Linear Algebra

Lecture Notes 4

Euclidean Vector Spaces

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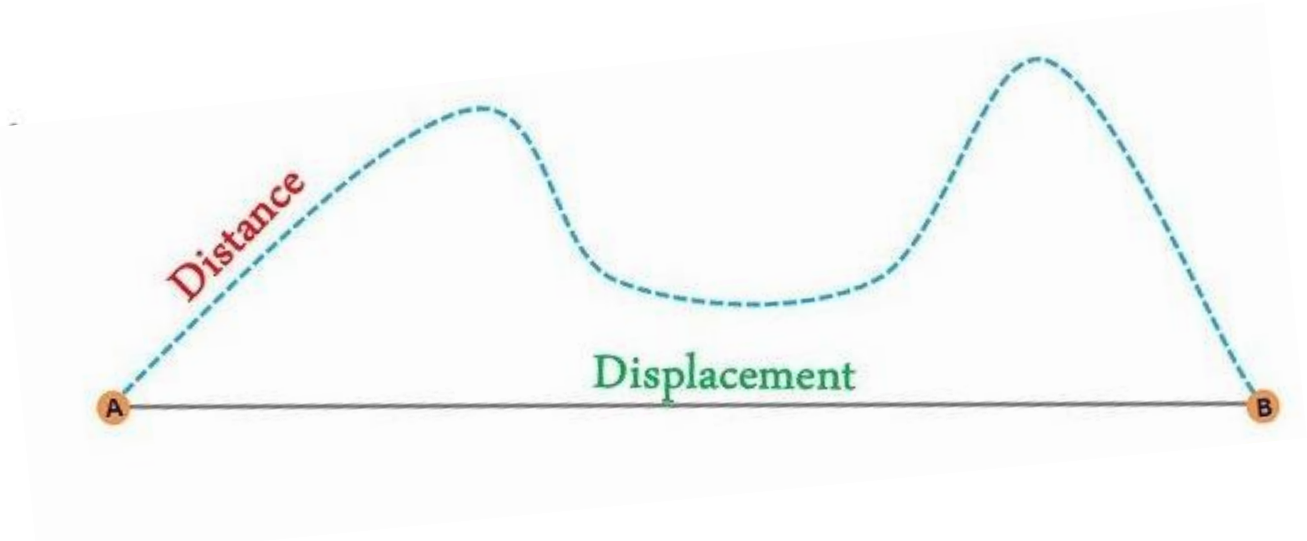
Scalar and Vector

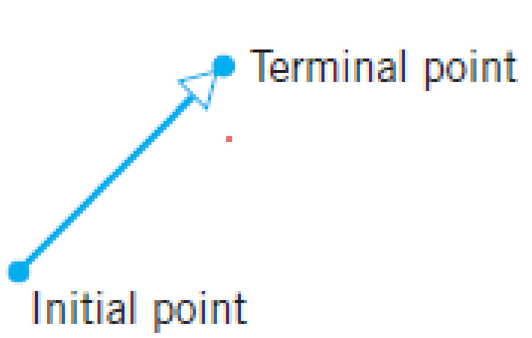
- **Scalar**: A scalar only tells us **how much** of something there is.
- **Vector**: A vector tells us both **how much** and **which direction**.

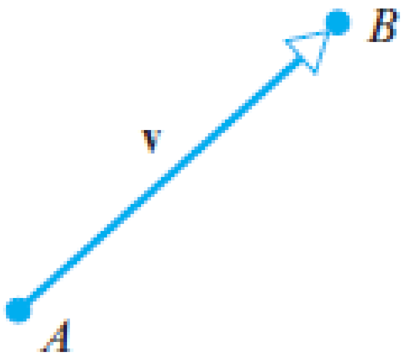
Scalar	vs	Vector
Distance		Displacement
Speed		Velocity
Mass		Force
Temperature		Accelerat



Vectors in 2-Space

A vector $\boldsymbol{v}$ with initial point A and terminal point B can be shown below



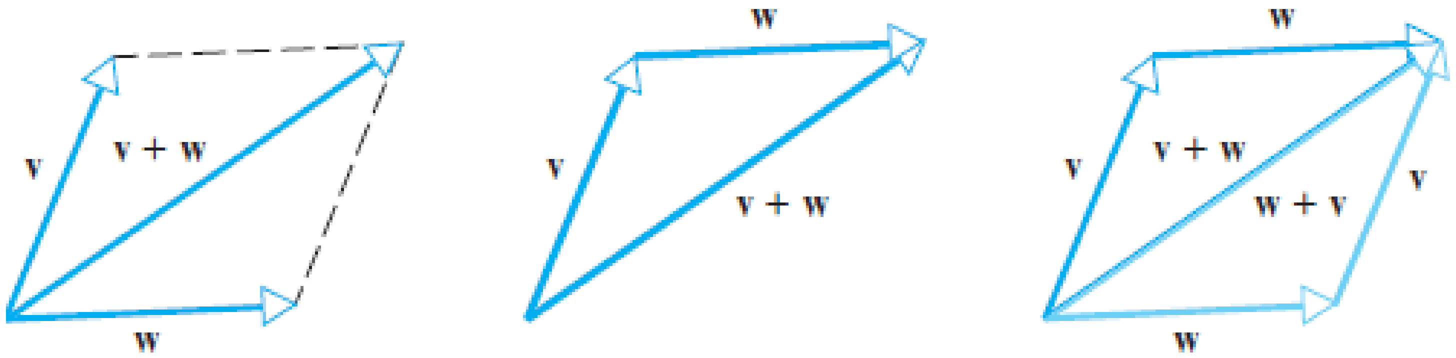


$$\mathbf{v} = \overrightarrow{AB}$$

Vector Addition

If **v** and **w** are vectors in 2-space

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

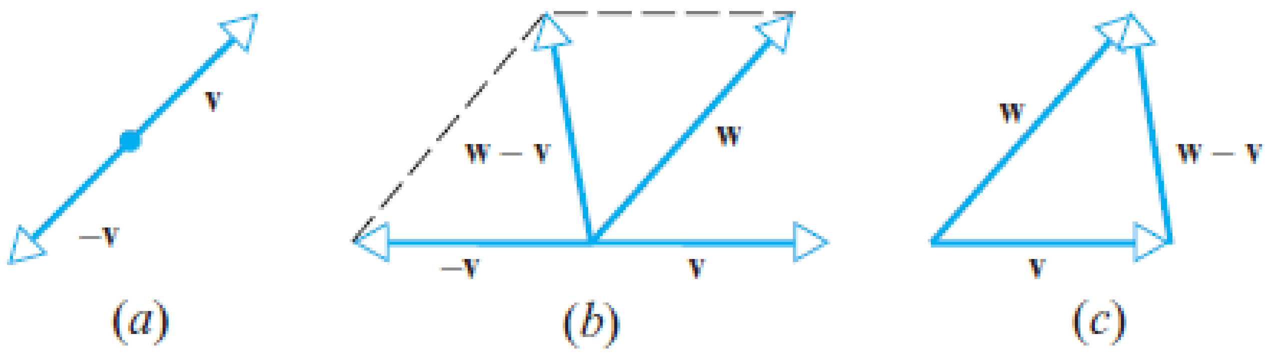


(a) (b) (c)

Vector Subtraction

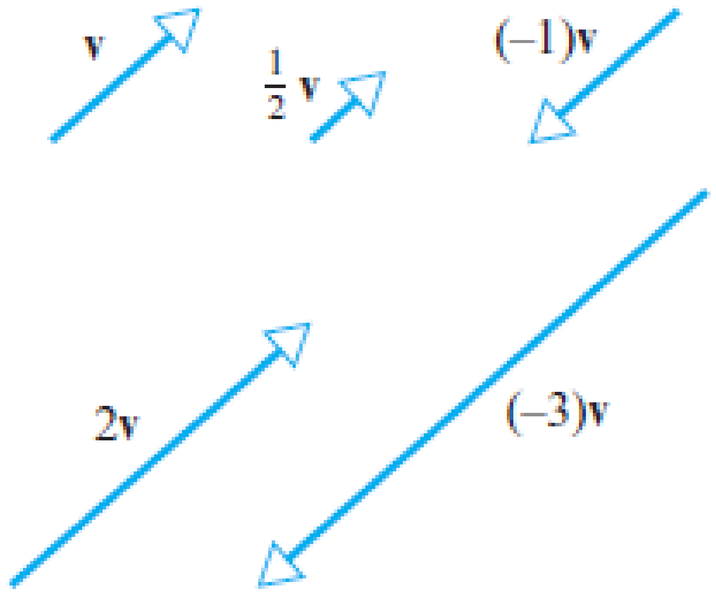
The ***negative*** of a vector $\boldsymbol{v}$, denoted by $-\boldsymbol{v}$, is the vector that has the same length as $\boldsymbol{v}$ but is oppositely directed, and the ***difference*** of $\boldsymbol{v}$ from $\boldsymbol{w}$, denoted by $\boldsymbol{w} - \boldsymbol{v}$, is taken to be the sum

$$\boldsymbol{w} - \boldsymbol{v} = \boldsymbol{w} + (-\boldsymbol{v})$$



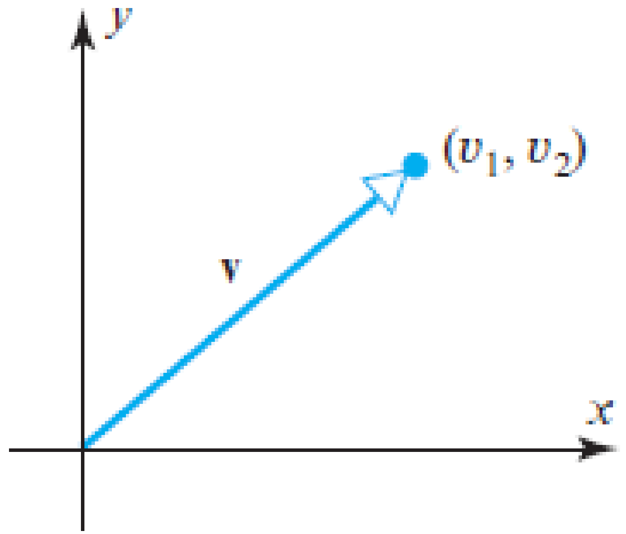
Scalar Multiplication

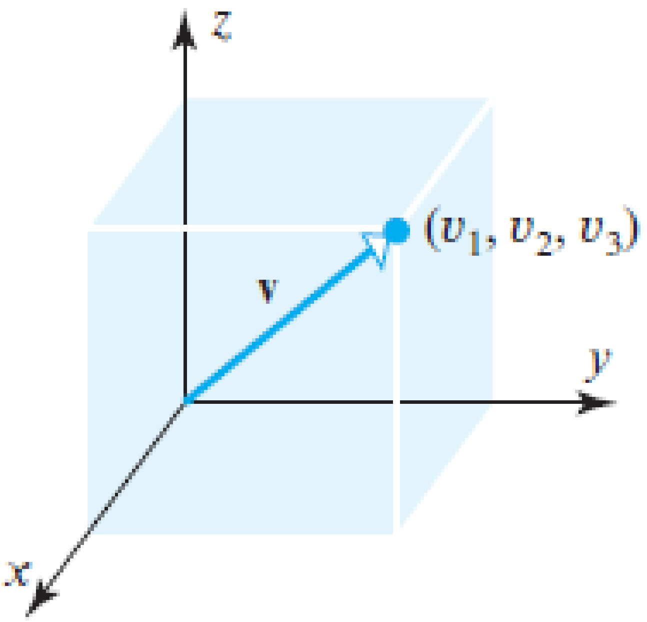
If $\boldsymbol{v}$ is a nonzero vector in 2-space or 3-space, and if k is a nonzero scalar, then we define the ***scalar product of $\boldsymbol{v}$ by $\boldsymbol{k}$***



Vectors in Coordinate

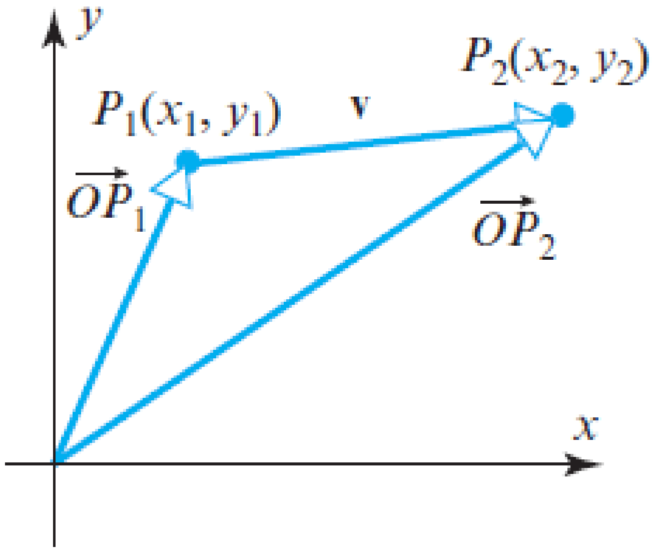
- $\boldsymbol{v} = (v_1, v_2)$ *is* denoted a vector $\boldsymbol{v}$ in 2-space with components (v_1, v_2) , and $\boldsymbol{v} = (v_1, v_2, v_3)$ *is* denoted a vector $\boldsymbol{v}$ in 3-space with components (v_1, v_2, v_3)
- The component forms of the zero vector are 0 = (0,0) in 2-space and 0 = (0, 0, 0) in 3-space.





It is sometimes necessary to consider vectors whose initial points are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$



$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

As you might expect, the components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ are given by

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

DEFINITION If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by R^n.

DEFINITION If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n)$$

Example:

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4)$$

$$-\mathbf{w} = (-4, -2, -1), \qquad \mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)$$

THEOREM

If $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in R^n, and if k and m are scalars, then:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(f) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(g) $k(m\mathbf{u}) = (km)\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

(a) $0\mathbf{v} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{v} = -\mathbf{v}$

DEFINITION If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

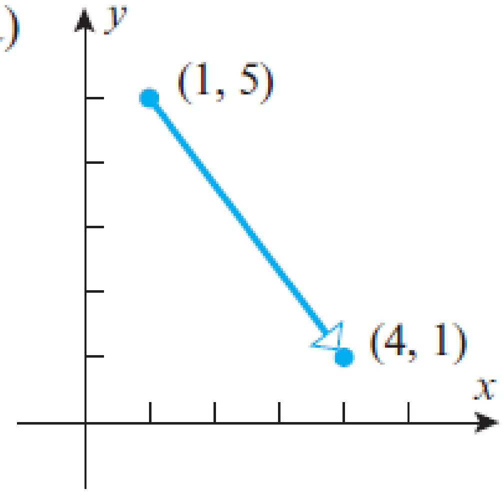
$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

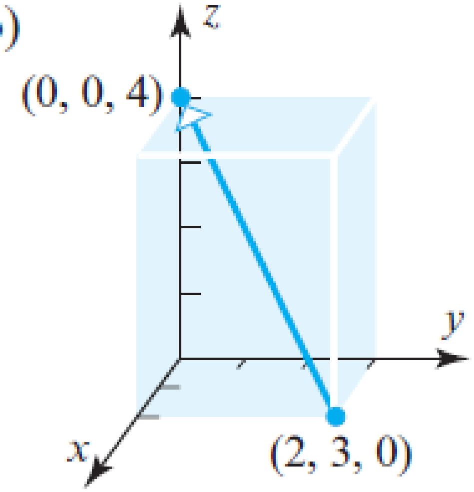
Exercises:

A) Find the components of the vector.

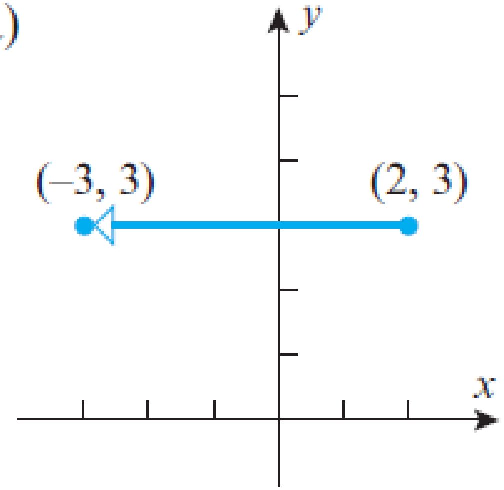
1. (a)



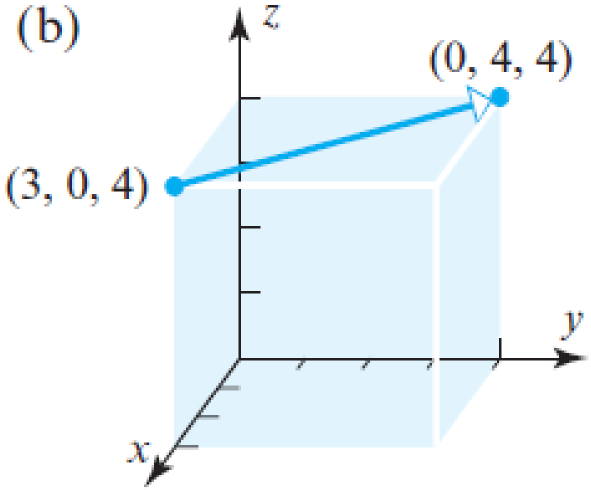
(b)



2. (a)



(b)



B)

(a) Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1, 2)$ and whose initial point is $A(1, 1)$.

(b) Find the initial point of the vector that is equivalent to $\mathbf{u} = (1, 1, 3)$ and whose terminal point is $B(-1, -1, 2)$.

C) Let $\mathbf{u} = (-3, 2, 1, 0)$, $\mathbf{v} = (4, 7, -3, 2)$, and $\mathbf{w} = (5, -2, 8, 1)$. Find the components of

(a) $\mathbf{v} - \mathbf{w}$

(b) $-\mathbf{u} + (\mathbf{v} - 4\mathbf{w})$

(c) $6(\mathbf{u} - 3\mathbf{v})$

(d) $(6\mathbf{v} - \mathbf{w}) - (4\mathbf{u} + \mathbf{v})$

D) Find scalars c_1, c_2, and c_3 for which the equation is satisfied.

a) $c_1(1, -1, 0) + c_2(3, 2, 1) + c_3(0, 1, 4) = (-1, 1, 19)$

b) $c_1(-1, 0, 2) + c_2(2, 2, -2) + c_3(1, -2, 1) = (-6, 12, 4)$