

# Lecture 4

## System of Linear Equations , Quadratics and Circles

- Solving a System of Linear Equations
- Quadratics and Parabolas
- Methods of solving quadratic equations
  - Quadratic Formula
- Circles

## Solving a System of Linear Equations using Substitution Method

To solve a system of equations by substitution, solve one of the equations for a variable, for example  $x$ . Then replace that variable in the other equation with the terms you deemed equal and solve for the other variable,  $y$ . The solution to the system of equations is always an ordered pair  $(x, y)$ .

**Example: Solve the following system by substitution.**

$$2x - 3y = -2$$

$$4x + y = 24$$

Solving the second equation for  $y$

$$y = -4x + 24$$

Plugging this in the first equation, and solve the value of  $x$ :

$$2x - 3(-4x + 24) = -2$$

$$2x + 12x - 72 = -2$$

$$14x = 70$$

$$x = 5$$

$$y = -4(5) + 24 = -20 + 24 = 4$$

Then my solution is the following point:

$$\text{solution: } (x, y) = (5, 4)$$

## Solving Systems of Equations by Substitution Worksheet

Solve.

$$\begin{cases} \text{1} & x = -y \\ & 3x + 4y = -8 \end{cases}$$

$$\begin{cases} \text{2} & y = -3x + 4 \\ & y = -5x - 17 \end{cases}$$

$$\begin{cases} \text{3} & y = -x - 6 \\ & y = x - 4 \end{cases}$$

$$\begin{cases} \text{4} & 2x - 3y = 1 \\ & -6x + 9y = -3 \end{cases}$$

$$\begin{cases} \text{5} & 7x + 2y = -19 \\ & -x + 2y = 21 \end{cases}$$

$$\begin{cases} \text{6} & x = y - 4 \\ & -2x + 3y = 6 \end{cases}$$

# Graphically Solving a System of Linear Equations

**Step 1:** Plot both lines on the same graph.

**Step 2:** Find the point of intersection of the two lines.

**Example:** Find the solution to the following system of linear equations by graphing:

$$\begin{aligned}x + y &= 5 \\x - 2y &= 2\end{aligned}$$

Line 1: Solve for the  $x$ -intercept by setting  $y$  equal to 0 and solving

$$\begin{aligned}x + y &= 5 \\x + 0 &= 5 \\x &= 5\end{aligned}$$

This gives us the coordinate point  $(5,0)$  as the  $x$ -intercept for Line 1.

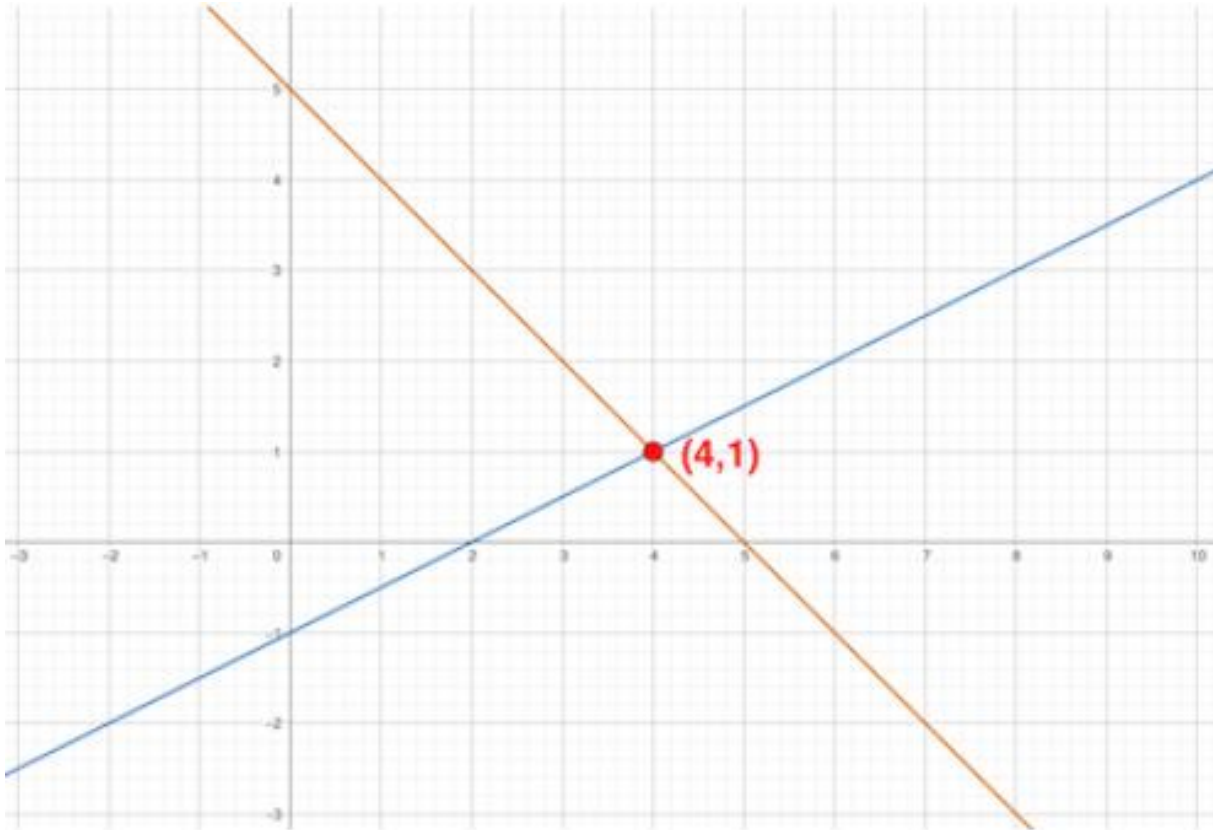
Solve for the  $y$ -intercept by setting  $x$  equal to 0 and solving.

$$\begin{aligned}x + y &= 5 \\0 + y &= 5 \\y &= 5\end{aligned}$$

This gives us the coordinate point  $(0, 5)$  as the  $y$ -intercept for Line 1

**Line 2:** We can also solve the second equation to find the following coordinates  $(2, 0)$  and  $(0, -1)$

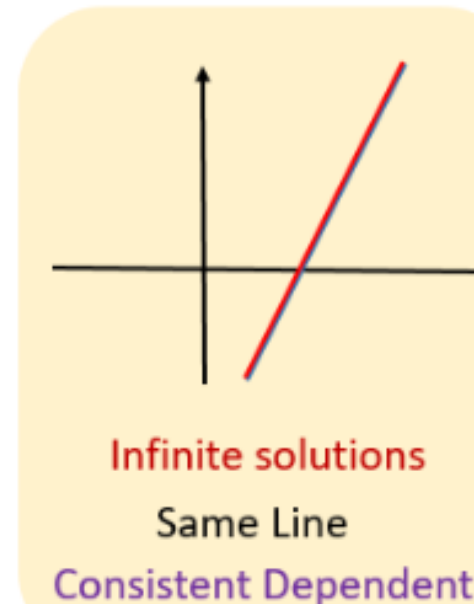
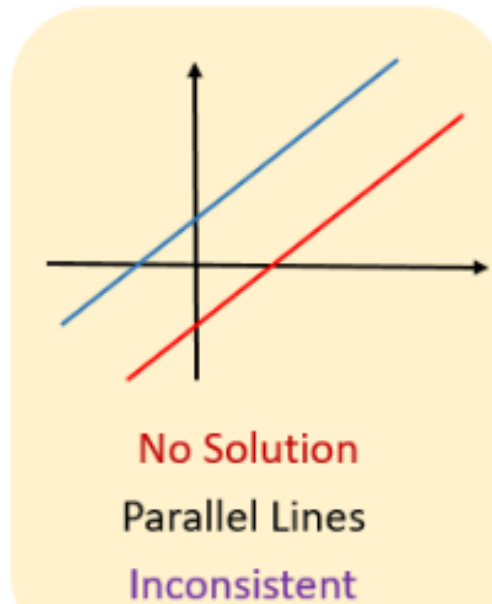
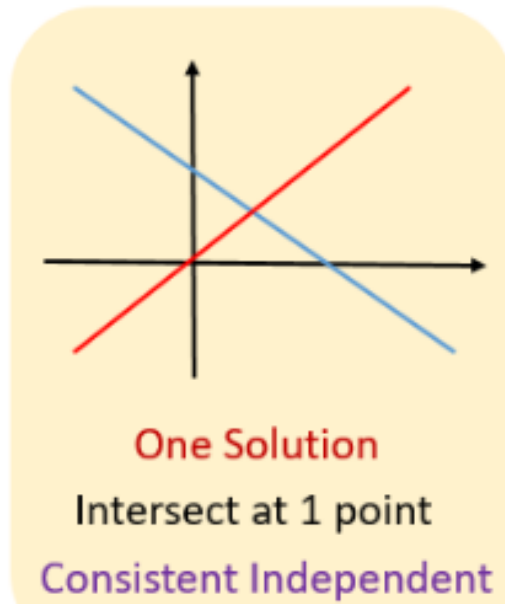
Let's graph the two lines using the intercepts:



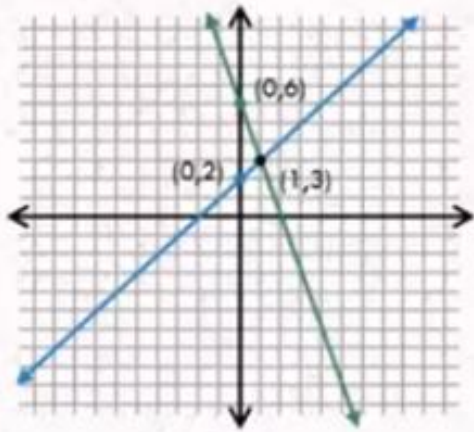
Therefore, the solution to the system of the linear equations is  $(4, 1)$  .

## Types of Solutions:

- **One intersection point:** The system has one unique solution. Also called a consistent, independent system.
- **Parallel lines:** The lines never intersect, meaning the system has no solution. Also called an inconsistent system.
- **Same line:** The lines overlap completely, meaning the system has infinitely many solutions. Also called a consistent, dependent system.



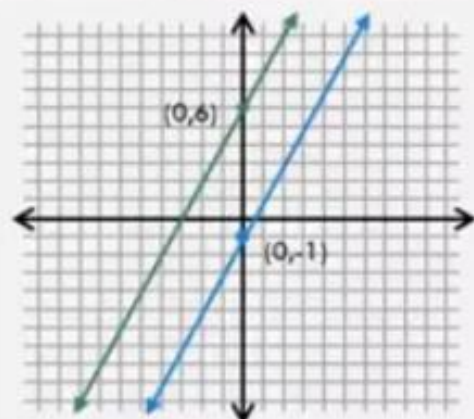
### Graphing – 3 Possibilities



$$y = -3x + 6$$

$$y = x + 2$$

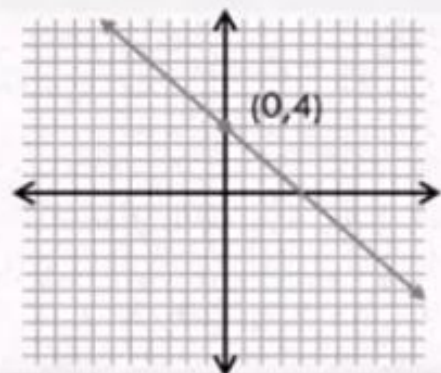
One Solution:  
 $(1, 3)$



$$y = 2x + 6$$

$$y = 2x - 1$$

No Solution



$$y = -x + 4$$

$$2y = -2x + 8$$

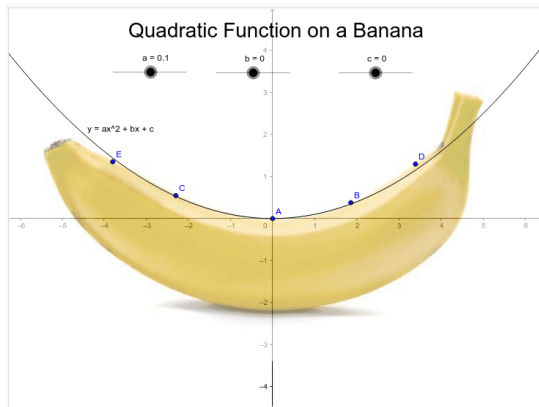
Infinite  
Solutions

# Quadratics

Root of x square make it quadratic

$$2x^2 - 3x + 1 = 0$$

- Quadratic expressions: A *quadratic expression (in x)* has general form:

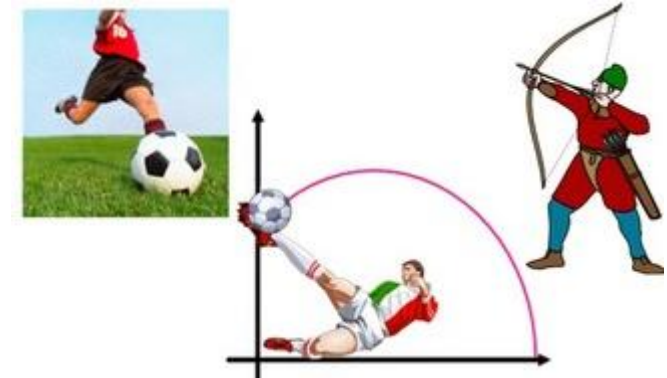
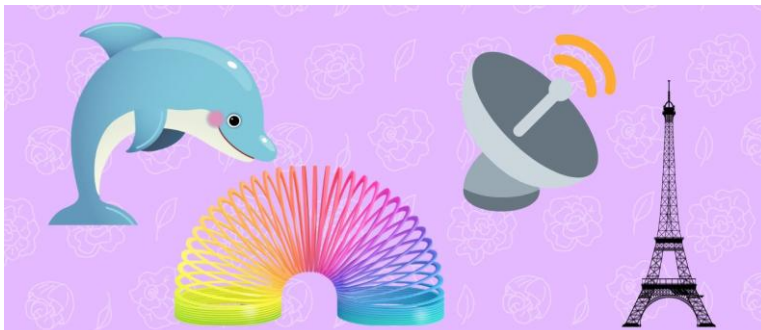


$$ax^2 + bx + c$$

Ex.  $3x^2 - 16x + 36$

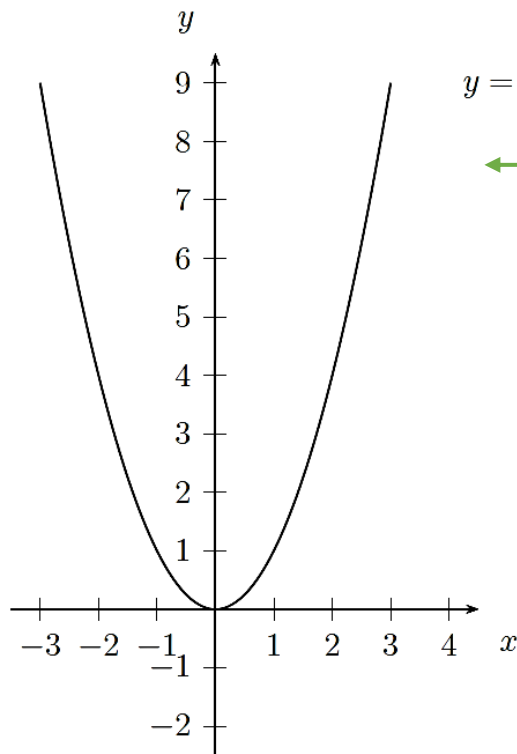
Ex.  $\frac{4}{7}x^2 - x + 5$

.....



# Parabolas

- Parabola is a curve in the plane that is result of any combinations of (positive or negative) dilations, rotations or translations of the curve



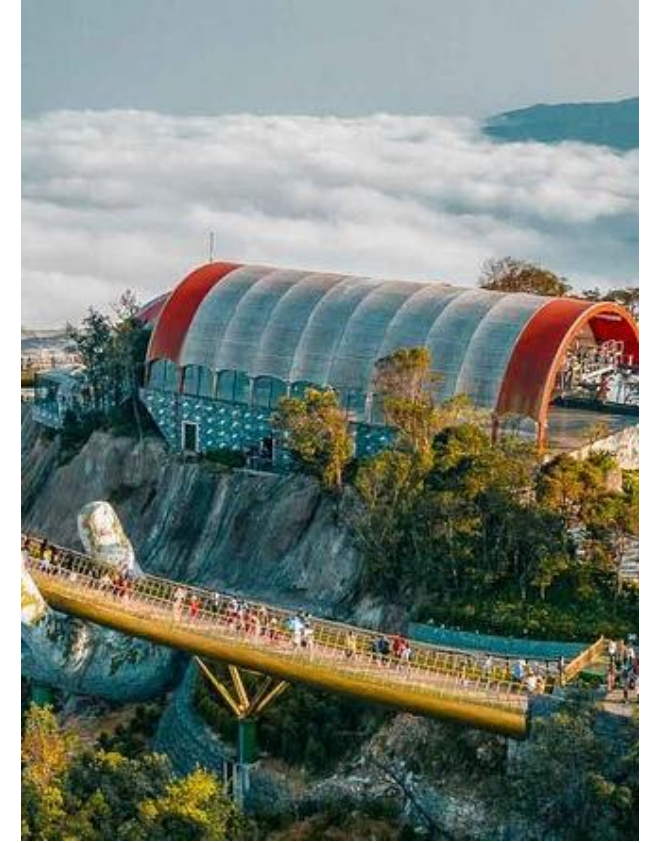
$$y = x^2$$

$$\underline{y = ax^2}$$

Parent function

Facing upwards,  $a > 0$   
Facing downwards,  $a < 0$





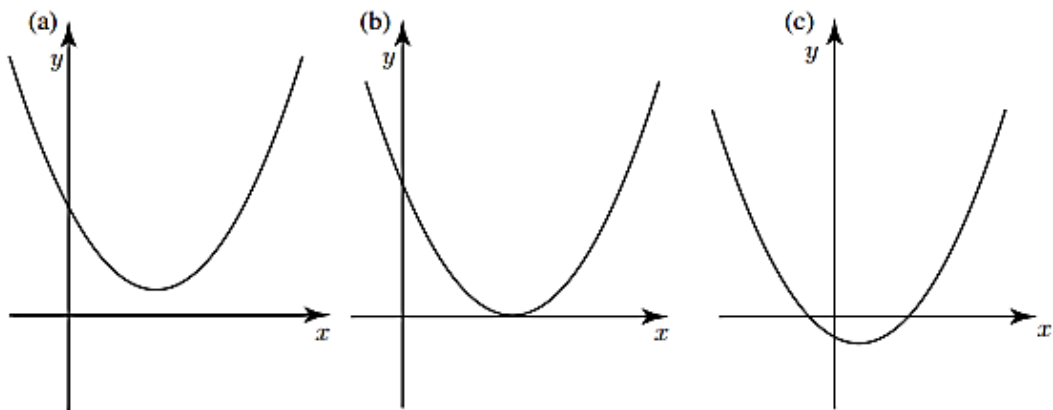
# Parabola

- Consider the satellite dish. These structures have a parabolic shape, allowing the reflection and focus of radio waves (parabolic shape helps receive and transmit signals).

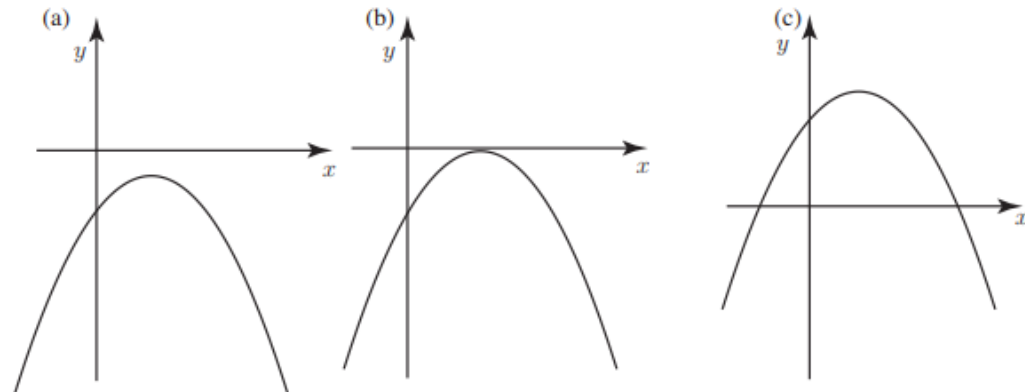




**Graphs of  $y = ax^2 + bx + c$  when  $a$  is positive**

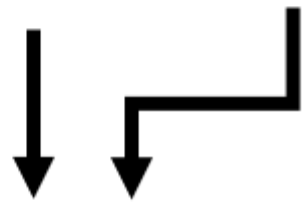


**Graphs of  $y = ax^2 + bx + c$  when  $a$  is negative**



# Vertex Form of Parabolas

$$y = a(x - h)^2 + k$$



**vertex:**  $(h, k)$

$h$  = x-coordinate of vertex

$k$  = y-coordinate of vertex

$$f(x) = a(x - h)^2 + k$$

➤  $a \neq 0$ , if  $a$  is ...

- Positive: opens up  $\cup$
- Negative: opens down  $\cap$

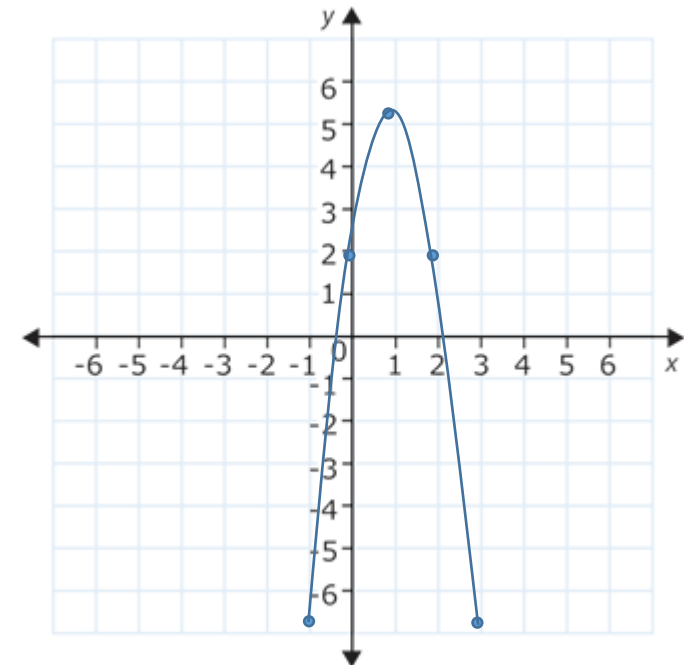
➤ Vertex  $(h, k)$

$$y = -3(x - 1)^2 + 5$$

- $a = -3$  – opens down
  - $h = 1$
  - $k = 5$
- } Vertex
- So, vertex is at point  $(1, 5)$

↑  
Vertex  
↓

$x$	$y$
-1	-7
0	2
1	5
2	2
3	-7



$$f(x) = ax^2 + bx + c$$

### Find the Vertex

$$f(x) = ax^2 + bx + c$$

$$x = \frac{-b}{2a}$$

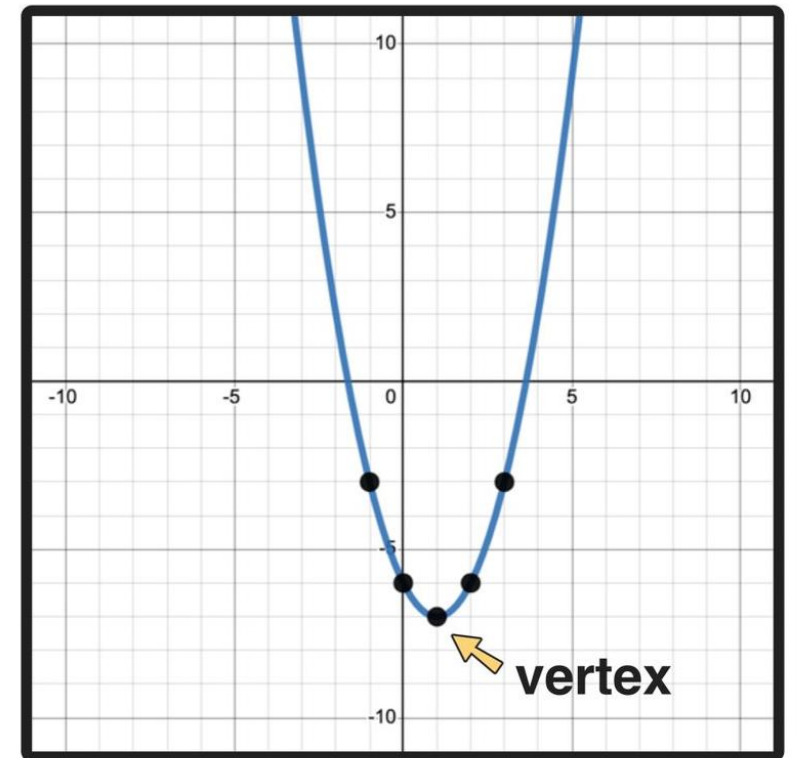
**(1, -7)**

### Build a Function Table

$$f(x) = x^2 - 2x - 6$$

x	y
-1	-3
0	-6
1	-7
2	-6
3	-3

### Plot Points and Graph



$$f(x) = ax^2 + bx + c$$

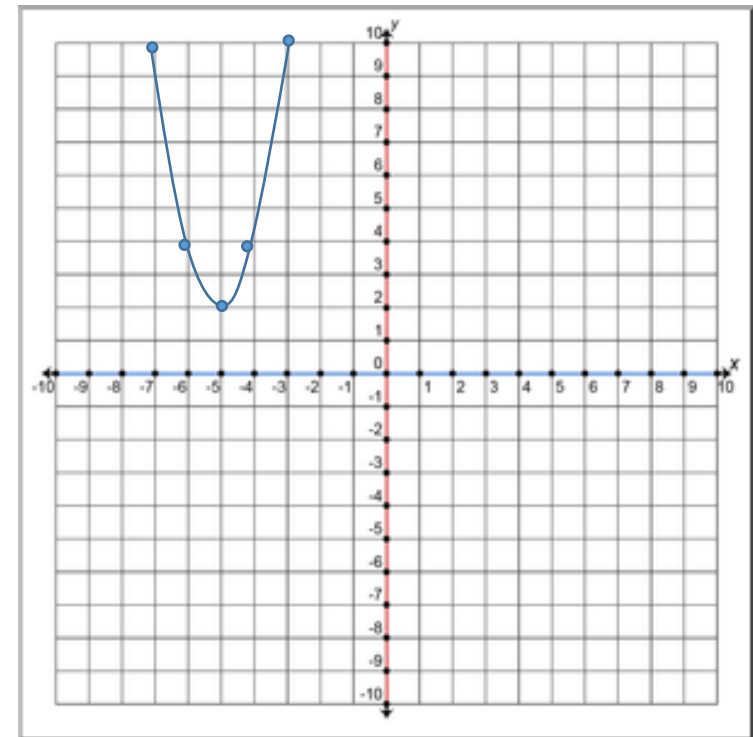
$$\text{Vertex: } \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$y = 2x^2 + 20x + 52$$

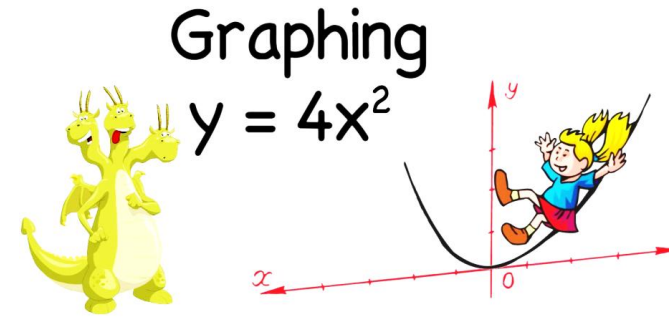
- $-\frac{b}{2a} = -\frac{20}{4} = -5$
- $f\left(-\frac{b}{2a}\right) = (2 \times (-5)^2) + 20 \times (-5) + 52 = 2$

↑  
*Vertex*  
↓

$x$	$y$
-7	10
-6	4
-5	2
-4	4
-3	10



# Problems



(i)  $y = \frac{1}{2}x^2$

1)  $y = x^2 + 12x + 38$

(ii)  $y = -3x^2$

2)  $y = 2x^2 - 16x + 30$

(iii)  $y = x^2 + 1$

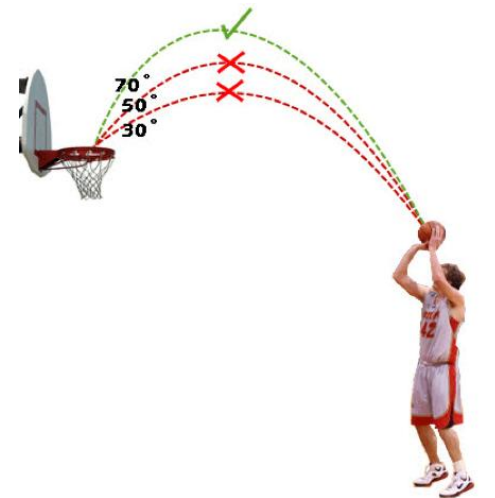
3)  $y = 2x^2 + 6x$

(iv)  $y = (x + 1)^2$

4)  $y = -x^2 - 10x - 26$

(v)  $y = -(x - 2)^2$

(vi)  $y = (x - 1)^2 - 2$



# THE QUADRATIC FORMULA

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If  $ax^2 + bx + c = 0$  but  $a \neq 0$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

DISCRIMINANT

→  $b^2 - 4ac > 0$  two real solutions

→  $b^2 - 4ac = 0$  one real solutions

→  $b^2 - 4ac < 0$  zero real solutions



Example 1:

$$\text{Solve: } x^2 + 2x - 15 = 0$$

Solution using Quadratic Formula:  $a = 1$ ;  $b = 2$ ;  $c = -15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-15)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 + 60}}{2} = \frac{-2 \pm \sqrt{64}}{2} = \frac{-2 \pm 8}{2}$$

$$x = \frac{-2 + 8}{2} = \frac{6}{2} = 3$$

$$x = \frac{-2 - 8}{2} = \frac{-10}{2} = -5$$

## Example 2: Solve the following equation

•  $\frac{2}{x+1} - \frac{2}{x+2} = 7$   $\implies$  In order to solve this type of questions, we have to get the form of quadratic equation:  $ax^2 + bx + c$

check for the denominator, should be the same

$$\frac{2 \times (x+2)}{(x+1)(x+2)} - \frac{2 \times (x+1)}{(x+1)(x+2)} = 7$$

$$\frac{2x+4-2x-2}{(x+1)(x+2)} = 7$$

$$2 = 7(x+1)(x+2)$$

$$2 = 7(x^2 + 3x + 2)$$

$$7x^2 + 21x + 14 = 2$$

$$7x^2 + 21x + 12 = 0$$

$$a = 7$$

$$b = 21$$

$$c = 12$$

Quadratic Formula:

$$x = \frac{-21 \pm \sqrt{21^2 - 4 \times 7 \times 12}}{2 \times 7}$$

$$x = \frac{-21 \pm \sqrt{105}}{14}$$

**Write each quadratic equation in standard form then solve them.**

1.)  $3m^2 + 2m = 7$

2.)  $3x^2 - 6 = -2x$

3.)  $x^2 = -3x + 2$

4.)  $(x + 2)(x + 9) = 0$

5.)  $(x + 7)(x - 3) = 0$

6.)  $(x - 8)(x - 1) = -7$

7.)  $x^2 = 81$

8.)  $4m^2 + 3m = 1$

9.)  $4x^2 - 9 = -3x$

10.)  $x^2 = -4x + 9$

11.)  $2m^2 + 3m = 5$

12.)  $7x^2 - 1 = -3x$

13.)  $x^2 = -2x + 1$

14.)  $(x + 1)(x + 3) = 0$

15.)  $(x + 10)(x - 1) = 0$

16.)  $(x - 11)(x - 2) = -4$

17.)  $x^2 = 49$

18.)  $3m^2 + 7m = 4$

19.)  $4x^2 - 1 = -7x$

20.)  $x^2 = -3x + 10$



*Try this!*

1) A right triangle has one leg that is 3 m shorter than the other leg. The triangle has area of  $54 \text{ m}^2$ . Find the lengths of the legs.

2) The hypotenuse of a right triangle is 3 in. longer than the longer leg. The shorter leg is 3 in. shorter than the longer leg. Find the lengths of the sides of the triangle.

# Circle

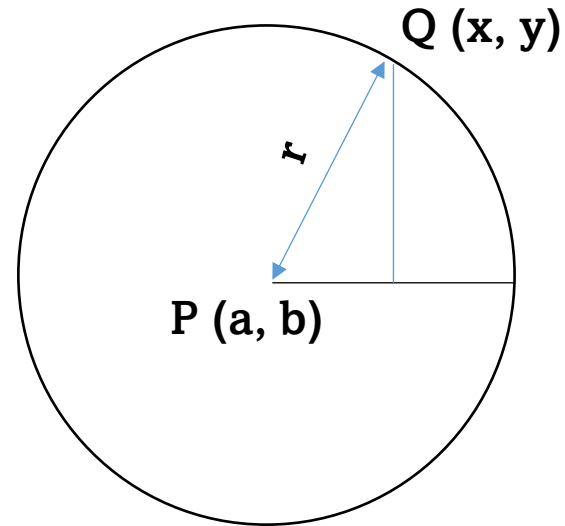
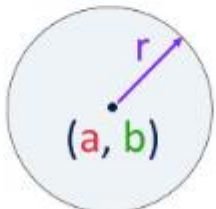


- **r - radius**

The Equation of a Circle

$$(x - a)^2 + (y - b)^2 = r^2$$

With centre of  $(a, b)$  and a radius of  $r$



By Pythagoras:

$$r^2 = (x - a)^2 + (y - b)^2$$

Equation of the circle

### Circle whose center is at the origin

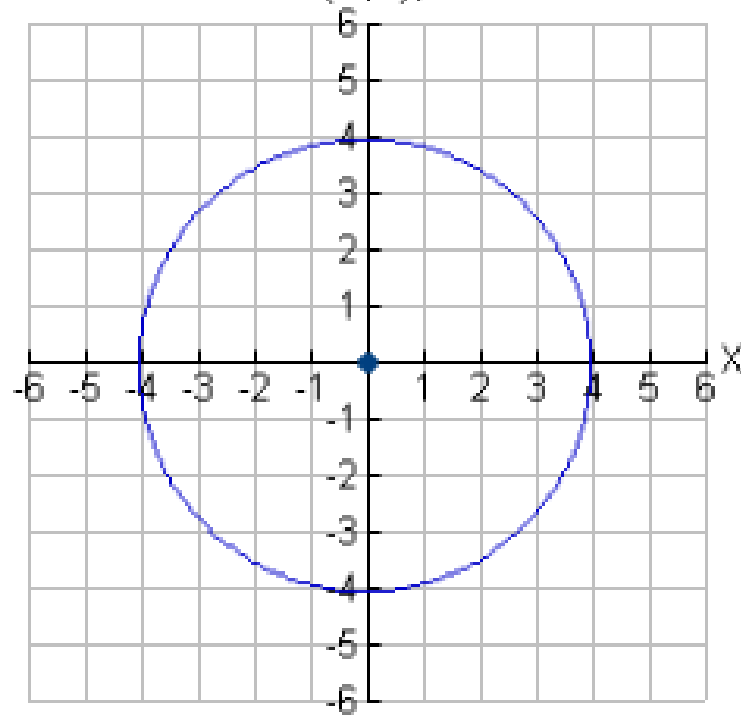
Equation:  $x^2 + y^2 = r^2$

Example: Circle with center (0,0), radius 4

$$x^2 + y^2 = 16$$

Graph:

Center at (0,0), radius 4



### Circle whose center is at (h,k)

(This will be referred to as the "center-radius form". It may also be referred to as "standard form".)

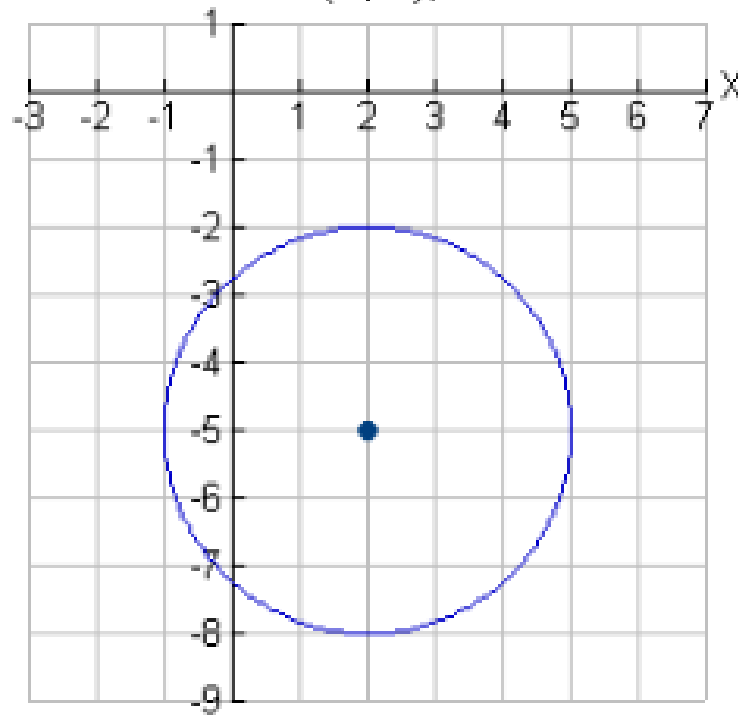
Equation:  $(x-h)^2 + (y-k)^2 = r^2$

Example: Circle with center (2,-5), radius 3

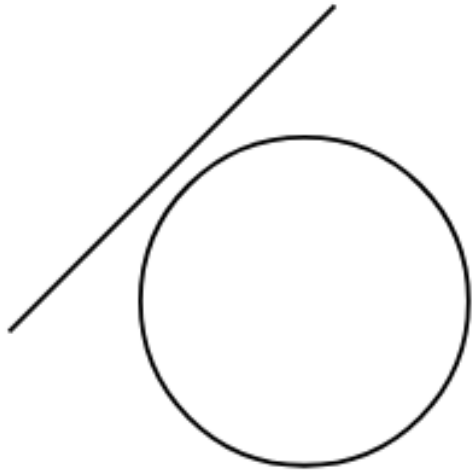
$$(x-2)^2 + (y+5)^2 = 9$$

Graph:

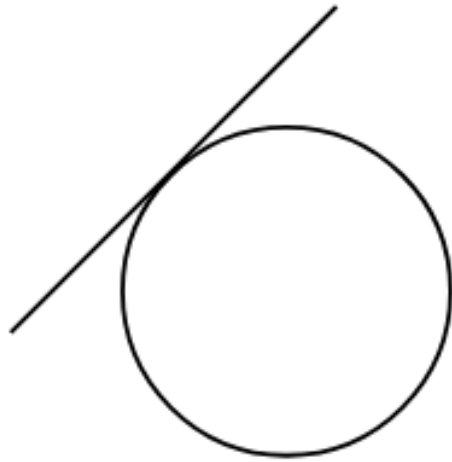
Center at (2,-5), radius 3



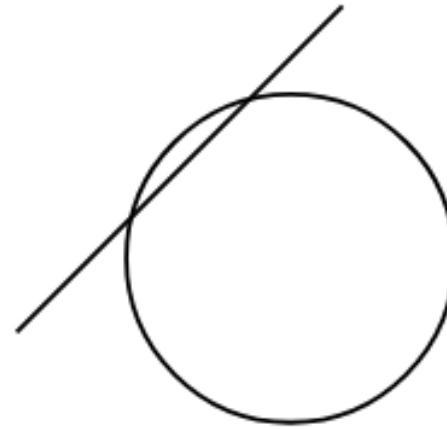
# Circle-Line Intersection



Line outside the circle



Line touches the circle



Line through the circle

## Example: Find x and y

$$(x - 2)^2 + (y + 3)^2 = 4$$
$$2x + 2y = -1$$

**Step 1.**  $2x + 2y = -1$

$$2y = -2x - 1$$

$$y = -x - \frac{1}{2}$$

**Step 2.**  $(x - 2)^2 + (y + 3)^2 = 4$

$$y \Leftrightarrow x$$

$$(x - 2)^2 + \left(-x - \frac{1}{2} + 3\right)^2 = 4$$

$$(x - 2)^2 + \left(-x + \frac{5}{2}\right)^2 = 4$$

$$2x^2 - 9x + \frac{25}{4} = 0$$

**Step 3.** Any method of solving quadratic equations:

$$x = \frac{9 \pm \sqrt{81 - 50}}{4} = \frac{9 \pm \sqrt{31}}{4}$$

(quadratic formula)

$$x_1 = \frac{9 + \sqrt{31}}{4}$$

$$x_2 = \frac{9 - \sqrt{31}}{4}$$



$$y_1 = -\frac{9 + \sqrt{31}}{4} - \frac{1}{2} = \frac{-11 - \sqrt{31}}{4}$$

$$y_2 = -\frac{9 - \sqrt{31}}{4} - \frac{1}{2} = \frac{-11 + \sqrt{31}}{4}$$

➤ At what points given equations' graphs intersect?

$$\begin{cases} x + y = 1 \\ (x - 2)^2 + (y + 1)^2 = 8 \end{cases}$$

$$\begin{cases} x^2 + y^2 = 9 \\ (x - 1) + y = 2 \end{cases}$$

➤ At what points does the line with equation  $y = 2x + 5$  intersect a circle with radius 2 and center  $(0, 5)$ ?



**Q1:** Consider the circle  $(x - 5)^2 + (y + 2)^2 = 25$ . Is the line  $y - 3 = 0$  tangent to, intersecting, or disjoint from the circle?

- A disjoint from the circle
- B tangent to the circle
- C intersecting the circle

**Problem:**  $-4y - 3x - 2 = 0$  meets the circle  $x^2 + y^2 + 6x + 4y = 0$  in points A and B. Find the coordinates of A and B.

1.) Find the coordinates of all points where the given parabola and line intersect each other.

a) 
$$\begin{cases} y = x^2 + 6x - 10 \\ y + 5 = 2x \end{cases}$$

b) 
$$\begin{cases} y = 2x^2 - 8x + 1 \\ y = -4x - 1 \end{cases}$$

c) 
$$\begin{cases} y = -x^2 + 3x - 5 \\ y = x + 1 \end{cases}$$

2.) Find the coordinates of all points where the given circle and line intersect each other.

a) 
$$\begin{cases} (x - 4)^2 + (y + 1)^2 = 20 \\ x + 3y = 11 \end{cases}$$

c) 
$$\begin{cases} (x - 3)^2 + (y - 1)^2 = 20 \\ x - y = -5 \end{cases}$$

b) 
$$\begin{cases} (x + 5)^2 + (y - 6)^2 = 10 \\ y = -\frac{1}{3}x + 1 \end{cases}$$

d) 
$$\begin{cases} (x + 1)^2 + (y - 2)^2 = 25 \\ 4y = 3x + 11 \end{cases}$$