Linear Algebra

Lecture Notes 2

Matrices

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Matrix multiplication

In general, if **a** is the row vector

$$[a_{11} \quad a_{12} \quad a_{13} \quad \ldots \quad a_{1s}]$$

and **b** is the column vector

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{s1} \end{bmatrix}$$

then we define the matrix product

$$\mathbf{ab} = [a_{11} \quad a_{12} \quad a_{13} \quad \ldots \quad a_{1s}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{s1} \end{bmatrix}$$

to be the 1×1 matrix

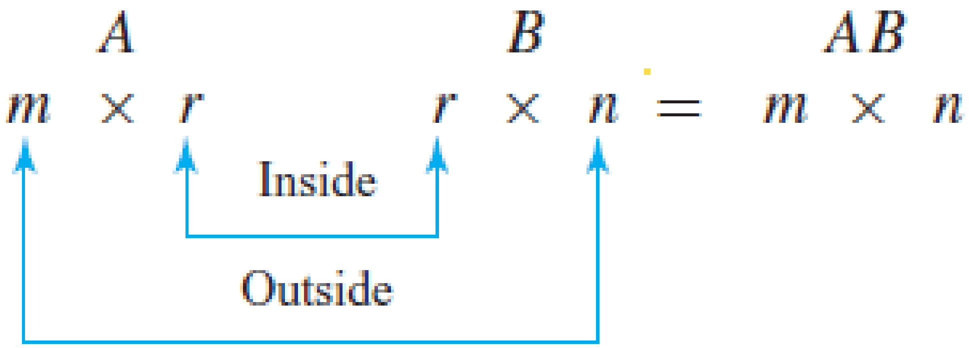
$$[a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \ldots + a_{1s}b_{s1}].$$

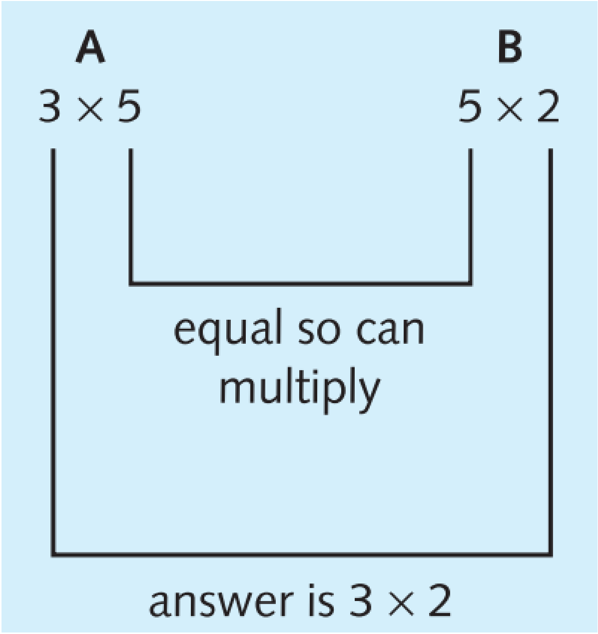
If

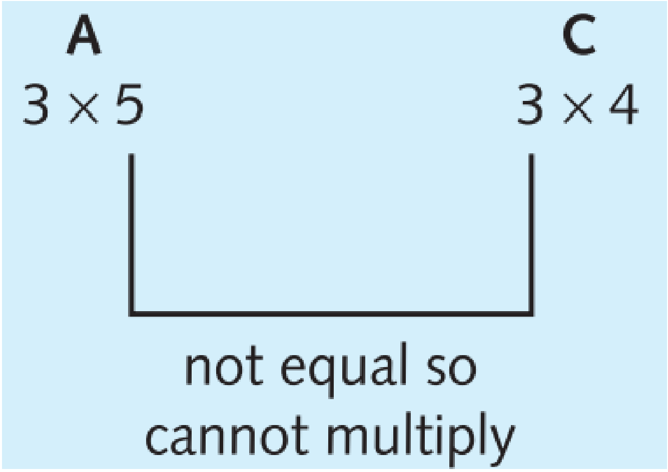
$$\mathbf{a} = [1 \quad 2 \quad 3 \quad 4], \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{ab} = [1 \quad 2 \quad 3 \quad 4] \begin{bmatrix} 2 \\ 5 \\ -1 \\ 0 \end{bmatrix} = [1(2) + 2(5) + 3(-1) + 4(0)] = [9]$$

The product exists if the inner numbers are the same and the order of the answer is given by the outer numbers: that is,







Example: Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 26 & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & 13 \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned}
(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\
(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\
(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\
(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\
(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\
(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12
\end{aligned}
\qquad
AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Example:

If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

evaluate **AB** and **BA.**

Solution

It is easy to check that it is possible to form both products **AB** and **BA** and that they both have order 2×2. In fact

$$\mathbf{AB} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 8 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$$

so $\mathbf{AB} \neq \mathbf{BA}$.

Practice problems:

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 1 & 0 \\ -1 & 1 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

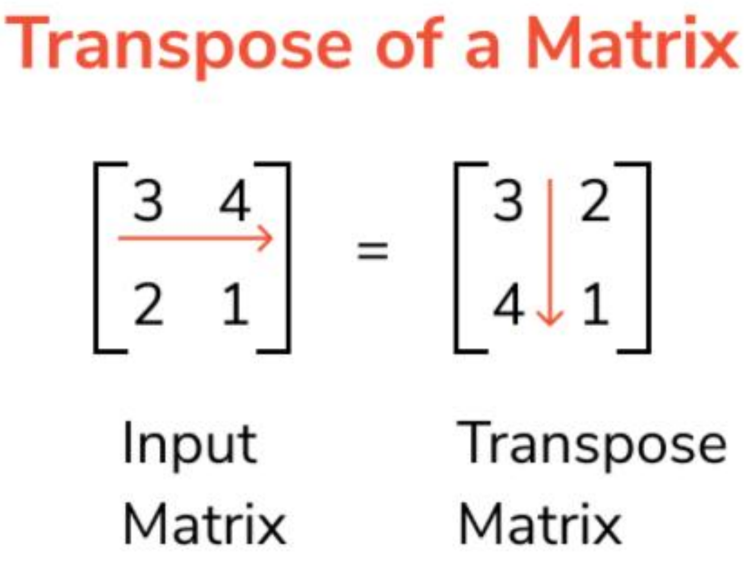
Find (where possible)

(a) $\mathbf{AB}$ **(b)** $\mathbf{BA}$ **(c)** $\mathbf{CD}$ **(d)** $\mathbf{DC}$

(e) $\mathbf{AE}$ **(f)** $\mathbf{EA}$ **(g)** $\mathbf{DE}$ **(h)** $\mathbf{ED}$

Transposition

The transpose of a matrix is found by **replacing rows by columns**, so that the first row becomes the first column, the second row becomes the second column, and so on. The number of rows of A is then the same as the number of columns of A^T and vice versa. Consequently, if A has order $m \times n$ then A^T has order $n \times m$.



Transpose of a Matrix- examples

A	A^T
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
$[5]$	$[5]$
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$	$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Exercises:

A) Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

Determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

1. (a) BA (b) AB^T (c) $AC + D$

(d) $E(AC)$ (e) $A - 3E^T$ (f) $E(5B + A)$

2. (a) CD^T (b) DC (c) $BC - 3D$

(d) $D^T(BE)$ (e) $B^TD + ED$ (f) $BA^T + D$

B) In each part, find a 4×4 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

(a) $a_{ij} = 0 \quad \text{if} \quad i \neq j$

(b) $a_{ij} = 0 \quad \text{if} \quad i > j$

(c) $a_{ij} = 0 \quad \text{if} \quad i < j$

(d) $a_{ij} = 0 \quad \text{if} \quad |i - j| > 1$

C) Find a 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i + j$

(b) $a_{ij} = i^{j-1}$

(c) $a_{ij} = \begin{cases} 1 & \text{if} \quad |i - j| > 1 \\ -1 & \text{if} \quad |i - j| \le 1 \end{cases}$

DEFINITION If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_r$ with ***coefficients*** $c_1, c_2, \ldots, c_r$.

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and $\mathbf{x}$ an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix equation

$$\begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The $m \times 1$ matrix on the left side of this equation can be written as a product to give

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, $\mathbf{x}$, and $\mathbf{b}$, respectively, then we can replace the original system of m equations in n unknowns by the single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

Example: Find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given linear system as a single matrix equation $A\mathbf{x} = \mathbf{b}$

(a)
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 \qquad - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 \qquad - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

Examples: Express the matrix equation as a system of linear equations

(a) $\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

Example: Solve the matrix equation for *a*, *b*, *c*, and *d*.

$$\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$$

$$\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$