Tishk International University
Mechatronics Engineering Department
Finite Element Method ME 323
Lecture 4: 2025

Stress Analysis of Truss Element

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Stiffness matrix of inclined bar element or trusess.

$$\underline{k} = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Example

For the plane truss composed of the three elements shown in Figure 3–15 subjected to a downward force of 10,000 lb applied at node 1, determine the x and y displacements at node 1 and the stresses in each element. Let $E = 30 \quad 10^6$ psi and $A = 2$ in.^{2} for all elements. The lengths of the elements are shown in the figure.

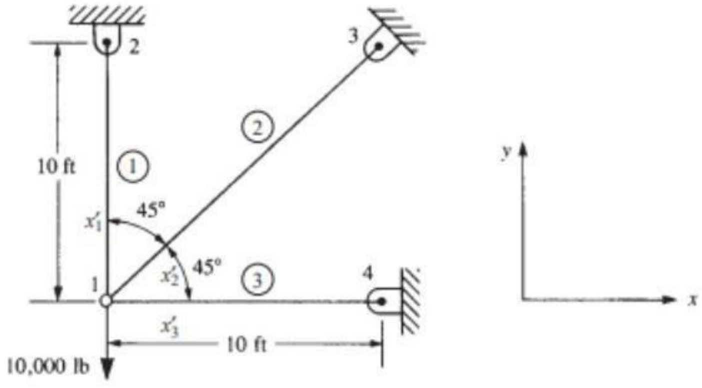

Figure 3–15 Plane truss

Table 3–1 Data for the truss of Figure 3–15

Element	θ	C	S	C^2	S^2	CS
1	90	0	1	0	1	0
2	45	$\sqrt{2}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
3	0	1	0	1	0	0

$$[k^{(1)}] = \frac{(30 \quad 10^6)(2)}{120} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\underline{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Similarly, for element 2, we have

$$[k^{(2)}] = \frac{(30 \quad 10^6)(2)}{120 \quad \sqrt{2}} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} \qquad (3.6.2)$$

and for element 3, we have

$$[k^{(3)}] = \frac{(30 \quad 10^6)(2)}{120} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (3.6.3)$$

The common factor of $30 \quad 10^6 \quad 2/120$ (= 500,000) can be taken from each of Eqs. (3.6.1) through (3.6.3), where each term in the square bracket of Eq. (3.6.2) is now multiplied by $1/\sqrt{2}$. After adding terms from the individual element stiffness matrices into their corresponding locations in $[K]$, we obtain the total stiffness matrix as

$$
[K] = (500{,}000)
\begin{array}{c}
\begin{array}{cccccccc}
u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4
\end{array}\\
\begin{bmatrix}
1.354 & 0.354 & 0 & 0 & \times 0.354 & \times 0.354 & \times 1 & 0\\
0.354 & 1.354 & 0 & \times 1 & \times 0.354 & \times 0.354 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \times 1 & 0 & 1 & 0 & 0 & 0 & 0\\
\times 0.354 & \times 0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0\\
\times 0.354 & \times 0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0\\
\times 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\qquad (3.6.4)
$$

The global $[K]$ matrix, Eq. (3.6.4), relates the global forces to the global displacements. We thus write the total structure stiffness equations, accounting for the applied force at node 1 and the boundary constraints at nodes 2–4 as follows:

$$
\begin{Bmatrix}
0\\
\times 10{,}000\\
F_{2x}\\
F_{2y}\\
F_{3x}\\
F_{3y}\\
F_{4x}\\
F_{4y}
\end{Bmatrix}
= (500{,}000)
\begin{bmatrix}
1.354 & 0.354 & 0 & 0 & \times 0.354 & \times 0.354 & \times 1 & 0\\
0.354 & 1.354 & 0 & \times 1 & \times 0.354 & \times 0.354 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \times 1 & 0 & 1 & 0 & 0 & 0 & 0\\
\times 0.354 & \times 0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0\\
\times 0.354 & \times 0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0\\
\times 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
u_1\\
v_1\\
u_2 = 0\\
v_2 = 0\\
u_3 = 0\\
v_3 = 0\\
u_4 = 0\\
v_4 = 0
\end{Bmatrix}
$$

Solution

$$\begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} = (500{,}000)\begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \qquad (3.6.6)$$

$$u_1 = 0.414 \quad 10^{-2}\ \text{in.} \qquad v_1 = -1.59 \quad 10^{-2}\ \text{in.}$$

$$\sigma^{(1)} = \frac{30 \quad 10^6}{120}[0 \quad -1 \quad 0 \quad 1]\begin{Bmatrix} u_1 = 0.414 \quad 10^{\times 2} \\ v_1 = -1.59 \quad 10^{\times 2} \\ u_2 = 0 \\ v_2 = 0 \end{Bmatrix} = 3965 \text{ psi}$$

$$\sigma^{(2)} = \frac{30 \quad 10^6}{120\sqrt{2}}\left[\frac{-\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2}\right]\begin{Bmatrix} u_1 = 0.414 \quad 10^{\times 2} \\ v_1 = -1.59 \quad 10^{\times 2} \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$= 1471 \text{ psi}$$

$$\sigma^{(3)} = \frac{30 \quad 10^6}{120}[-1 \quad 0 \quad 1 \quad 0]\begin{Bmatrix} u_1 = 0.414 \quad 10^{\times 2} \\ v_1 = -1.59 \quad 10^{\times 2} \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} = -1035 \text{ psi}$$

Solution

$$\sum F_x = 0 \qquad (1471\ \text{psi})(2\ \text{in}^2)\frac{\sqrt{2}}{2} - (1035\ \text{psi})(2\ \text{in}^2) = 0$$

$$\sum F_y = 0 \qquad (3965\ \text{psi})(2\ \text{in}^2) + (1471\ \text{psi})(2\ \text{in}^2)\frac{\sqrt{2}}{2} - 10{,}000 = 0$$

To illustrate how we can combine spring and bar elements in one structure, we now solve the two-bar truss supported by a spring shown in Figure 3–17. Both bars have $E = 210$ GPa and $A = 5.0 \quad 10^{\times 4}\ \text{m}^2$. Bar one has a length of 5 m and bar two a length of 10 m. The spring stiffness is $k = 2000$ kN/m.

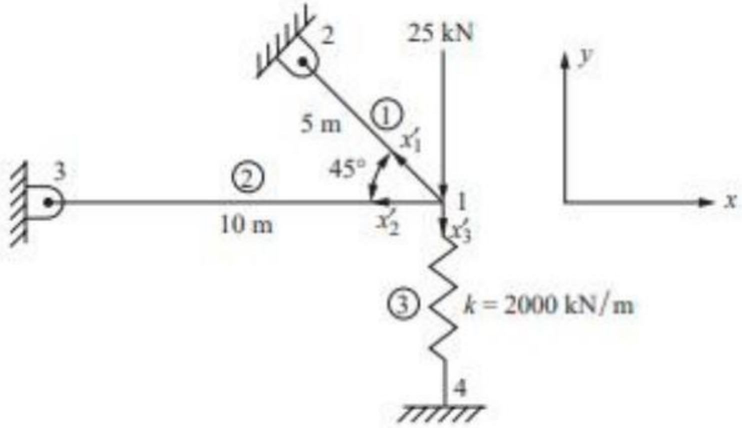

Figure 3–17 Two-bar truss with spring support

Element 1

$$\theta^{(1)} = 135\ , \qquad \cos\theta^{(1)} = \times\sqrt{2}/2, \qquad \sin\theta^{(1)} = \sqrt{2}/2$$

$$[k^{(1)}] = \frac{(5.0 \quad 10^{\times 4}\,\mathrm{m}^2)(210 \quad 10^6\,\mathrm{kN/m}^2)}{5\,\mathrm{m}} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Simplifying Eq. (3.6.20), we obtain

$$[k^{(1)}] = 105 \quad 10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Element 2

$$\theta^{(2)} = 180\,, \qquad \cos\theta^{(2)} = -1.0, \qquad \sin\theta^{(2)} = 0$$

$$[k^{(2)}] = \frac{(5 \quad 10^{\times 4}\,\text{m}^2)(210 \quad 10^6\,\text{kN/m}^2)}{10\,\text{m}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Simplifying Eq. (3.6.22), we obtain

$$[k^{(2)}] = 105 \quad 10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 3

$$\theta^{(3)} = 270\ , \qquad \cos\theta^{(3)} = 0, \qquad \sin\theta^{(3)} = -1.0$$

Using Eq. (3.4.23) but replacing AE/L with the spring constant k, we obtain the stiffness matrix of the spring as

$$[k^{(3)}] = 20 \quad 10^2 \begin{array}{c} \begin{matrix} u_1 & v_1 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Total stiffness matrix

- $$\begin{bmatrix} 210 & -105 & -105 & 105 & -105 & 0 & 0 & 0 \\ -105 & 125 & 105 & -105 & 0 & 0 & 0 & -20 \\ -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\ 105 & -105 & -105 & 105 & 0 & 0 & 0 & 0 \\ -105 & 0 & 0 & 0 & 105 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -105 & 0 & 0 & 0 & 0 & 0 & 105 \end{bmatrix}$$

Equation of stress

Applying the boundary conditions, we have

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

- $\sigma = \frac{E}{L}\begin{bmatrix} -C & -S & C & S \end{bmatrix}\begin{bmatrix} U1 \\ V1 \\ U2 \\ V2 \end{bmatrix}$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -25\,\text{kN} \end{Bmatrix} = 10^2 \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$u_1 = -1.724 \quad 10^{-3}\,\text{m} \qquad v_1 = -3.448 \quad 10^{-3}\,\text{m}$$

$$\sigma^{(1)} = \frac{210 \quad 10^3\,\text{MN/m}^2}{5\,\text{m}} [0.707 \quad -0.707 \quad -0.707 \quad 0.707] \begin{Bmatrix} -1.724 \quad 10^{-3} \\ -3.448 \quad 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying, we obtain

$$\sigma^{(1)} = 51.2\,\text{MPa}\,(T)$$

Similarly, we obtain the stress in element two as

$$\sigma^{(2)} = \frac{210 \quad 10^3\,\text{MN/m}^2}{10\,\text{m}} [1.0 \quad 0 \quad -1.0 \quad 0] \begin{Bmatrix} -1.724 \quad 10^{-3} \\ -3.448 \quad 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying, we obtain

$$\sigma^{(2)} = -36.2\,\text{MPa}\,(\text{C})$$

Ansys Adding new material Exercise

	Structural Steel 13 240	***Aluminium Alloy 6061***	***Plastic ABS***
Density ($kg.m^{-3}$)	7850	2770	1050
Young's Modulus (MPa)	$2,1.10^5$	$0,71.10^5$	$0,024.10^5$
Poisson's Ratio (-)	0,3	0,33	0,4078
Tensile Yield Strength (MPa)	250	164,8	36,13
Tensile Ultimate Strength (MPa)	460	246,1	38,73