Linear Algebra

Lecture Notes 3

Matrices

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Determinant of the matrix is expressed in mathematical symbol and can be calculated as follows

Symbol of Determinant

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If the matrix has a non-zero determinant, it is said to be **non-singular**; otherwise it is said to be **singular.**

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$\det A = ad - bc$

Examples:

- The determinant of $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = (3 \times 5) - (2 \times 1) = 15 - 2 = 13.$
- The determinant of $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = (1 \times 2) - (-1 \times 0) = 2 - 0 = 2.$

Determinants by Cofactor Expansion

If A is a square matrix, then the ***minor of entry a_{ij}*** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. Then the ***cofactor of entry a_{ij}*** is defined below:

$$C_{ij} = (-1)^{i+j}\, M_{ij}$$

Example: Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

Similarly, the minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Remark Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example,

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

Example:1.

Find all minors and cofactors of the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

DEFINITION If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

[cofactor expansion along the *j*th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

[cofactor expansion along the *i*th row]

Example:

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

Evaluate det(A) by cofactor expansion along the first column of A.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

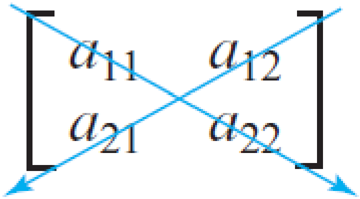
***Theorem**: If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then **det(A)** is the product of the entries on the main diagonal of the matrix; that is,*

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn}$$

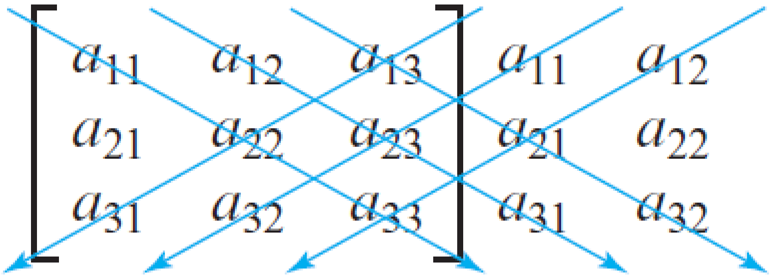
For example:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}$$

Note : The ***arrow technique*** works only for determinants of 2 × 2 and 3 × 3 matrices. It *does not* work for matrices of size 4 ×4 or higher.



$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$



$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Examples:

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

Inverse of a Matrix

When we **multiply a number** by its **reciprocal** we get **1**

$$8 \times (1/8) = \mathbf{1}$$

When we **multiply a matrix** by its **inverse** we get the **Identity Matrix** :

$$A \times A^{-1} = \mathbf{I}$$

The inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Sometimes there is no inverse at all.

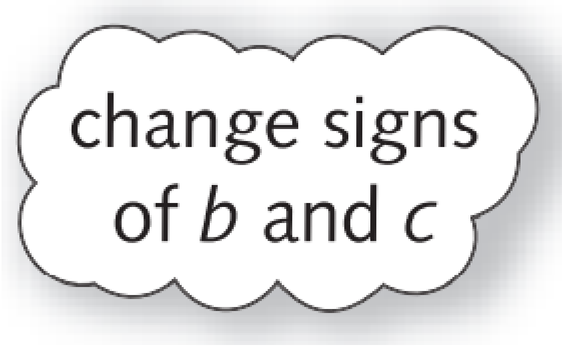
Inverse Formula

For a given 2x2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of A is defined by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



Example: Find (where possible) the inverse of the following matrices. Are these matrices singular or nonsingular?

$$\mathbf{A} = \begin{bmatrix} 6 & 4 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$$

Inverse by method of Cofactors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det \mathbf{A} \neq 0.$$

Step:1. **Find Matrix of cofactors**

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. **Find Adjoint of matrix A , adj(A)**

$$\mathbf{Adj(A)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A) $\neq$ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

Example: 3 . Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$ by the method of cofactors.

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \ C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Matrix of cofactors, $C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

Adjoint of matrix A, $\text{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 2(-12)+0(-4)+3(6)$$
$$=-24 + 18 = -6 \neq 0$$

Inverse of the matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

Solving a Linear System Using A^{-1}

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix **b**, the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution, namely,

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

Example:

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$x_1 = 1, x_2 = -1, x_3 = 2.$

Cramer's Rule

If $\mathbf{Ax} = \mathbf{b}$ *is a system of n linear equations in n unknowns such that* $\det(A) \neq 0$, *then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j *is the matrix obtained by replacing the entries in the* j*th column of* A *by the entries in the matrix*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example: Using Cramer's Rule to Solve a Linear System

$$\begin{aligned} x_1 + + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

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$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$