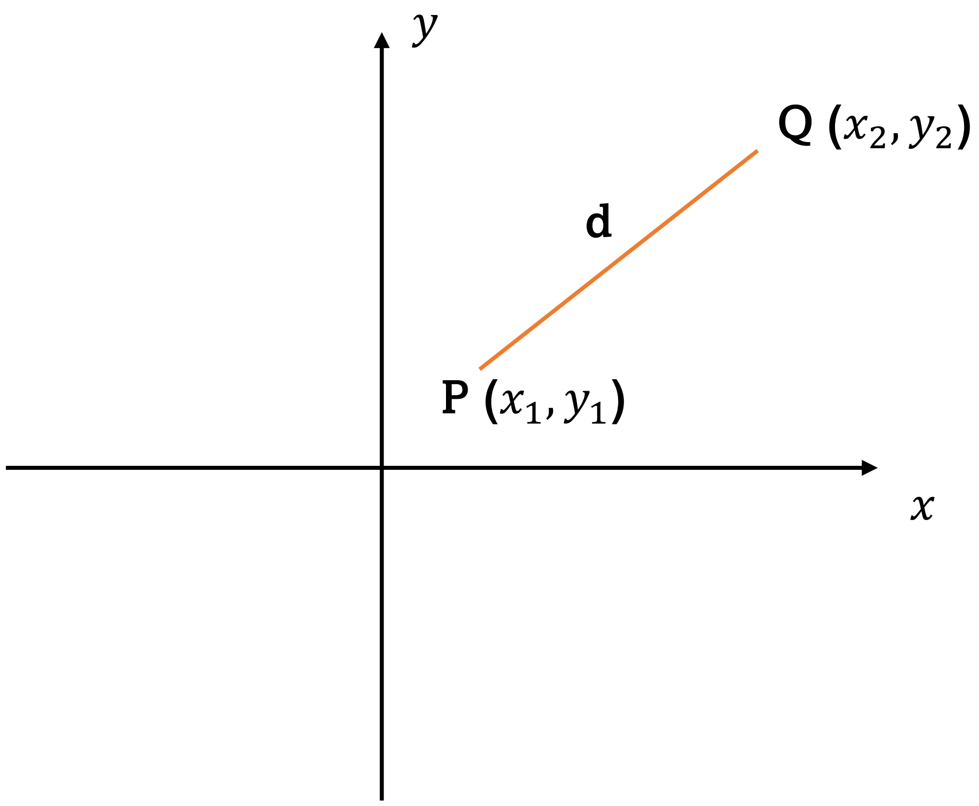
Plot points on a graph:

$P\ (2, 1)$
$Q\ (6, 4)$
$R\ (6, 1)$

Distance

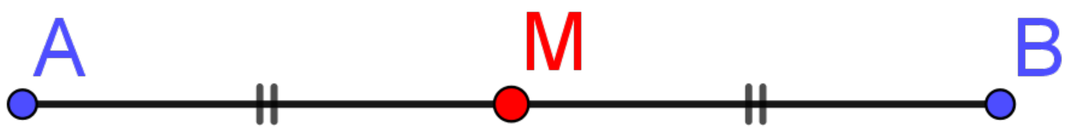
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$





Midpoint

m $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$



Find midpoint of a segment PQ, where $P\ (-2, 4)$ and $Q\ (4, -2)$.

P (1, 2)
Q (5, 6)
R (a, 2) – at this point occurs right-angle
Find a. Find the length of PQ, QR and PR.

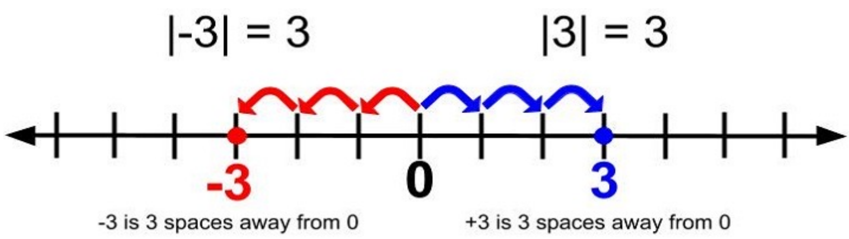
A (0, 2)
B (4, 5)
C (4, 2)
Find the length of AB, BC, and AC.

Find distance between two points (1, 3) and (3, 12).

Find distance between two points (2, 3) and (8, 27). Find midpoint of the line segment.

Find midpoint of the line segment between two points (1, 4) and (3, 10).

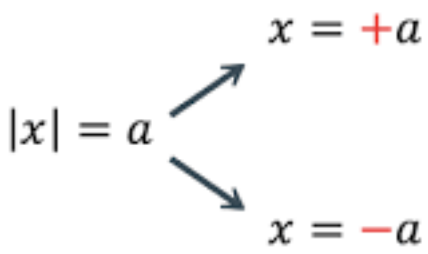
Distance and Absolute Value

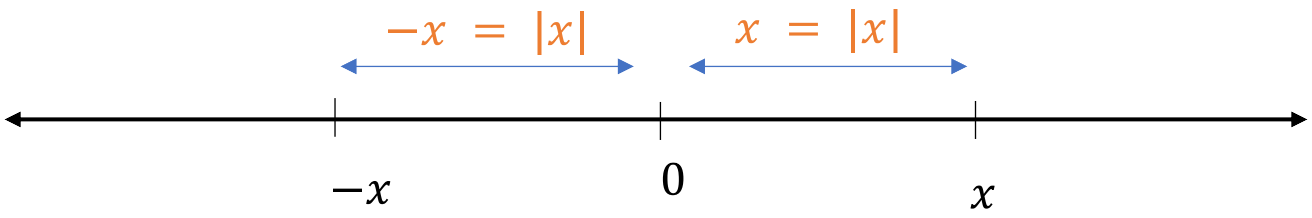


- Let $x \in \mathbb{R}$
- Define the absolute value or magnitude of x to be

$$|x| = \begin{cases} x, if\ x \geq 0 \\ x, if\ x < 0 \end{cases}$$

= distance between x and 0 on the real line.

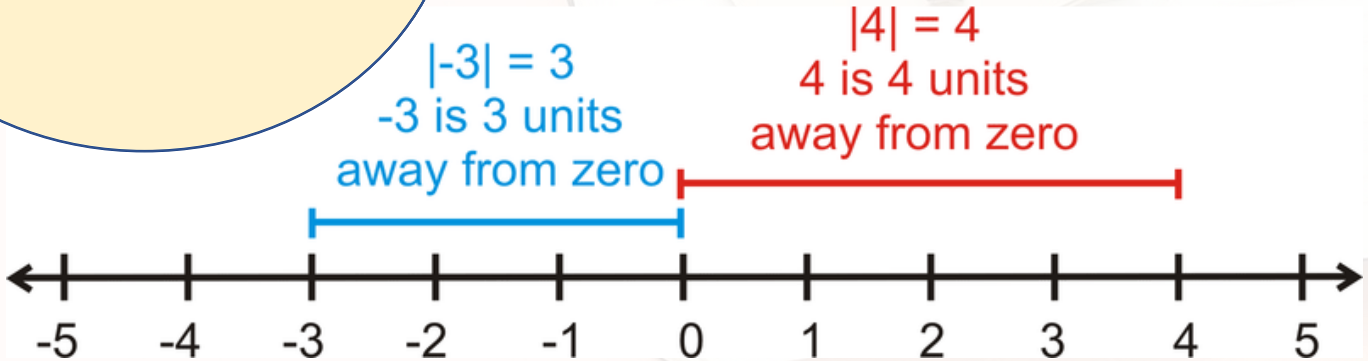


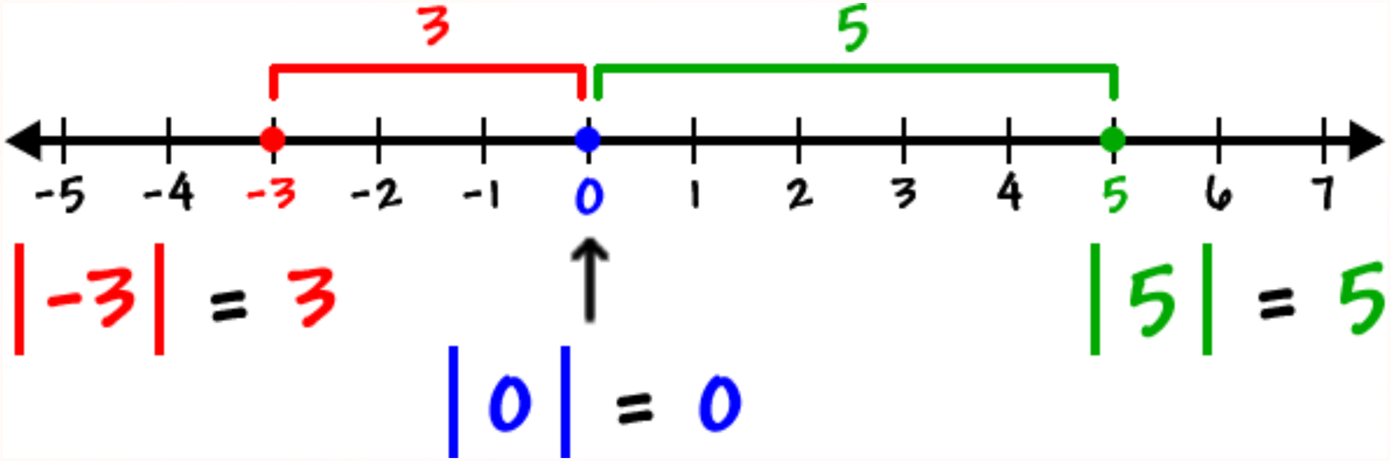


Absulute value is a distance.

Distance is always positive

Absolute value always positive





Absolute Value Equation

$|5 - 2x| - 11 = 0$

$|5 - 2x| = 11$ Isolate the absolute value

Split the equation up into two separate equations

$5 - 2x = 11$	$5 - 2x = -11$
$-2x = 6$	$-2x = -16$
$x = -3$	$x = 8$

Solve each of the equations

Therefore, $x = 1$ and $x = -\frac{7}{3}$ which are the two solutions to the given equation.

Example: Solve for x: $|3x + 2| = 5$.

Use **case analysis**:

Case 1:

$2x - 1 = 4x + 32x - 1 = 4x + 3 - 1 - 3 = 4x - 2x - 1 - 3 = 4x - 2x - 4 = 2x \Rightarrow x = -2 - 4 = 2x \Rightarrow x = -2.$

Case 2:

$2x - 1 = -(4x + 3)2x - 1 = -(4x + 3)2x - 1 = -4x - 32x - 1$

$= -4x - 32x + 4x = -3 + 12x + 4x = -3 + 16x = -2 \Rightarrow x = -136x$

$= -2 \Rightarrow x = -31.$

Thus, the solutions are $x = -2$ or $x = -13$.

Absolute Value on both sides

$$|2x + 1| = |3x - 5|$$

$$|\text{number}| = |\text{number}|$$

Check each case that is true:

$|\text{number}| = |\text{number}|$ $|\text{number}| = |-\text{number}|$

$|-\text{number}| = |-\text{number}|$ $|-\text{number}| = |\text{number}|$

Rules for Absolute Values

1. $|a| = \pm a, \quad |a| > 0, \text{ if } a \neq 0, \; |0| = 0.$

2. $|-a| = |a|.$

3. $|a| = \sqrt{a^2}.$

4. $|ab| = |a||b|.$

5. $|a + b| \leq |a| + |b|.$

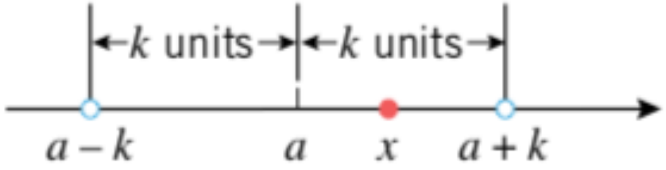
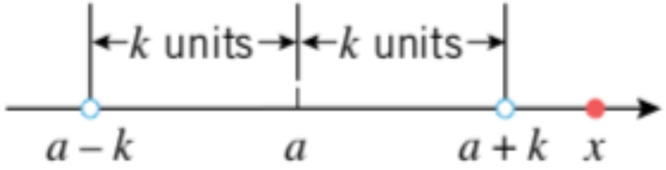
6. $|a - b| \leq |a| + |b|.$

7. $|a - b| \geq |a| - |b|.$

8. $||a| - |b|| \leq |a - b|.$

Inequalities with absolute value

Inequalities of the form $|x - a| < k$ and $|x - a| > k$ arise so often that we have summarized the key facts about them in Table below

INEQUALITY ($k > 0$)	GEOMETRIC INTERPRETATION	FIGURE	ALTERNATIVE FORMS OF THE INEQUALITY
$\|x - a\| < k$	x is within k units of a.		$-k < x - a < k$ $a - k < x < a + k$
$\|x - a\| > k$	x is more than k units away from a.		$x - a < -k$ or $x - a > k$ $x < a - k$ or $x > a + k$

Absolute Value Inequality

$$|x - 4| < 7$$

$$x - 4 < 7$$
$$x - 4 + 4 < 7 + 4$$
$$x < 11$$
$$|9 - 4| = |5| = 5$$
$$5 < 7$$

$$x - 4 > -7$$
$$x - 4 + 4 > -7 + 4$$
$$x > -3$$
$$|-2 - 4| = |-6| = 6$$
$$6 < 7$$

$$-3 < x < 11$$

Remember: If $a < b$ and $c < 0$, then $ac > bc$.

IMPORTANT

$|2x+1| \geq -5$

All real numbers. The absolute value will always be greater than zero.

$|8-x| \leq -3$

No solution. The absolute value will never be less than zero. Just like absolute value cannot be = to a negative number.

Example:

$$|x+3| < -2$$

$$|5+3| < -2$$

$$|8| < -2$$

$$8 < -2 \quad \text{False}$$

If an absolute value equation is equal to zero,

$$|x+5| = 0$$

there is one solution.

$$|x| = -3$$

NO SOLUTION

An absolute value can never equal a negative number

Two Solutions	One Solution	No Solutions
$\|x\| = 6$	$\|x\| = 0$	$\|x\| = -6$
$\|2x-5\| = 8$	$\|2x-5\| = 0$	$\|2x-5\| = -8$
$\left\|\frac{2}{3}x-7\right\| = 23$	$\left\|\frac{2}{3}x-7\right\| = 0$	$\left\|\frac{2}{3}x-7\right\| = -23$

Example: Solve the following inequality

(a) $|x-3|<4$ (b) $|x+4|\geq 2$ (c) $\frac{1}{|2x-3|}>5$

Solution: (a) Given $|x-3|<4$ and it can be written as

$$-4<x-3<4 \implies -1<x<7$$

It is clear the solution set is given by $(-1,7)$.

(b) The given inequality can be written as

$$x+4\geq 2 \quad \text{or} \quad x+4\leq -2 \implies x\geq -2 \quad \text{or} \quad x\leq -6$$

The solution is then $(-\infty,-6]\cup[-2,\infty)$.

(c) First of all, keep in your mind that $x = \frac{3}{2}$ is excluded from the set of solutions because this value of x results in a division by zero. Now,

$$\frac{1}{|2x - 3|} > 5 \implies |2x - 3| < \frac{1}{5} \implies 2x - 3 < 1/5 \text{ and } 2x - 3 > -1/5$$

After simplification we see that

$$x < \frac{8}{5} \quad \text{and} \quad x > \frac{7}{5}.$$

The solution set is $\left(\frac{7}{5}, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \frac{8}{5}\right)$.

Solve the equation.

$\frac{1}{4}|2x-6|+1=2$

$-3|x-1|-6=3$

$|x-7|+2=2$

$|3x+2|=|x-6|$

$|x-4|=|4-x|$

Solve the inequality.

$2|x-9|+6>6$

$-4|3x-1|\geq 8$

$-5|2x+2|-3\geq -3$

$-10+\frac{1}{2}|x-4|\geq -10$

$3\left|\frac{1}{2}x+2\right|+6<15$

$\frac{1}{4}\|2x-6\|+1=2$	$\{1,5\}$
$-3\|x-1\|-6=3$	No Solution
$\|x-7\|+2=2$	$\{7\}$
$\|3x+2\| = \|x-6\|$	$\{-4,1\}$
$\|x-4\| = \|4-x\|$	$\mathbb{R}$

$2\|x-9\|+6>6$	$(-\infty,9) \cup (9,\infty)$
$-4\|3x-1\| \geq 8$	No Solution
$-5\|2x+2\|-3 \geq -3$	$\{-1\}$
$-10+\frac{1}{2}\|x-4\| \geq -10$	$\mathbb{R}$
$3\left\|\frac{1}{2}x+2\right\|+6<15$	$(-10,2)$

- ❑ $|6x + 4| = 8x + 10$
- ❑ $|3x - 1| + 5 = 3$
- ❑ $|x - 2| = 2x - 3$
- ❑ $|12 - 6x| = 42$
- ❑ $|2b - 4| = 2b - 4$
- ❑ $|2a + 1| = a + 5$
- ❑ $-3|x + 5| + 1 = 7|x + 5| + 8$
- ❑ $5|c - 2| = 30$

- ❑ $|x - 5| > 7$
- ❑ $|2x + 3| \leq 12$
- ❑ $2|x - 6| \geq 24$
- ❑ $|4x - 1| - 11 < 20$
- ❑ $-3|x + 2| \leq -9$
- ❑ $\frac{|3x-3|}{-5} > -12$
- ❑ $8 + |4v - 7| \geq 17$

Lines

- A line is a one-dimensional figure (a straight path), which has length but no width and no endpoints.

- A line is an infinite number of points extends infinitely in both directions.

The general form of an equation is

$$ax + by = c \qquad a, b, c \ - \text{constants}$$

variables

How do we find the equation of a line?

- $ax + by = c$
- $by = -ax + c$
- $y = \frac{-ax+c}{b}, \qquad b \neq 0$
- $y = \left(-\frac{a}{b}\right)x + \frac{c}{b}$

$$y = mx + k$$

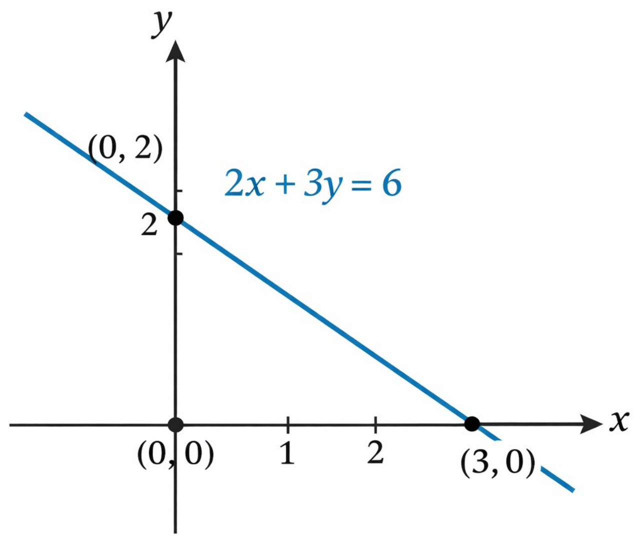
Lines

Sketch the line.

$$2x + 3y = 6$$

$$3y = 6 - 2x$$

$$\boldsymbol{y = -\frac{2}{3}x + 2}$$



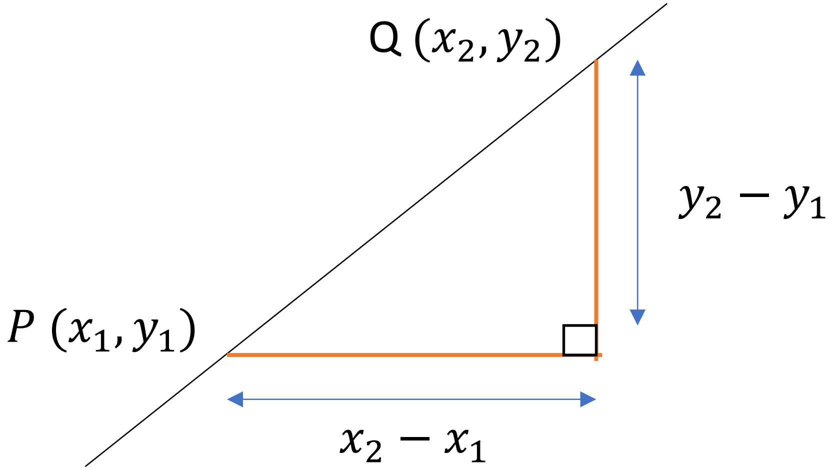
- A line is determined by two points.

x	y
0	2
3	0

Slope

The **slope** or **gradient** of a line is a number that describes both the direction and the steepness of the line.

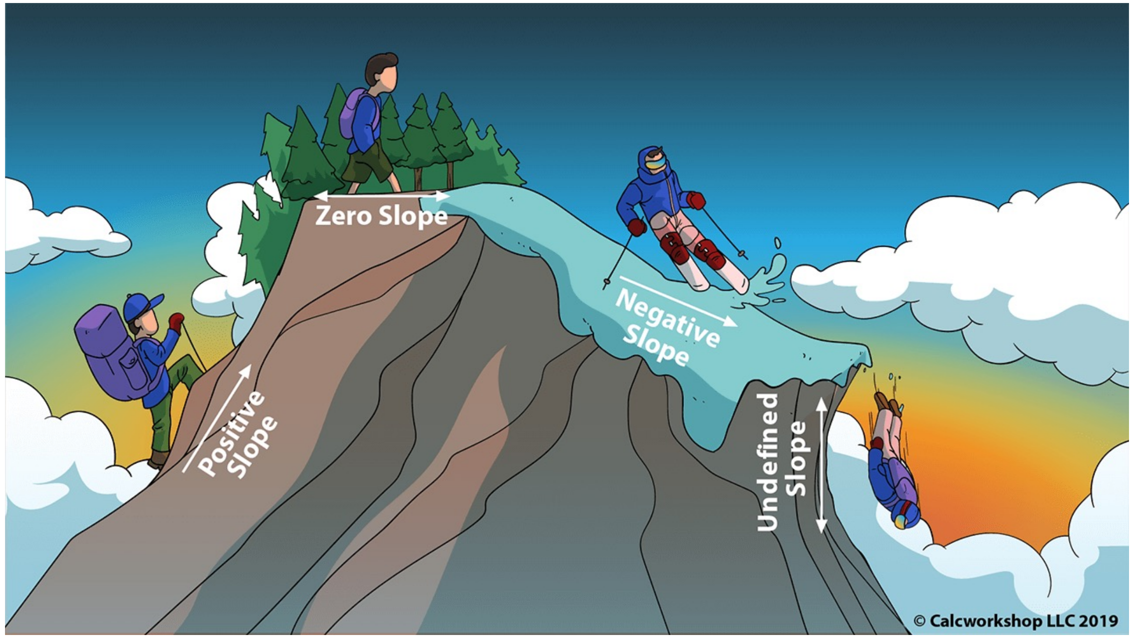
$$m = \text{slope} = \frac{\text{vertical rise}}{\text{horizontal run}}$$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In mathematics, the slope or gradient of a line is a **number that describes both the direction and the steepness of the line.**

Slope tells you how steep a line is, or how much y increases as x increases. The slope is constant (the same) anywhere on the line.



Some real life examples of slope include:

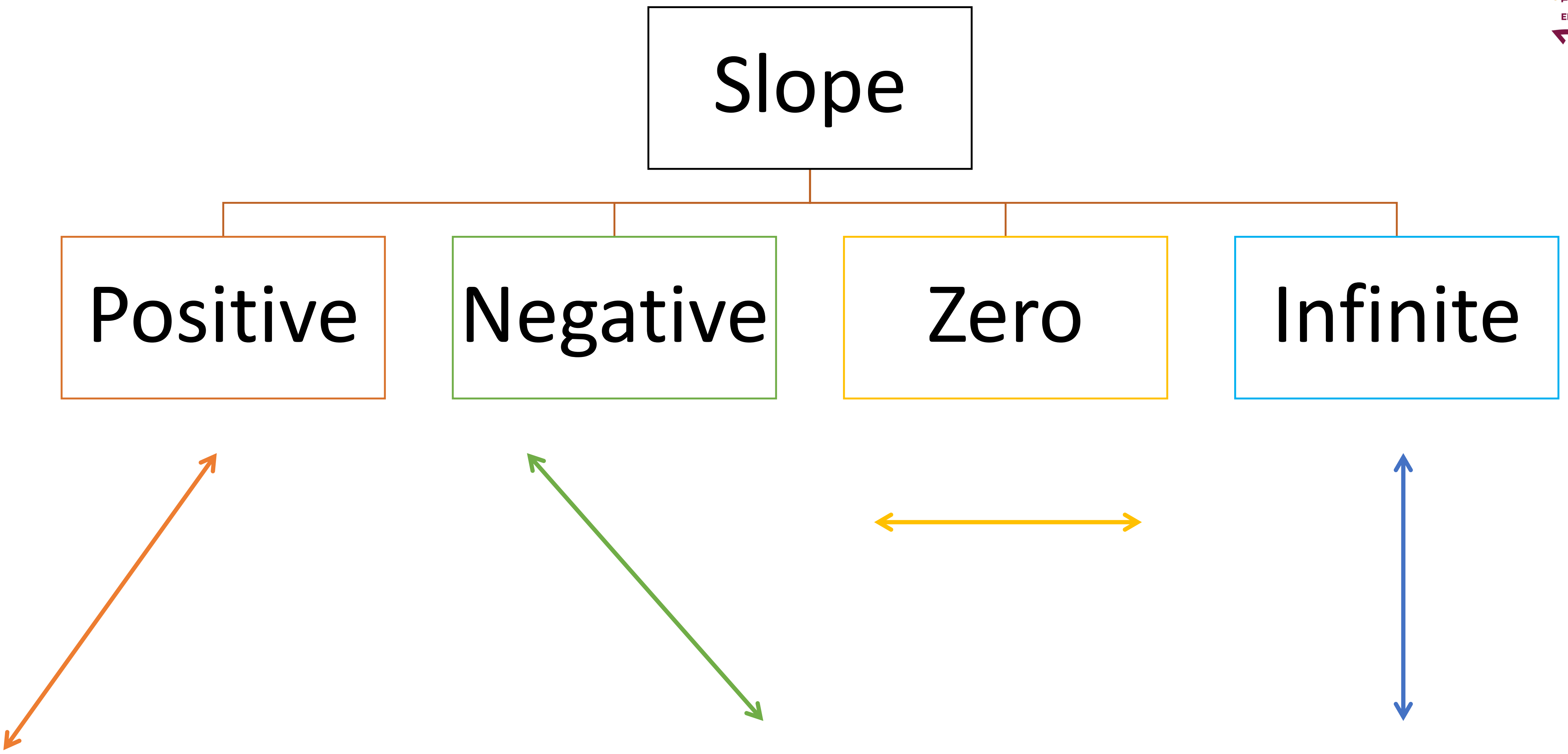
- ✓ In building roads one must figure out how steep the road will be.
- ✓ Skiers/snowboarders need to consider the slopes of hills in order to judge the dangers, speeds, etc.
- ✓ When constructing wheelchair ramps, slope is a major consideration.

- Find the slope of the line passing through P (2, 2) and Q (5, 6)

P(x_1, y_1) Q (x_2, y_2)

$$slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{5-2} = \frac{4}{3}$$

- Find the slope where P (-1, 2) and Q (5, -4)

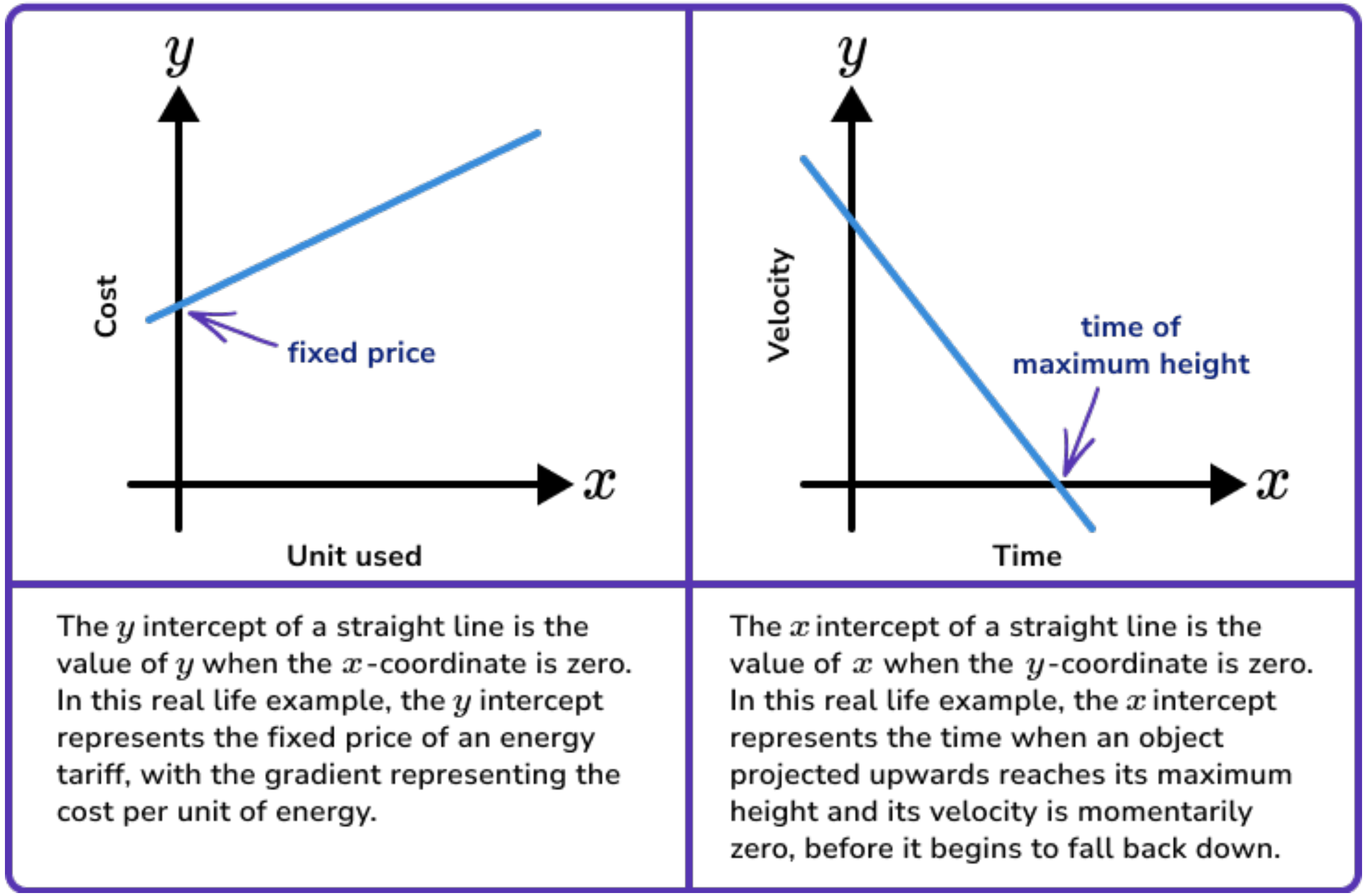

Slope
Positive
Negative
Zero
Infinite

What is the y intercept and the x intercept?

A $\boldsymbol{y}$**-intercept** is the point where a graph crosses the $\boldsymbol{y}$**-axis** (set $x = 0$).

An $\boldsymbol{x}$**-intercept** is the point where a graph crosses the $\boldsymbol{x}$**-axis** (set $y = 0$).

- **Intersect** means when **two or more lines/curves meet or cross each other**.
- An **intercept** is a **specific point where a graph crosses (intersects) an axis**.



Find the equation of line L passing through P(2, 3) and Q (7, 13)

1) Find a slope.

2) Put coordinates of any of given points and find constant ***b***.

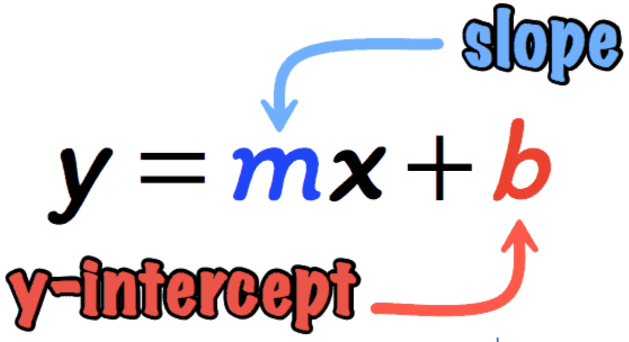
1. $m = \frac{13-3}{7-2} = \frac{10}{5} = 2$ (slope)

2. $y = mx + b$ – line equation

- $y = 2x + b$ $\quad m = 2$
- P(2,3): $3 = 2\times2 + b$ $\quad 3 = 4 + b$ $\quad b = 3 - 4$

  $b = -1$
- $m = 2, b = -1$

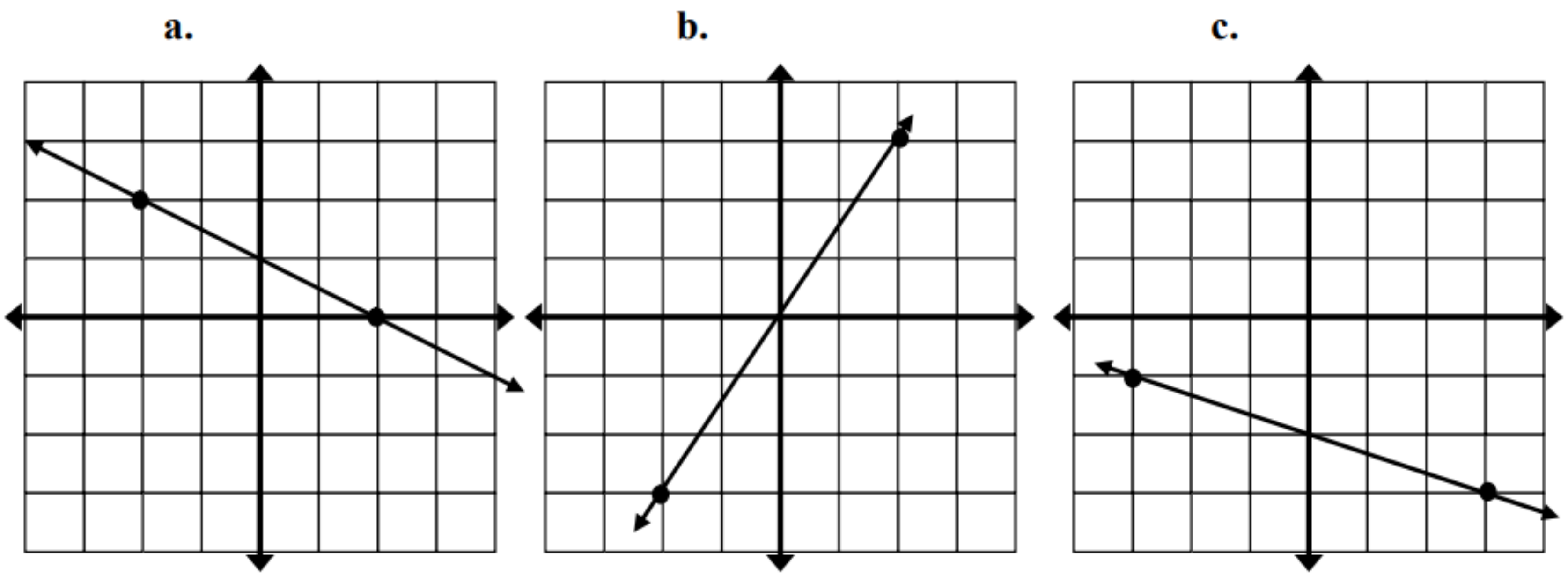
$y = 2x - 1$



- P (2, 5) and Q (4, 9) – find a line equation.
- A (2, 0) and B (8, 3) – find a line equation.
- W (-2, 5) and Y (4, -9) – find slope.

- What is the slope of a line that runs through points: (-2, 5) and (1, 7)? Write the line equation.
- A line passes through the points (–3, 5) and (2, 3). What is the slope of this line and line equation?

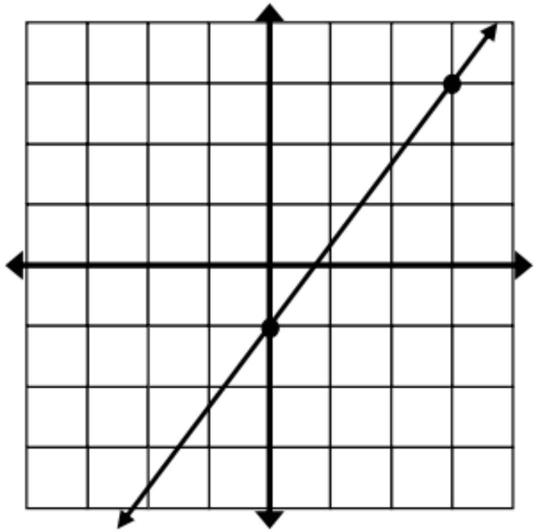
1) **<u>For each graph: Write the equation of the line in SLOPE-INTERCEPT FORM.</u>**



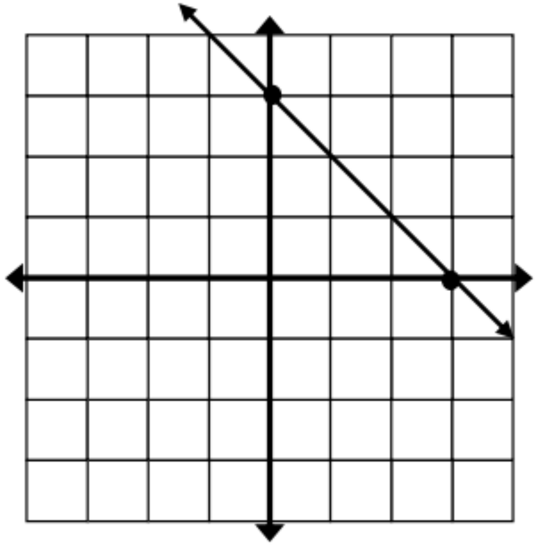
a.	b.	c.
Slope = ___________	**Slope =** ___________	**Slope =** ___________
y-intercept = __________	**y-intercept =** __________	**y-intercept =** __________
equation: _____________	**equation:** _____________	**equation:** _____________

Homework

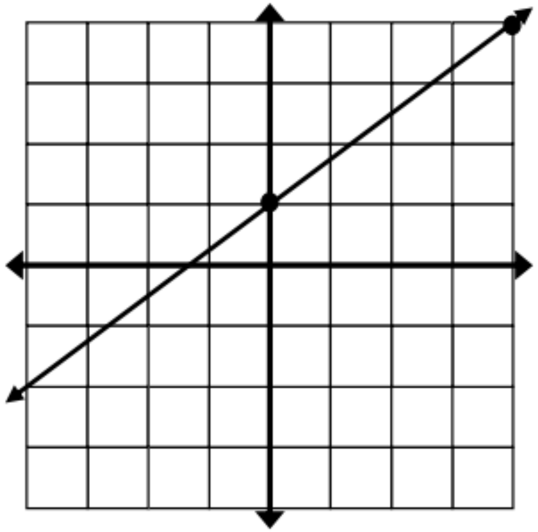
d.



e.



f.



Slope = ___________ Slope = __________ Slope = __________

y-intercept = __________ y-intercept = __________ y-intercept = __________

equation: _____________ equation: _____________ equation: _____________

Homework

2) Find the slope of the line through each pair of points. $Slope = \frac{y_2 - y_1}{x_2 - x_1}$

a. (8, -7) and (5, -3)

b. (-5, 9) and (5, 11)

c. (-8, -4) and (-4, -9)

d. (-4, 3) and (-6, -8)

e. (-7, -1) and (-7, 2)

f. (9, 4) and (-6, 4)

3) Tell whether each slope is positive, negative, zero, or undefined.

a.

____

b.

____

c.

____

d.

____

4) <u>For each linear equation, identify the slope (m) and the y-intercept (b)</u>

a. $y = 4x - 5$

b. $y = 11 + \frac{2}{3}x$

c. $y = \frac{2}{3} - x$

d. $6 - \frac{9}{2}x = y$

e. $y = \frac{5}{2}x - \frac{19}{8}$

f. $-\frac{5}{4} - \frac{2}{7}x = y$

Practice Problems

1. Find the equation of the line that passes through the point (1, 4) and has a slope of 12.
2. Find the equation of the line that passes through the point (1, 4) and has a slope of 2.
3. Find the equation of the line that passes through the point (27, 4) and has a slope of $\frac{-2}{9}$.
4. Find the equation of the line that passes through the point $(-11, 2)$ and has a slope of $\frac{-5}{11}$.
5. Find the equation of the line that passes through the point (10, 6) and has a slope of $\frac{1}{5}$. What is the y-intercept of the line?
6. Find the equation of the line that passes through the point (3, 29) and has a slope of 6. What is the y-intercept of the line?

Systems of Equations

$$\begin{cases} 2x + 3y = -12 \\ -x - 3y = 18 \end{cases}$$

$$\begin{cases} x - y = 11 \\ 2x + y = 19 \end{cases}$$

$$\begin{cases} 7x - y = -10 \\ -7x + 5y = -6 \end{cases}$$

$$\begin{cases} x + 3y = 18 \\ -x - 4y = -25 \end{cases}$$

$$\begin{cases} -6x + 5y = 1 \\ 6x + 4y = -10 \end{cases}$$

$$\begin{cases} -7x - y = 13 \\ 8x + y = -14 \end{cases}$$

Systems of Equations

① $\begin{cases} -6x - 8y = -28 \\ 9x + 5y = -14 \end{cases}$

② $\begin{cases} -9x + 3y = 27 \\ -3x + 4y = 27 \end{cases}$

③ $\begin{cases} -30x + 4y = 2 \\ 15x - 12y = -81 \end{cases}$

④ $\begin{cases} -5x + 5y = -25 \\ 3x + 2y = 10 \end{cases}$

⑤ $\begin{cases} -6x = 18 - y \\ 9 = -3x - 8y \end{cases}$

⑥ $\begin{cases} 10x + 12y = -26 \\ -6x + 6y = -24 \end{cases}$

⑦ $\begin{cases} 18x - 6y = 30 \\ -9x - y = -19 \end{cases}$

⑧ $\begin{cases} 3x - 5y = -17 \\ 2x + 15y = 7 \end{cases}$

[1] $\begin{cases} 5x + y = 9 \\ 10x - 7y = -18 \end{cases}$

[2] $\begin{cases} 5x - 3y = 2 \\ -5x + 3y = 8 \end{cases}$

[3] $\begin{cases} 2x = -3y + 16 \\ 5x - 4y = -6 \end{cases}$

[4] $\begin{cases} 6x + 6y = -6 \\ 5x + y = -13 \end{cases}$

[5] $\begin{cases} -4x + 9y = 9 \\ x - 3y = -6 \end{cases}$

[6] $\begin{cases} -2x - 9y = -25 \\ -4x - 9y = -23 \end{cases}$

[7] $\begin{cases} 4x + 8y = 12 \\ 2x + 4y = -6 \end{cases}$

[8] $\begin{cases} -7x + y = -19 \\ -2x + 3y = -19 \end{cases}$

Solve and show a solution set for each problem.

- $|-6x + 3| = 27$
- $2|3x - 1| - 1 = 7$
- $2|3x - 1| - 1 \leq 7$
- $|-2x + 7| + 5 \geq 14$

- $A(5, -6)$ *and* $B\ (-3, 1)$

find midpoint coordinates and distance between given two points.