

Set Theory

First Stage

Lecture No.2

Computer Science Department

Basic Concepts

- A **set** is any well-defined **collection** of **objects**, called the **elements** or **members** of the set.
- These **elements** may be **anything**: numbers, points in geometry, letters of alphabets, etc. The following are examples of the set:
 1. *The rivers in London;*
 2. *The vowels of alphabets;*

Methods of Describing Sets

- Mainly there are **two ways** of describing a set:
 1. Roster Method
 2. Set Selector Method (or Set Builder Form or Property Method)

Roster Method

- A set may be described by listing all its elements.
- e.g. $N = \{1, 2, 3, \dots\}$, Set of natural numbers.
- This method is called roster method or listing method.

Set Selector Method (or Set Builder Form)

- Sometimes a set is described by means of **some property** which **characterizes all the elements** of the set.
- A set S characterized by a property p may be written as:
- $S = \{(x : p(x))\}$
- Example
 - $A = \{x ; x^2 - 4 = 0, X \in \mathfrak{R}\}$, A is set of those elements which satisfies equation $x^2 - 4 = 0$ and x is a real number.

Subsets

- Let A and B be two sets, If **each element** of set A is an **element** of set B then 'A is called subset of B' and is denoted by: ' $A \subset B$ '.
- Sometime it is said to be 'A is contained in B'.
- If 'A is not subset of B', we write $A \not\subset B$

Proper Subset

- If 'A is a subset of B' and number of elements in A and B are not same i.e. $A \neq B$ then A is known as **proper subset** of B.
- Examples:
- $A = \{a, b, c, d\}$
- $C = \{a, b, c, d, x, y, z\}$
- $D = \{a, b, 1, 2\}$
- $A \subset C$ and $D \not\subset C$

Equality of Sets

- Two sets A and B are equal sets if ' $A \subset B$ ' and ' $B \subset A$ '.

The Empty Set

- The **set** having **no element** is called empty set or null set or void set. Empty set is denoted by \emptyset or $\{ \}$.
- **Example:**

$$A = \{ x ; x^2 + 4 = 0 \text{ and } x \in \mathbb{R} \}$$

$$A = \emptyset$$

as no real number satisfies equation $x^2 + 4 = 0$.

Universal Set

- In any application of set theory, all the sets under consideration will likely be subsets of a fixed set.
- We call this set the universal set and denote it by capital letter X .
- Thus a non empty set of which all the sets under consideration are subsets is called universal set.
- **Example :**
- (i) In a study of human population, all people in the world may be assumed to form the Universal set. The people of any continent, country, religion is a subset of this universal set.
- (ii) The set of letters in alphabets is the universal from which the letters of any word may be chosen to form a set.

Power Set

- If S is any set, then set of all subsets of S is called power set of S and is denoted by $P(S)$.

Example

$$S = \{1, 2, 3\}$$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

So if S contains 3 elements power set of S , i.e. $P(S)$ will contain 2^3 , i.e. 8 elements.

- If a finite set S has n elements, then power set of S has 2^n elements.

Venn-Euler Diagrams or Venn Diagrams

- The relation between sets can be conveniently illustrated by certain diagrams called **Venn diagrams**.
- In Venn diagrams a **universal set** X is represented by a large rectangle and subsets of X by regions enclosed within a closed curve generally circles, lying in the region enclosed by the rectangle.
- If a sets B is a subset of A , the circle representing B is drawn inside the circle representing A . U A and B are disjoint sets , then the circles representing A and B are drawn in such a way that they have no common area.
- If A and B are not disjoint, then the circles representing A and B are drawn in such a way that they have some common area.

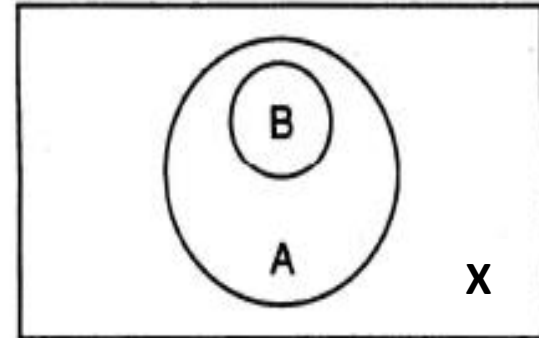
Set Operations

Symbol

Venn Diagram

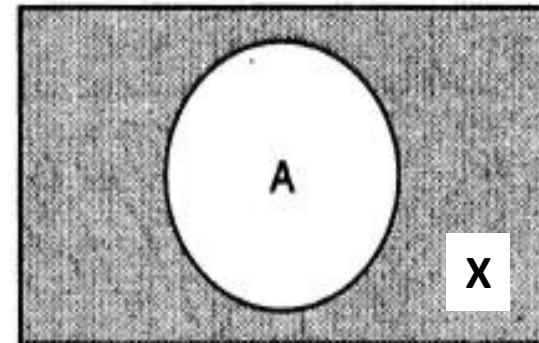
Set B is a proper subset of A

$$B \subset A$$



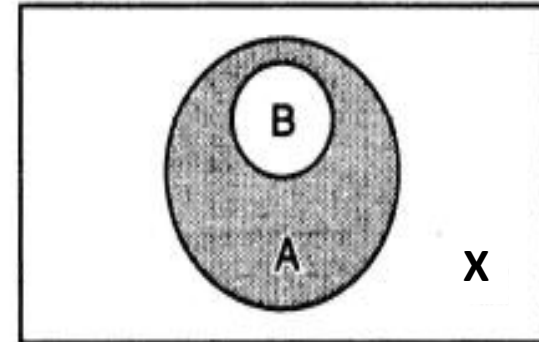
The complement of Set A

$$A'$$



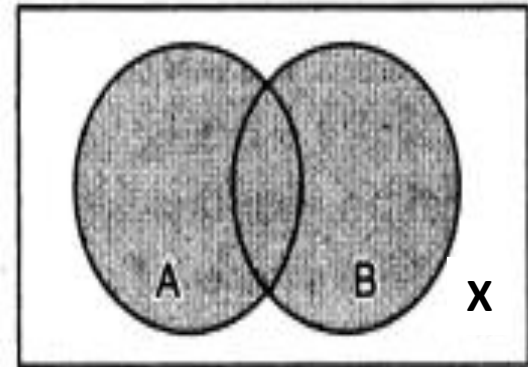
The difference of set A and B

$$A - B$$



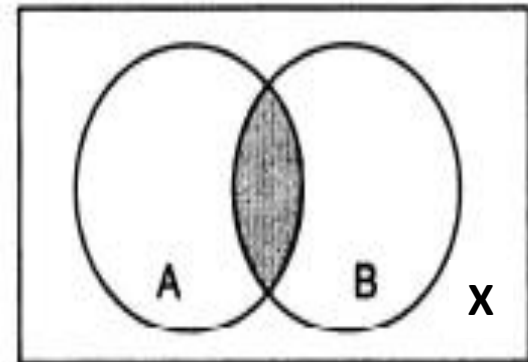
The union of sets A and B

$$A \cup B$$



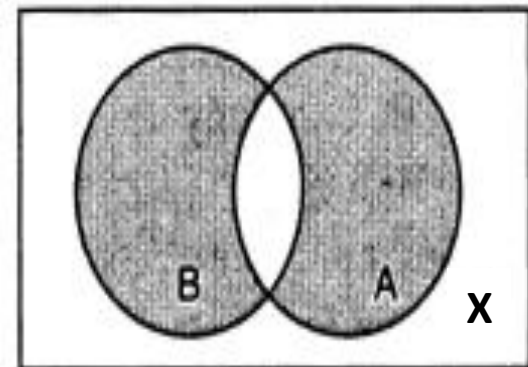
The intersection of sets A and B

$$A \cap B$$



The symmetrical difference of sets A and B

$$A \Delta B$$



Operations on Sets

- Sets can be combined in many different ways so as to produce new sets.
- **Union of Two Sets**
- Let A and B be two sets. Then the union of A and B is set of all those elements which are either in set A or in set B or in both sets.

Union of sets A and B is denoted by $A \cup B$.

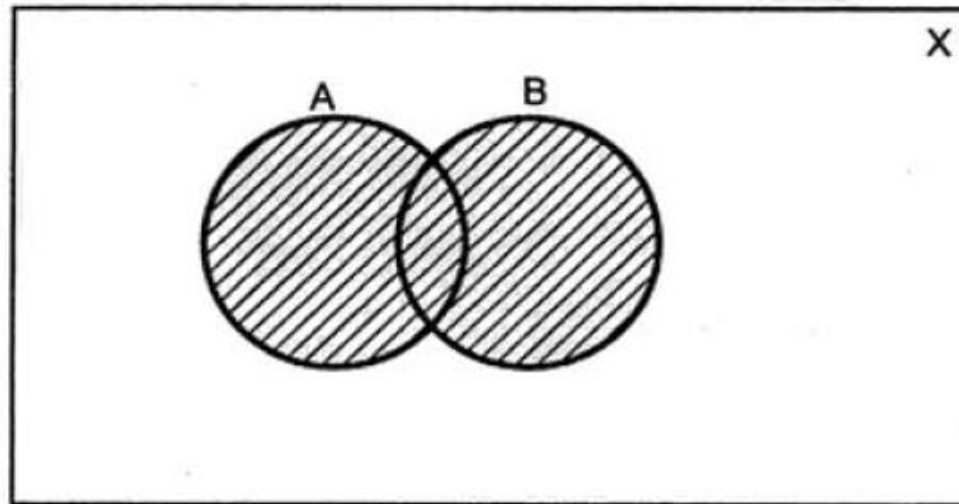
Which is read as 'A union B'.

Symbolically,

$$A \cup B = \{x ; x \in A \text{ or } x \in B\}$$

Operations on Sets (Cont....)

- In mathematics, when we use 'or', i.e. $x \in A$ or $x \in B$, we do not exclude the possibility that x is an element of both A and B .



$A \cup B$ 

Operations on Sets (Cont....)

1. $A = \{1, 2, 3\}, B = \{x, y, z\}$

$$A \cup B = \{1, 2, 3, x, y, z\}$$

2. $A = \{a, b, c, d\}, B = \{a, b, 1, 2\}$

$$A \cup B = \{a, b, c, d, 1, 2\}$$

3. $A = \{0, 1, 2, 3, 4\}, B = \{2, 4, 6, 8\}$

$$A \cup B = \{0, 1, 2, 3, 4, 6, 8\}$$

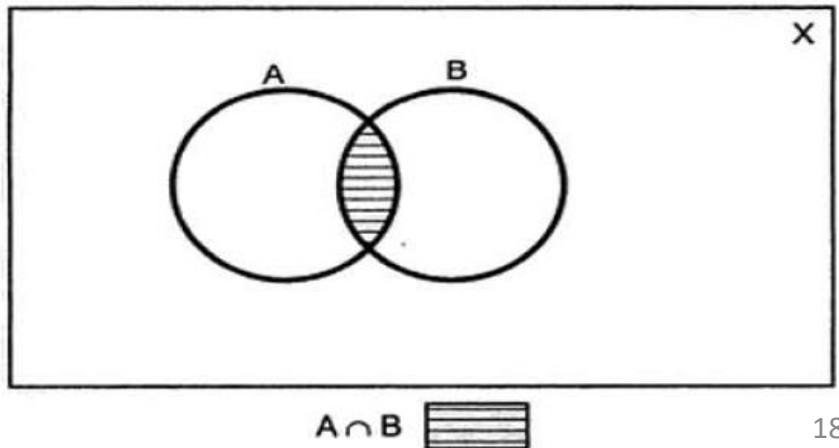
4. $A = \{1, 2, 3, 4, \}, B = \{1, 3\}, B \subseteq A$

$$A \cup B = \{1, 2, 3, 4\} = A$$

Operations on Sets (Cont....)

- Intersection of Two Sets
- Let A and B be two sets. The intersection of A and B is the set of all elements which are in A and also in B.
- We denote the intersection of A and B by $A \cap B$

$$A \cap B = \{x ; x \in A \text{ and } x \in B\}$$



Operations on Sets (Cont....)

Examples

1. $A = \{1, 2, 3, 4\}, B = \{x, y, z\}$

$$A \cap B = \emptyset$$

intersection is null set as no element is common in A and B.

2. $A = \{1, 2, 3, 4, 5\}, B = \{1, 2, 6, 7\}$

$$A \cap B = \{1, 2\}$$

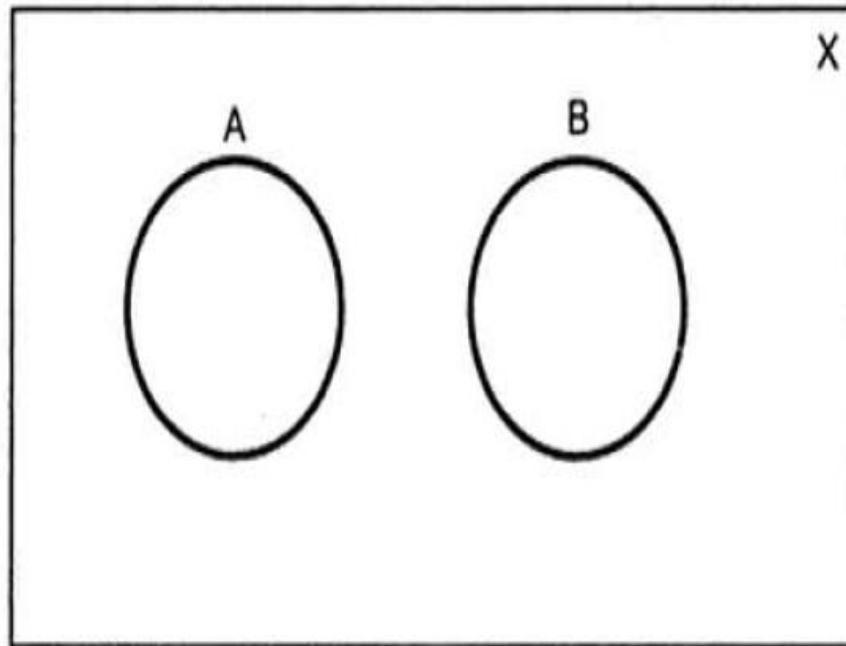
as $1 \in A$ and $1 \in B$ also $2 \in A$ and $2 \in B$.

3. $A = \{1, 2, 3, 4\}, B = \{1, 3\}, B \subset A$ then $A \cap B = \{1, 3\} = B$

Operations on Sets (Cont....)

- Disjoint Sets

Set A and set B are called disjoint sets if no element is common to A and B. i.e. A and B are disjoint then $A \cap B = \emptyset$.

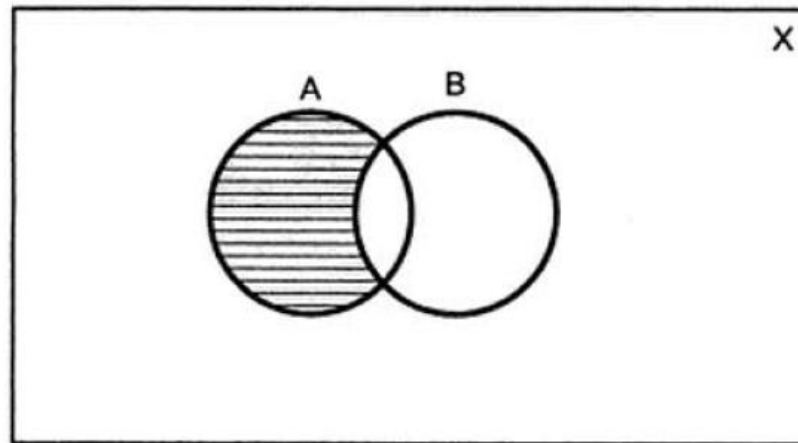


$$A \cap B = \emptyset$$

Operations on Sets (Cont....)

- Difference of Two Sets
- Let A and B be two sets. The difference of A and B is denoted by 'A - B' is set of all those elements of A which are not in B.

$$A - B = \{x ; x \in A \text{ and } x \notin B\}$$



$$A - B = \text{shaded region}$$

Example

$$A = \{1, 2, 3, 4\}, B = \{4, 5, 6, 7\}$$

$$A - B = \{1, 2, 3\}$$

$$B - A = \{5, 6, 7\}$$

$$A \cap B = \{4\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

Operations on Sets (Cont....)

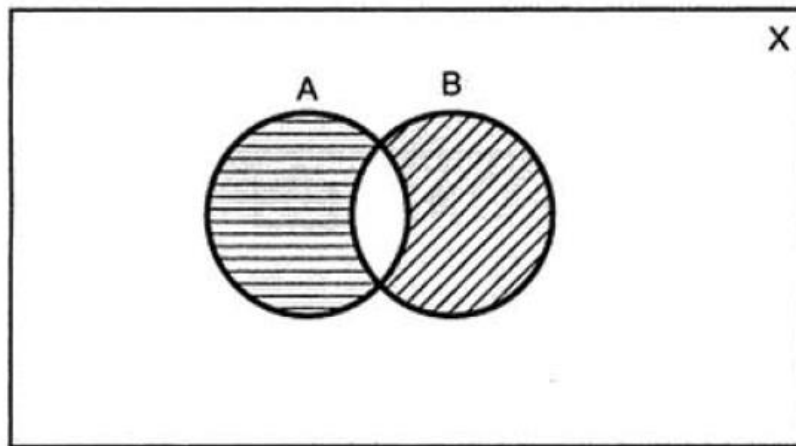
- Symmetric Difference of Two Sets

Let A and B be two sets then symmetric difference of two sets A and B is denoted by $A \Delta B$ or $A \oplus B$ [read as A ring sum B].

And

$$A \oplus B = (A - B) \cup (B - A)$$

$$A \oplus B = (A \cup B) - (A \cap B)$$



$$A \oplus B = \text{[shaded region]}$$

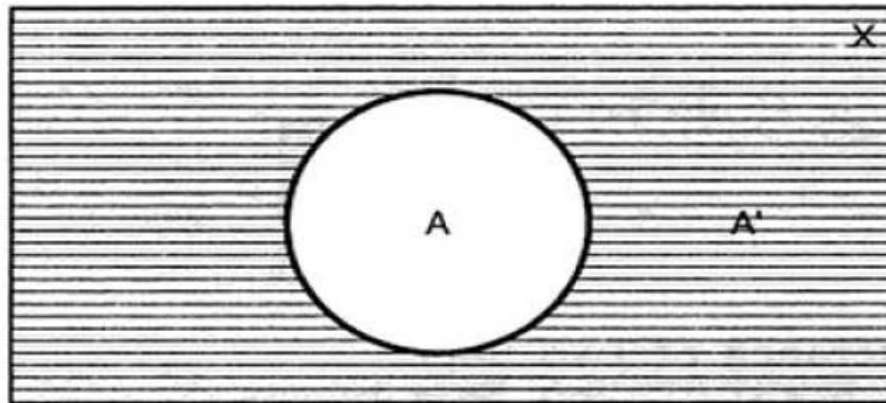
Operations on Sets (Cont....)

- Complement of a Set

Let A be any set, then the complement of set A is denoted by A' or \tilde{A} or A^c .
Complement of A is set of those elements which are in universal set X but are not in set A .

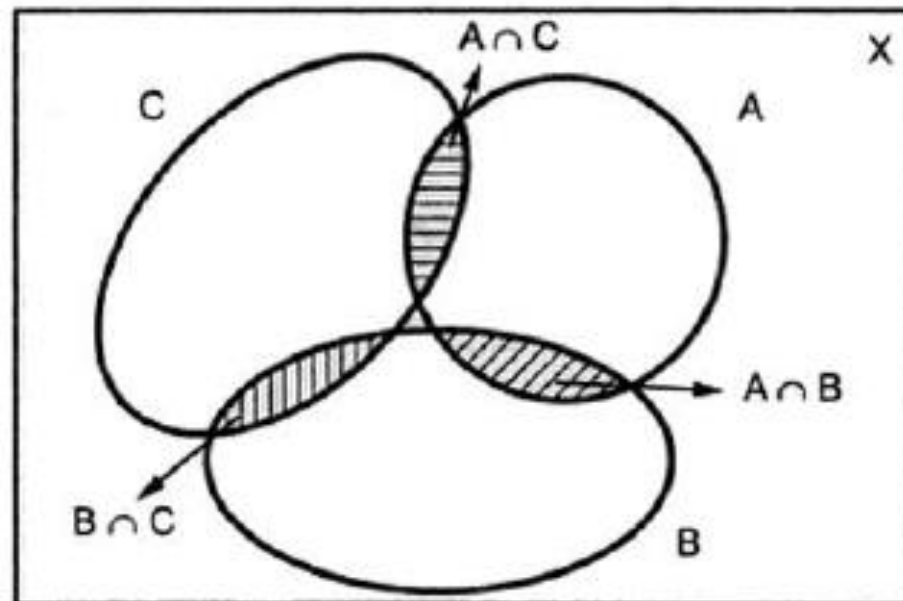
$$\therefore A' = X - A$$

$$A' = \{x; x \in X \text{ and } x \notin A\}$$




Example 3.3 : Let A , B and C be sets such that $(A \cap B \cap C) = \emptyset$, $(A \cap B) \neq \emptyset$, $(A \cap C) \neq \emptyset$, $(B \cap C) \neq \emptyset$. Draw the corresponding Venn diagram.

Solution :



$A \cap B$  , $A \cap C$ 

$B \cap C$  and $A \cap B \cap C = \emptyset$

Homework.....

1. Draw Venn diagram for the sets P and Q if:

I. $P \cap Q = P$

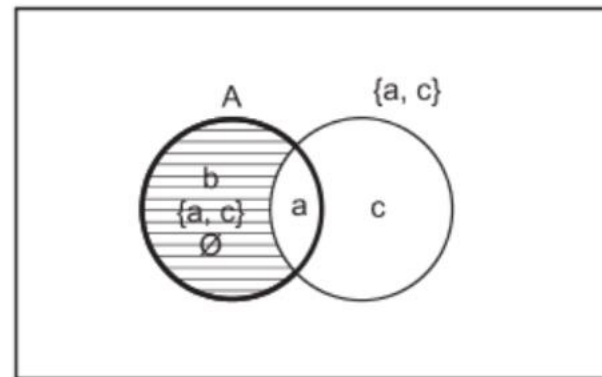
II. $P \cup Q = P$

2. Let $A = \{a, b, \{a, c\}, \emptyset\}$. determine the following sets.

I. $A - \{a\}$

II. $A - \emptyset$

III. $A - \{a, c\}$



$$A - \{a, c\} = \text{[shaded area of circle A]}$$

as $\{a, c\}$ is a set containing two elements a and c .

Algebra of Set Operations

- Let A, B, C be any subsets of a set X , then

1. Idempotent Laws

$$\text{i) } A \cup A = A$$

$$\text{ii) } A \cap A = A$$

2. Commutative Laws

$$\text{i) } A \cup B = B \cup A$$

$$\text{ii) } A \cap B = B \cap A$$

3. Associative Laws

$$\text{i) } A \cup (B \cup C) = (A \cup B) \cup C$$

$$\text{ii) } A \cap (B \cap C) = (A \cap B) \cap C$$

4. Distributive Laws

' \cap ' distributes over ' \cup ' from the left and also from the right.

$$\text{i) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

' \cup ' Distributes over ' \cap ' from the left and also from the right.

$$\text{ii) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

5. DeMorgan's Laws

$$\text{i) } (A \cup B)' = A' \cap B'$$

$$\text{ii) } (A \cap B)' = A' \cup B'$$

6. Absorption Laws

$$\text{i) } A \cup (A \cap B) = A$$

$$\text{ii) } A \cap (A \cup B) = A$$

7. Double complement

$$(A')' = A$$

Cardinality of Finite Set

- Cardinality of a set is number of distinct elements in the set.
- Cardinality of a set A is denoted by $n(A)$ or $|A|$.

Examples

1. $A = \{2, 3, 4, 5\}$

$$n(A) \text{ or } |A| = 4$$

2. $A = \{x ; 2 < x < 10\}$

$$A = \{3, 4, 5, 6, 7, 8, 9\}$$

$$|A| = 7$$

Principle of Inclusion-Exclusion

- we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

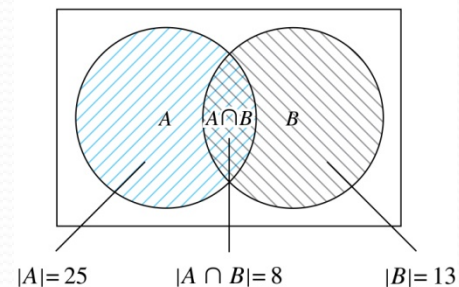
- We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

Solution: $|A \cup B| = |A| + |B| - |A \cap B|$
 $= 25 + 13 - 8 = 30$

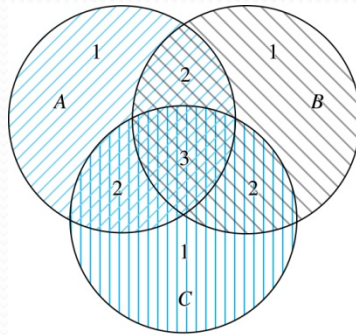
$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$



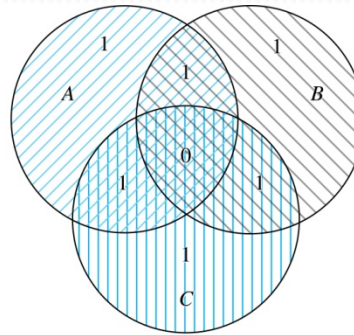
Three Finite Sets

$$|A \cup B \cup C| =$$

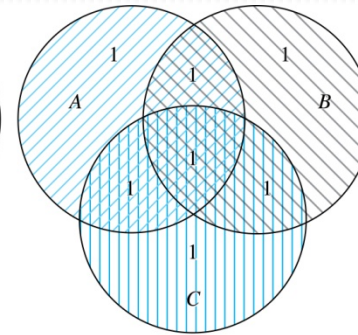
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by
 $|A| + |B| + |C|$



(b) Count of elements by
 $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$



(c) Count of elements by
 $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

- **Example** : Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 and 5 and 7? Also indicate how many are divisible by 3 and 5 but not by 7.

Solution : Let A, B, C denote the set of integers from 1 to 250 divisible by 3 or 5 or 7.

$$|A| = \left[\frac{250}{3} \right] = 83$$

$$|B| = \left[\frac{250}{5} \right] = 50$$

$$|C| = \left[\frac{250}{7} \right] = 35$$

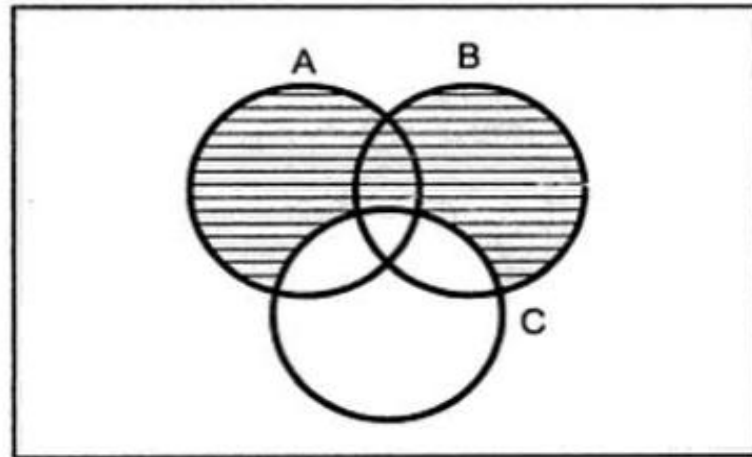
$$|A \cap B| = \left[\frac{250}{3 \times 5} \right] = 16$$

$$|A \cap C| = \left[\frac{250}{3 \times 7} \right] = 11$$

$$|B \cap C| = \left\lfloor \frac{250}{5 \times 7} \right\rfloor = 7$$

$$|A \cap B \cap C| = \left\lfloor \frac{250}{3 \times 5 \times 7} \right\rfloor = 2$$

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C| \\ &= 83 + 50 + 35 - [16 + 11 + 7] + 2 \\ &= 136 \end{aligned}$$



$$\equiv |A \cup B \cup C| - |C|$$

Number of elements divisible by 3 or 5 but not by 7 is

$$|A \cup B \cup C| - |C| = 136 - 35 = 101$$

- **Homework:**

1. How many integers between 1 to 2000 are divisible by 2, 3, 5 or 7.
2. The 60,000 fans who attended the home coming football game bought up all the paraphernalia for their cars. Altogether 20,000 bumpers stickers, 36,000 window decals and 12,000 key rings were sold. We know that 52,000 fans bought at least one item and no one bought more than one of a given item. Also 6000 fans bought both decals and key rings, 9000 bought both decals and bumpers stickers and 5000 bought both key rings and bumper stickers.
 - How many fans bought all three items ?
 - How many fans bought exactly one items ?
 - Someone questioned the accuracy of the total number of purchasers; 52000 (given that all the other numbers have been confirmed to be correct.) This person claimed the total number of purchasers to be either 60,000 or 44,000. How do you dispel the claim ?

- **Example:** Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects.
- a) Find the number of students studying all three subjects.
- b) Find the number of students studying exactly one of the three subjects.

Solution : Let A, B, C denotes the set of students studying mathematics, physics and biology respectively.

$$\text{And } |X| = 100$$

$$|A| = 32$$

$$|B| = 20$$

$$|C| = 45$$

$$|A \cap C| = 15$$

$$|A \cap B| = 7$$

$$|B \cap C| = 10$$

$$|A' \cap B' \cap C'| = 30$$

$$|A' \cap B' \cap C'| = 100 - |A \cup B \cup C|$$

$$\text{Or } |A \cup B \cup C| = 100 - 30$$

$$= 70$$

$$\text{a) } |A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$70 = 32 + 20 + 45 - [7 + 15 + 10] + |A \cap B \cap C|$$

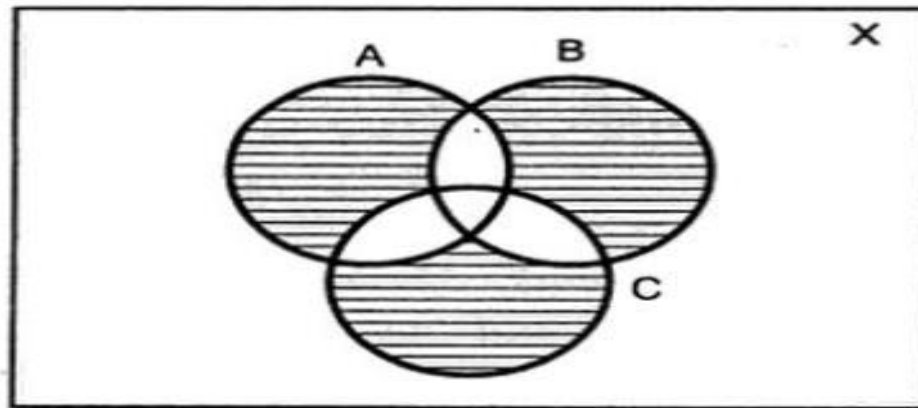
$$70 = 97 - [32] + |A \cap B \cap C|$$

$$70 - 65 = |A \cap B \cap C|$$

$$\Rightarrow |A \cap B \cap C| = 5$$

5 students study all 3 subjects.

b) Number of students studying exactly one subject.



Number of students studying only mathematics is

$$\begin{aligned} & |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\ &= 32 - 7 - 15 + 5 \\ &= 15 \end{aligned}$$

Number of students studying only physics is

$$\begin{aligned} & |B| - |B \cap A| - |B \cap C| + |A \cap B \cap C| \\ &= 20 - 7 - 10 + 5 \\ &= 8 \end{aligned}$$

Number of students studying only biology is

$$\begin{aligned} & |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 45 - 15 - 10 + 5 \\ &= 25 \end{aligned}$$

∴ Number of students studying exactly one subject

$$= 15 + 8 + 25 = 48$$

- **Homework.....**

A survey was conducted among 1000 people of these 595 are Democrats, 595 wear glasses, and 550 like ice-cream. 395 of them are Democrats who wear glasses, 350 of them are Democrats who like ice-cream, and 400 of them wear glasses and like ice-cream 250 of them are Democrats who wear glasses and like ice-cream. How many of them are not Democrats do not wear glasses, and do not like ice-cream ? How many of them are Democrats who do not wear glasses and do not like ice-cream ?

Multisets

- Set is a collection of distinct objects. Multiset is a collection of objects that are not necessarily distinct.
- **Example:**
- $A = \{a, a, a, b, b, e\}$, $B = \{a, a, a, a, b, b, b, d, d\}$ are multisets.
- The multisets A and B can also be written as $A = \{3.a, 2.b, 1.e\}$ and $B = \{4.a, 3.b, 2.d\}$

Multiplicity of an element In a Multiset

- Multiplicity of an element in a multiset is the number of times the element present in the multiset.

Examples

1. $\{1, 1, 2, 2, 3, 3, 3, 3\}$
2. $\{1, 2, 3, 4\}$
3. $\{1, 2, 3, 4, 4\}$
4. \emptyset

- Hence the multiplicity of 2 in example 1 above is 2 and multiplicity of 3 in example 1 is 4, multiplicity of 1 in example 1 is 2.

Cardinality of multiset

- The total number of elements in a multiset, including the repeated memberships, is the cardinality of the multiset.
- **Example:**
- in the multiset $\{a, a, a, b, b, c, c, d\}$
- The multiplicities of the elements a, b, c, d are 3, 2, 2, 1 respectively, and the **cardinality of the multiset** is 8.

Union of multisets

- Let P and Q be two multisets. The union of P and Q , denoted by $P \cup Q$ is a multiset such that the multiplicity of an element in $P \cup Q$ is equal to the maximum of the multiplicities of the elements in P and in Q .
- **Example:**
- $P = \{a, a, a, b, b, c, c, d\}$
- $Q = \{a, a, b, c, d, e\}$
- $P \cup Q = \{a, a, a, b, b, c, c, d, e\}$

Intersection of multisets

- The intersection of P and Q denoted by $P \cap Q$, is a multiset such that the multiplicity of an element in $P \cap Q$ is equal to the minimum of the multiplicities of the element in P and in Q .
- $P = \{a, a, a, b, b, c, c, d\}$
- $Q = \{a, a, b, c, d, e\}$
- $P \cap Q = \{a, a, b, c, d\}$

Sum of multisets

- The sum of P and Q denoted by $P + Q$, to be a multiset such that the multiplicity of an element in $P + Q$ is equal to the sum of the multiplicities of the element in P and in Q .
- **Example:**
- $P = \{a, a, a, b, b, c, c, d\}$, $Q = \{a, a, b, c, d, e\}$
- $P + Q = \{a, a, a, a, a, b, b, b, c, c, c, d, d, e\}$.