Tishk International University
Mechatronics Engineering Department
Finite Element Method ME 323
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week3

Spring And Bar Analysis By stiffness method

Instructor: Sara Serwer Youns
Email: sara.sarwer@tiu.edu.iq

Previous lecture

- Spring analysis by direct stiffness method
- Shape function of the spring element
- Derivation of the spring stiffness matrix

Example 2.3

(a) Using the ideas presented in Section 2.3 for the system of linear elastic springs shown in Figure 2–14, express the boundary conditions, the compatibility or continuity condition similar to Eq. (2.3.3), and the nodal equilibrium conditions similar to Eqs. (2.3.4)–(2.3.6). Then formulate the global stiffness matrix and equations for solution of the unknown global displacement and forces. The spring constants for the elements are k_1, k_2, and k_3; P is an applied force at node 2.

(b) Using the direct stiffness method, formulate the same global stiffness matrix and equation as in part (a).

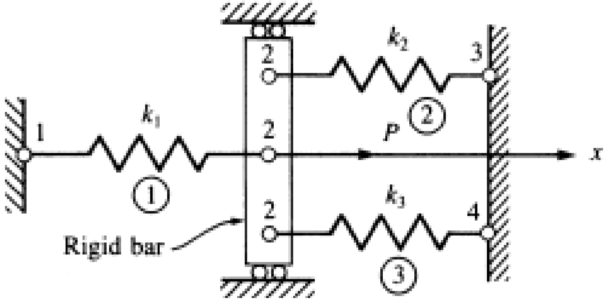


Figure 2–14 Spring assemblage for solution

(a) The boundary conditions are

$$d_{1x} = 0 \qquad d_{3x} = 0 \qquad d_{4x} = 0 \tag{2.5.44}$$

The compatibility condition at node 2 is

$$d_{2x}^{(1)} = d_{2x}^{(2)} = d_{2x}^{(3)} = d_{2x} \tag{2.5.45}$$

The nodal equilibrium conditions are

$$\begin{aligned} F_{1x} &= f_{1x}^{(1)} \\ P &= f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \\ F_{3x} &= f_{3x}^{(2)} \\ F_{4x} &= f_{4x}^{(3)} \end{aligned} \tag{2.5.46}$$

where the sign convention for positive element nodal forces given by Figure 2–2 was used in writing Eqs. (2.5.46). Figure 2–15 shows the element and nodal force free-body diagrams.

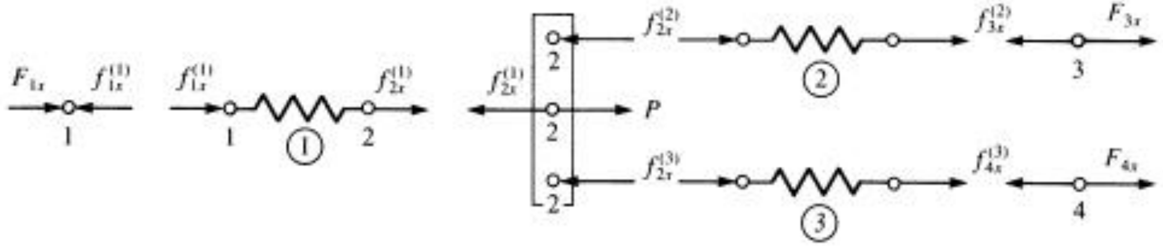


Figure 2–15 Free-body diagrams of elements and nodes of spring assemblage of Figure 2–14

Using the local stiffness matrix Eq. (2.2.17) applied to each element, and compatibility condition Eq. (2.5.45), we obtain the total or global equilibrium equations as

$$
\begin{aligned}
F_{1x} &= k_1 d_{1x} - k_1 d_{2x} \\
P &= -k_1 d_{1x} + k_1 d_{2x} + k_2 d_{2x} - k_2 d_{3x} + k_3 d_{2x} - k_3 d_{4x} \\
F_{3x} &= -k_2 d_{2x} + k_2 d_{3x} \\
F_{4x} &= -k_3 d_{2x} + k_3 d_{4x}
\end{aligned}
\tag{2.5.47}
$$

In matrix form, we express Eqs. (2.5.47) as

$$
\begin{Bmatrix} F_{1x} \\ P \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix} \tag{2.5.48}
$$

Therefore, the global stiffness matrix is the square, symmetric matrix on the right side of Eq. (2.5.48). Making use of the boundary conditions, Eqs. (2.5.44), and then considering the second equation of Eqs. (2.5.47) or (2.5.48), we solve for d_{2x} as

$$
d_{2x} = \frac{P}{k_1 + k_2 + k_3} \tag{2.5.49}
$$

We could have obtained this same result by deleting rows 1, 3, and 4 in the $\underline{F}$ and $\underline{d}$ matrices and rows and columns 1, 3, and 4 in $\underline{K}$, corresponding to zero displacement, as previously described in Section 2.4, and then solving for d_{2x}.

Using Eqs. (2.5.47), we now solve for the global forces as

$$F_{1x} = -k_1 d_{2x} \qquad F_{3x} = -k_2 d_{2x} \qquad F_{4x} = -k_3 d_{2x} \tag{2.5.50}$$

The forces given by Eqs. (2.5.50) can be interpreted as the global reactions in this example. The negative signs in front of these forces indicate that they are directed to the left (opposite the x axis).

(b) Using the direct stiffness method, we formulate the global stiffness matrix. First, using Eq. (2.2.18), we express each element stiffness matrix as

$$\underline{k}^{(1)} = \begin{array}{c} \begin{array}{cc} d_{1x} & d_{2x} \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \quad \underline{k}^{(2)} = \begin{array}{c} \begin{array}{cc} d_{2x} & d_{3x} \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \quad \underline{k}^{(3)} = \begin{array}{c} \begin{array}{cc} d_{2x} & d_{4x} \end{array} \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{array} \tag{2.5.51}$$

where the particular degrees of freedom associated with each element are listed in the columns above each matrix. Using the direct stiffness method as outlined in Section 2.4, we add terms from each element stiffness matrix into the appropriate corresponding row and column in the global stiffness matrix to obtain

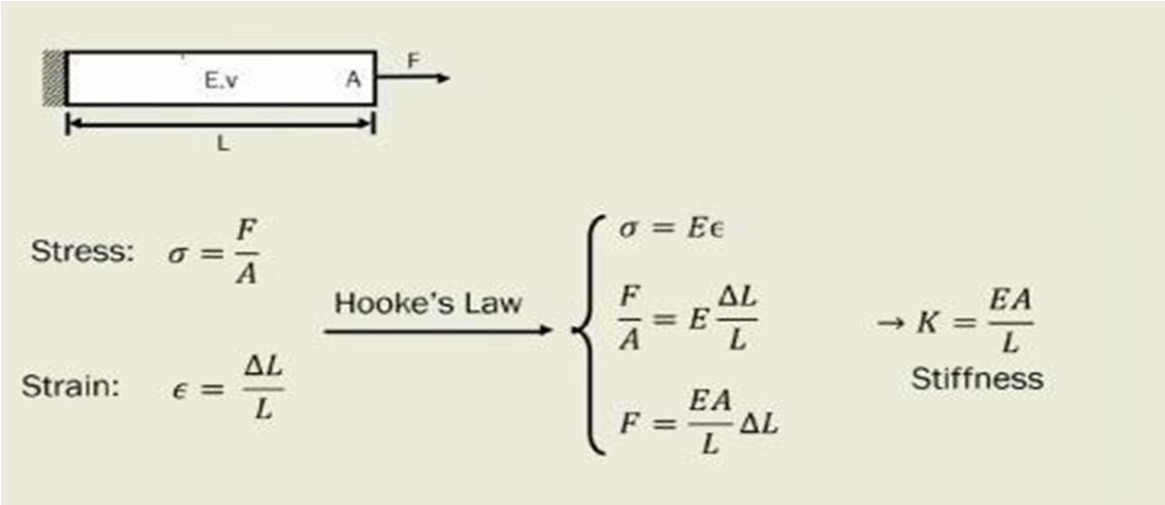
$$\underline{K} = \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{2x} & d_{3x} & d_{4x} \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \end{array} \tag{2.5.52}$$

Finite Element Basis

- Recall the process for matrix analysis:
  - Divide the structure into elements
  - Form stiffness matrix for each element
  - Assemble stiffness matrices
  - Apply boundary conditions
  - Solve for unknown displacements
  - Back substitute to solve for:
    - Reactions
    - Internal forces

Derive Stiffness matrix for Bar Element

- $E = \frac{\sigma}{\epsilon}$
- $\sigma = E\,\epsilon$
- $\frac{F}{A} = E\frac{change\ in length}{original\, length}$
- $\frac{F}{A} = E\frac{u}{L}$



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

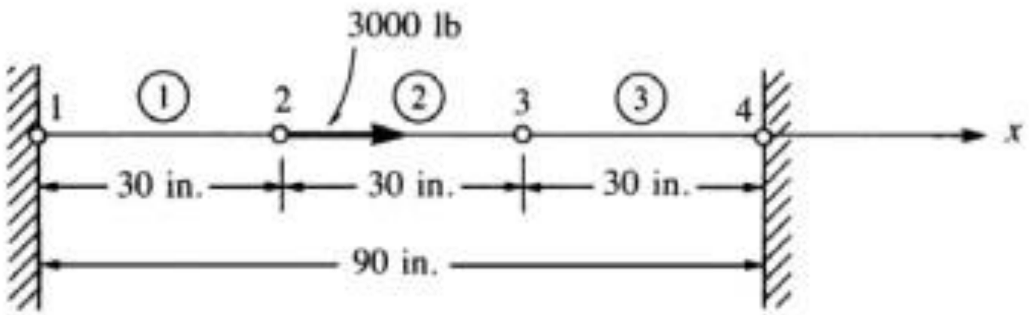
Now, because $\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$, we have, from Eq. (3.1.13),

$$\underline{\hat{k}} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Consider the following three-bar system shown below. Assume for elements 1 and 2: $A = 1\ in^2$ and $E = 30\ (10^6)\ psi$ and for element 3: $A = 2\ in^2$ and $E = 15\ (10^6)\ psi$.



Determine: (a) the global stiffness matrix, (b) the displacement of nodes 2 and 3, and (c) the reactions at nodes 1 and 4.

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

For elements 1 and 2:

1 2 node numbers for element 1

2 3 node numbers for element 2

$$\mathbf{k}^{(1)} = \mathbf{k}^{(2)} = \frac{(1)\left(30\times10^{6}\right)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} {}^{lb}\!/_{in} = 10^{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} {}^{lb}\!/_{in}$$

For element 3:

3 4 node numbers for element 3

$$\mathbf{k}^{(3)} = \frac{(2)\left(15\times10^{6}\right)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} {}^{lb}\!/_{in} = 10^{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} {}^{lb}\!/_{in}$$

As before, the numbers above the matrices indicate the displacements associated with the matrix.

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Assembling the global stiffness matrix by the direct stiffness methods gives:

E1 E2 E3

$$\mathbf{K} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Relating global nodal forces related to global nodal displacements gives:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

The boundary conditions are: $u_1 = u_4 = 0$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \left[\begin{array}{c|cc|c} 1 & -1 & 0 & 0 \\ \hline -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ \hline 0 & 0 & -1 & 1 \end{array}\right] \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

Applying the boundary conditions and the known forces ($\boldsymbol{F}_{2x}$ = 3,000 *lb*) gives:

$$\begin{Bmatrix} 3{,}000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Solving for u_2 and u_3 gives:

$$u_2 = 0.002\, in$$
$$u_3 = 0.001\, in$$

The global nodal forces are calculated as:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \\ 0.001 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2{,}000 \\ 3{,}000 \\ 0 \\ -1{,}000 \end{Bmatrix} lb$$

Example

Using direct stiffness method, determine the nodal displacement of stepped bar as shown in figure also determine reaction support and stresses in each element. Take E=200GPa

Solution

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$A_1 = A_2 = 200\ mm^2$

$A_3 = 100\ mm^2$

$A_1 = A_2 = 200\ mm^2$

$A_3 = 100\ mm^2$

$E = 200\ GPa$

$= 200 \times 10^3\ MPa$

$L_1 = L_2 = 100\ mm$

$L_3 = 150\ mm$

$$[K_1] = \frac{A_1 E}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{200 \times 200 \times 10^3}{100}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_3] = \frac{A_3 E_3}{L_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{100 \times 200 \times 10^3}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 1.33 \times 10^5 \begin{bmatrix} 1 & -1 \\ - & 1 \end{bmatrix}$$

$$= 10^5 \begin{array}{cc} \begin{array}{cc} \quad 3 & \quad\quad 4 \end{array} & \\ \begin{bmatrix} 1.33 & -1.33 \\ -1.33 & 1.33 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

Solution

$$[K] = 10^5 \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 5.33 & -1.33 \\ 0 & 0 & -1.33 & 1.33 \end{bmatrix}$$

$$[K]\{d\} = \{f\}$$

$$10^5 \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 5.33 & -1.33 \\ 0 & 0 & -1.33 & 1.33 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} f_1 = R_1 \\ f_2 = 20\times \\ f_3 \\ f_4 \end{Bmatrix} 10^3$$

$d_2 = 0.075$ mm

$d_3 = 0.1$ mm

$d_4 = 0.1752$ mm

Solution

$$10^5(4d_1 - 4d_2 + 0d_3 + 0d_4) = R_1$$

$$10^5(-4 \times 0.075) = R_1$$

$$R_1 = -30000\ N$$

$$= -30\ kN$$

Stress

$$= E \frac{1}{L_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$= 200 \times 10^3 \times \frac{1}{100} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.075 \end{Bmatrix}$$

$$= 2 \times 10^3 (0 + 0.075)$$

$$= 150\ N/mm^2 = 150\ MPa$$

Solution

$$\sigma_2 = \frac{E}{L_2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}d_2\\ d_3\end{Bmatrix}$$

$$= \frac{200\times 10^3}{100}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0.075\\ 0.1\end{Bmatrix}$$

Second stress

$$= 2\times 10^3(-0.075 + 0.1)$$

$$= 185 \text{ MPa}$$

Solution

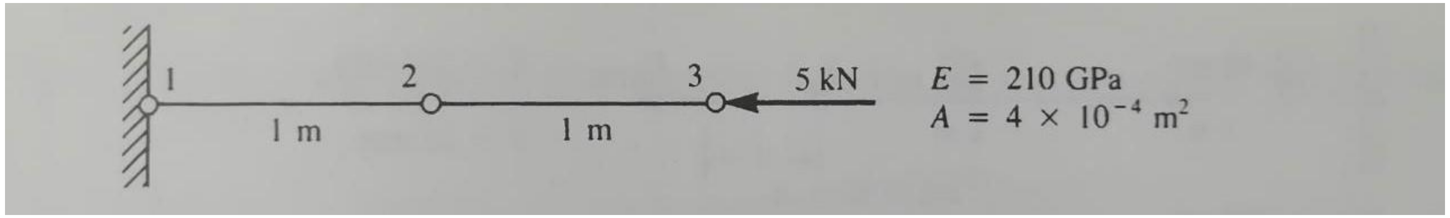
Stress

$$= \frac{200 \times 10^3}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0.1752 \end{Bmatrix}$$

$$= 33.33 \text{ MPa}$$

Class Activity

- For the Bar Assemblages shown in figure Determine the nodal Displacement, The forces in each element and the reaction use the direct stiffness method.



Solution

Element 1: u_1 u_2

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element 2: u_2 u_3

$$\begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

U1=0
Because it is fixed

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} F1 \\ F2 = 0 \\ F3 = 5\,000KN \end{bmatrix} = \begin{bmatrix} 840 * 10^6 & -840 * 10^6 & 0 \\ -840 * 10^6 & 1680 * 10^6 & -840 * 10^6 \\ 0 & -840 * 10^6 & 840 * 10^6 \end{bmatrix} \begin{bmatrix} u1 = 0 \\ u2 \\ u3 \end{bmatrix}$$

Solution

- $\begin{bmatrix} F2 = 0 \\ F3 = 5\,000KN \end{bmatrix}$=$\begin{bmatrix} 1680 * 10^6 & -840 * 10^6 \\ -840 * 10^6 & 840 * 10^6 \end{bmatrix} \begin{bmatrix} u2 \\ u3 \end{bmatrix}$

$0 = 1680 * 10^6$u2+$(-840 * 10^6)$u3

5000= $-840 * 10^6$u2+ $840 * 10^6$u3

$\begin{bmatrix} u2 \\ u3 \end{bmatrix}$= $\begin{bmatrix} 5.952 * 10^{-6} \\ 1.190 * 10^{-5} \end{bmatrix}$

- $\begin{bmatrix} F1 \\ F2 = 0 \\ F3 = 5\,000KN \end{bmatrix}$=$\begin{bmatrix} 840 * 10^6 & -840 * 10^6 & 0 \\ -840 * 10^6 & 1680 * 10^6 & -840 * 10^6 \\ 0 & -840 * 10^6 & 840 * 10^6 \end{bmatrix}\begin{bmatrix} u1 = 0 \\ u2 \\ u3 \end{bmatrix}$

- F1= $840 * 10^6$(0)+($-840 * 10^6$($5.952 * 10^{-6}$)=-4999.68 N

Nodal forces

- $\begin{bmatrix} f_1^1 \\ f_2^1 \end{bmatrix}$=$\begin{bmatrix} 840 * 10^6 & -840 * 10^6 \\ -840 * 10^6 & 840 * 10^6 \end{bmatrix} \begin{bmatrix} u1 = 0 \\ u2 = 5.952 * 10^{-6} \end{bmatrix}$= $\begin{bmatrix} -4999.68 \\ 4999.68 \end{bmatrix}$

- $\begin{bmatrix} f_2^2 \\ f_3^2 \end{bmatrix}$=$\begin{bmatrix} 840 * 10^6 & -840 * 10^6 \\ -840 * 10^6 & 840 * 10^6 \end{bmatrix} \begin{bmatrix} 5.952 * 10^{-6} \\ 1.190 * 10^{-5} \end{bmatrix}$= $\begin{bmatrix} -4996.32 \\ 4996.32 \end{bmatrix}$

Reference

- A First Course in the Finite Element Method, Daryl Logan, Fourth Edition.