



Linear Algebra

Lecture Notes 1

Matrices

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Linear Algebra

- Linear Algebra is the branch of mathematics that focuses on the study of vectors, vector spaces, matrices, solving equations, and linear transformations.
- It deals with linear equations, linear functions, and their representations through matrices and determinants.
- It forms the mathematical foundation for numerous algorithms and methods used in fields such as computer graphics, machine learning, data analysis, and more.

Foundations of Linear Algebra

Elementary linear algebra introduces the foundational concepts that form the building blocks of the subject. It covers basic operations on matrices, solving systems of equations, and understanding vectors.



Linear algebra in computer science

Scale, rotate, and move objects using matrices

Computer Graphics

Uses matrix multiplication to encrypt and decrypt messages.

Cryptography

Machine Learning

Data is stored as a vector and matrices, and uses eigenvalues/eigenvectors to reduce dimensions.

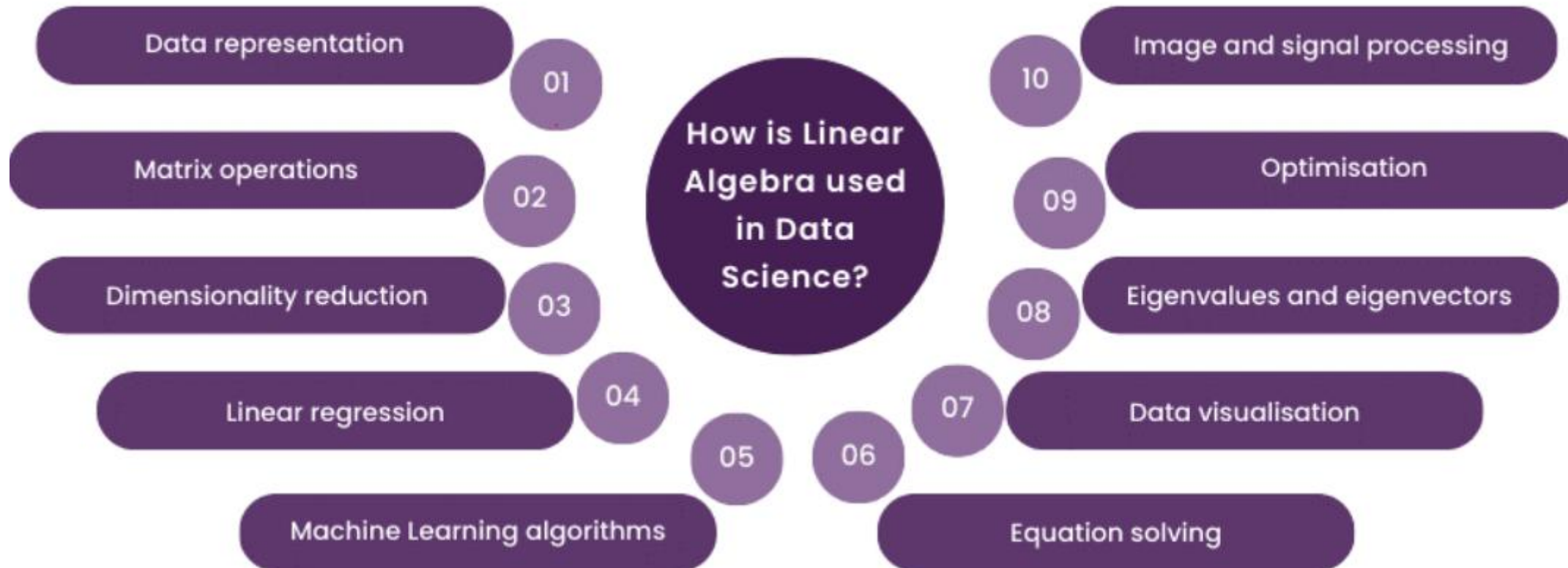
Computer Vision

Images can be represented as matrices of pixel values, and operations



Linear Algebra in Data Science

Linear Algebra serves as the foundational stone to perform big data analysis in the Data Science domain. Here are the few essential applications of Linear Algebra in Data Science.



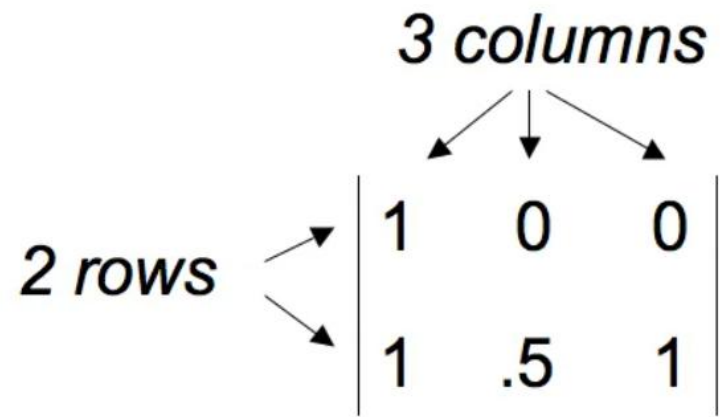
Matrix

In mathematics, a **matrix** is a rectangular *array* or *table* of numbers, symbols, or expressions, arranged in *rows* and *columns*. The individual numbers constituting the array are called **entries** or **elements**.

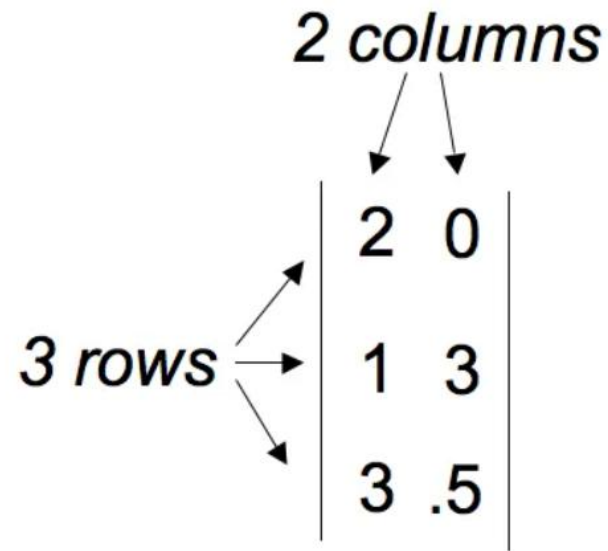
An **$m \times n$ matrix**: the **m rows** are horizontal and the **n columns** are vertical. Each element of a matrix is often denoted by a variable with two subscripts. For example, $a_{2,1}$ represents the element at the second row and first column of the matrix.

$$\begin{matrix} & \begin{matrix} 1 & 2 & \dots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{matrix} & \left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right] \end{matrix}$$

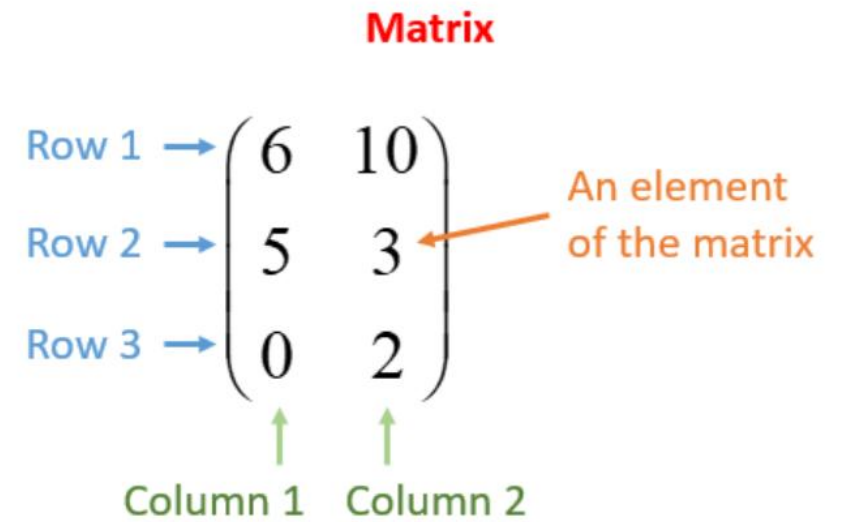
Dimension of the matrix is $m \times n$



2 x 3 Matrix



3 x 2 Matrix



Dimension of this matrix is 3 x 2

Examples:

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{B} = [1 \quad -1 \quad 0 \quad 6 \quad 2]$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 & 3 & 1 \\ 5 & 7 & 9 & 0 & 2 \\ 3 & 4 & 6 & 7 & 8 \end{bmatrix}$$

$$\mathbf{D} = [6]$$

(a) State the orders of the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} .

(b) Write down the values of

$$a_{11}, \quad a_{22}, \quad b_{14}, \quad c_{25}, \quad c_{33}, \quad c_{43}, \quad d_{11}$$

Types of Matrices

1. Column matrix or vector

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

2. Row matrix or vector

Any number of columns but only one row

$$[1 \quad 1 \quad 6]$$

$$[0 \quad 3 \quad 5 \quad 2]$$

$$[a_{11} \quad a_{12} \quad a_{13} \cdots a_{1n}]$$

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns, $m \neq n$

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $A_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The main diagonal of a square matrix is composed of all elements a_{ij} for which $i = j$

5. Diagonal matrix

A square matrix where all the elements are zero except those on the **main diagonal**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} \neq 0$ for all $i = j$

$a_{ij} = 0$ for some or all $i \neq j$

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all i, j

8. Triangular matrix

A **square matrix** whose elements above or below the **main diagonal** are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

8a. Upper triangular matrix

A square matrix whose elements below the **main diagonal** are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

8b. Lower triangular matrix

A square matrix whose elements above the **main diagonal** are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

9. Scalar matrix

A **diagonal matrix** whose main diagonal elements are equal to the same scalar.

A **scalar** is defined as a single number or constant.

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3 \times 3}$$

i.e. $a_{ij} = 0$ for all $i \neq j$
 $a_{ij} = a$ for all $i = j$

10. Symmetric matrix

Symmetric matrix is a type of square matrix where the top-right triangle is the same as the bottom-left triangle

i.e. $a_{ij} = a_{ji}$ for all $i \neq j$.

Example :

$B = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix}$ is (2×2) square matrix but not symmetric.

Example :

$C = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 1 & 0 \\ 6 & 0 & 3 \end{bmatrix}$ is (3×3) square and symmetric matrix by
($a_{12} = a_{21} = 4$), ($a_{13} = a_{31} = 6$) and ($a_{23} = a_{32} = 0$).

Matrices - Operations

Trace of a matrix

The trace of an $n \times n$ square matrix A is defined as a sum of the main diagonal elements

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Example: Let \mathbf{A} be a matrix, with

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 11 & 5 & 2 \\ 6 & 12 & -5 \end{pmatrix}$$

Then

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^3 a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 5 + (-5) = 1.$$

Equality of Matrices

Two matrices are said to be equal only when all corresponding elements are equal. Therefore, their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \text{then, } \mathbf{A} = \mathbf{B}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If $\mathbf{A} = \mathbf{B}$ then $a_{ij} = b_{ij}$

Addition and Subtraction of Matrices

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes **cannot** be added or subtracted

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \end{aligned}$$

Example:

Let

$$\mathbf{A} = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & 2 \\ -1 & 6 \\ 3 & 4 \end{bmatrix}$$

Find

(a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - \mathbf{B}$ (c) $\mathbf{A} - \mathbf{A}$

Solution

$$(a) \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -1 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -1 \\ 3 & 7 \\ 5 & 4 \end{bmatrix}$$

$$(b) \quad \mathbf{A} - \mathbf{B} = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ -1 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -5 \\ -1 & -4 \end{bmatrix}$$

$$(c) \quad \mathbf{A} - \mathbf{A} = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Commutative Law:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative Law:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

Additive identity

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$$

Additive inverse

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$$

Problem:

The monthly sales (in thousands) of burgers (B1) and bites (B2) in three fast-food restaurants (R1, R2, R3) are as follows:

	R1	R2	R3
B1	35	27	13
B2	42	39	24

January

	R1	R2	R3
B1	31	17	3
B2	25	29	16

February

- (a) Write down two 2×3 matrices \mathbf{J} and \mathbf{F} representing sales in January and February respectively.
- (b) By finding $\mathbf{J} + \mathbf{F}$, write down the matrix for the total sales over the two months.
- (c) By finding $\mathbf{J} - \mathbf{F}$, write down the matrix for the difference in sales for the two months.

Scalar multiplication

In general, to multiply a matrix A by a scalar k we simply multiply each element of A by k . For any $m \times n$ matrices A and B , and scalar k :

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & -1 \\ 2 & 7 \\ 1 & 6 \end{bmatrix}$$

Find $a) 2A$ $b) 3B$ $c) 2(A - B)$