Linear Algebra

Lecture Notes 5

Norm of Vectors and Dot Product

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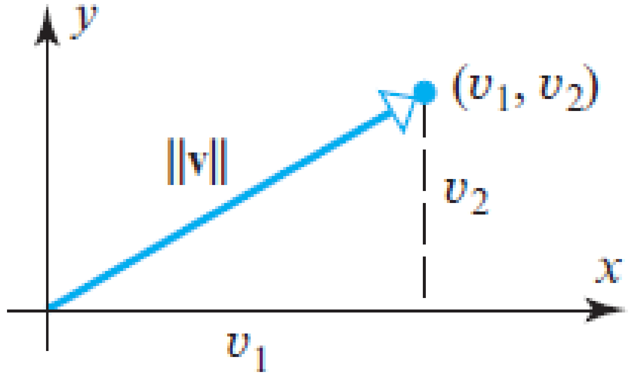
Norm of a Vector

The length of a vector $\boldsymbol{v}$ **is denoted by** $||\boldsymbol{v}||$, which is read as the *norm* of $\boldsymbol{v}$, the *length* of $\boldsymbol{v}$, or the *magnitude* of $\boldsymbol{v}$.

The norm of a vector (v_1, v_2) in R^2 is

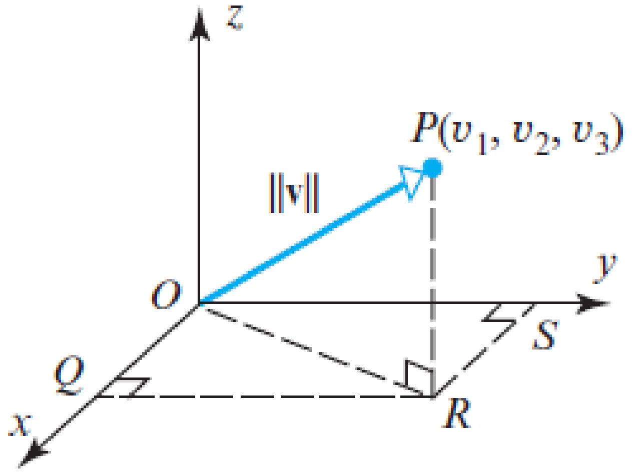
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$



(a)

Similarly, for a vector (v_1, v_2, v_3) in R^3

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$



(b)

DEFINITION If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Examples:

Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, $\boldsymbol{w} = (3, 6, -4)$.

(a) $\|\mathbf{u} + \mathbf{v}\|$

(b) $\|\mathbf{u}\| + \|\mathbf{v}\|$

(c) $\|-2\mathbf{u} + 2\mathbf{v}\|$

(d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$

Unit Vectors

A vector of norm 1 is called a ***unit vector.*** If v is any nonzero vector in R^n, then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

Example: Normalizing a Vector

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus,

$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

As a check, you may want to confirm that $\|\mathbf{u}\| = 1$.

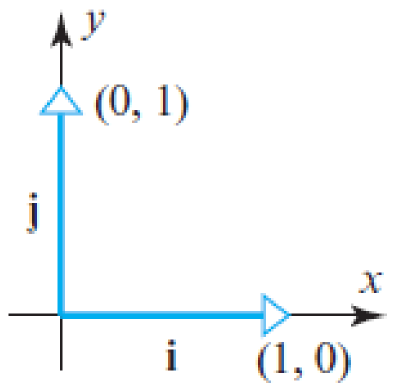
The Standard Unit Vectors

The unit vectors in the positive directions of the coordinate axes are called the ***standard unit vectors***. In R^2 these vectors are denoted by

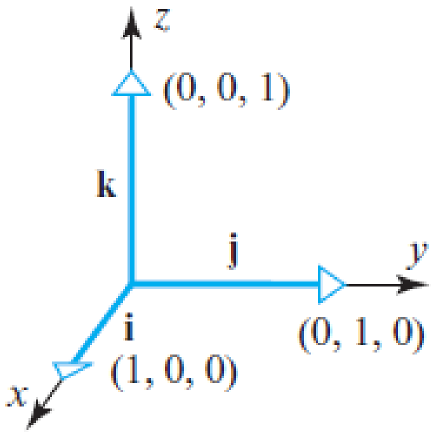
$$\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1)$$

and in R^3 by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1)$$



(a)



(b)

Moreover, we can generalize these formulas to R^n by defining the ***standard unit vectors in*** $\boldsymbol{R^n}$ to be

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

in which case every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

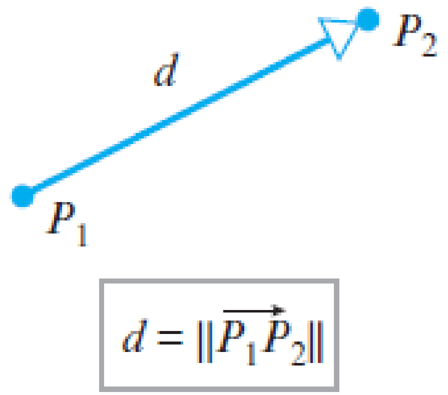
$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

Linear Combinations of Standard Unit Vectors

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$
$$(7, 3, -4, 5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4$$

Distance in R^n

If $P_1(x_1, y_1)$ and $P_2\ (x_2, y_2)$ are points in R^2



$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similarly, the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2\ (x_2, y_2, z_2)$ in 3-space is

$$d(\mathbf{u}, \mathbf{v}) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

DEFINITION If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the *distance* between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Example: If $\mathbf{u} = (1, 3, -2, 7)$ and $\mathbf{v} = (0, 7, 2, 2)$

then the distance between **u** and **v** is

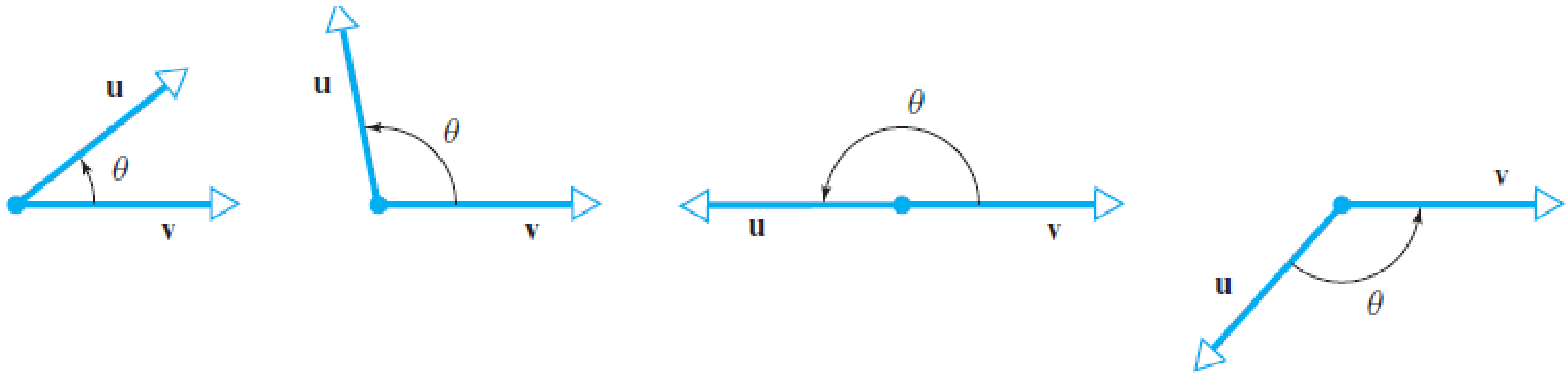
$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

Dot Product

DEFINITION If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the *dot product* (also called the *Euclidean inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

We define the ***angle between* u *and* v** to be the angle θ determined by **u** and **v** that satisfies the inequalities $0 \leq \theta \leq \pi$



DEFINITION If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the *dot product* (also called the *Euclidean inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The notion of "angle" between nonzero vectors **u** and **v** is given below

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)$$

Algebraic Properties

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2$$

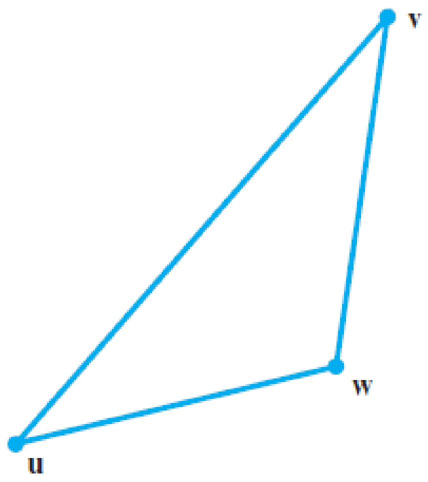
This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

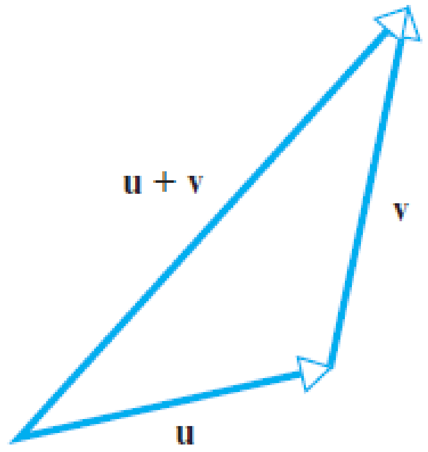
THEOREM *If* **u**, **v**, *and* **w** *are vectors in* R^n, *then:*

(a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ [Triangle inequality for vectors]

(b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ [Triangle inequality for distances]



$d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$



$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

DEFINITION Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be *orthogonal* (or *perpendicular*) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Examples:

(a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4.

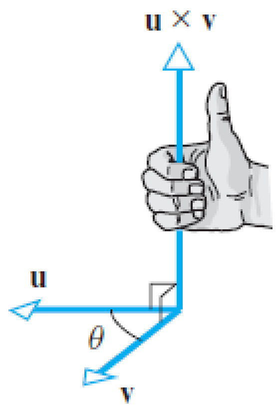
(b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3. Show that each ordered pair of vectors in S is orthogonal.

Cross Product

DEFINITION If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$



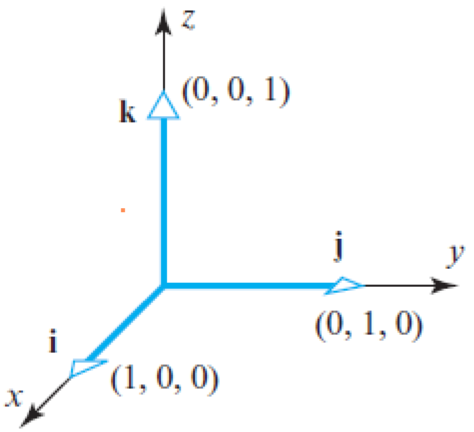
Example: Cross Products of the Standard Unit Vectors

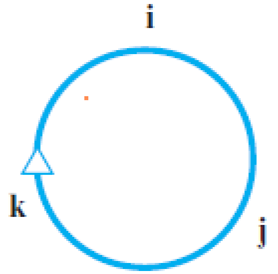
Show the following results

$$\mathbf{i}\times\mathbf{i}=\mathbf{0} \qquad \mathbf{j}\times\mathbf{j}=\mathbf{0} \qquad \mathbf{k}\times\mathbf{k}=\mathbf{0}$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$





$$\mathbf{i}\times\mathbf{j}=\left(\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}, -\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}, \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\right)=(0,0,1)=\mathbf{k}$$

Scalar triple product

The scalar triple product of $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ can be calculated from the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example: Calculate the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ of the vectors

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \quad \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

Solution

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5)\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 60 + 4 - 15 = 49$$

Exercises: Let $\mathbf{u} = (3, 2, -1)$, $\mathbf{v} = (0, 2, -3)$, and $\mathbf{w} = (2, 6, 7)$.

Compute the indicated vectors

1. (a) $\mathbf{v} \times \mathbf{w}$ (b) $\mathbf{w} \times \mathbf{v}$ (c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w}$

   (d) $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ (e) $\mathbf{v} \times \mathbf{v}$ (f) $(\mathbf{u} - 3\mathbf{w}) \times (\mathbf{u} - 3\mathbf{w})$

2. (a) $\mathbf{u} \times \mathbf{v}$ (b) $-(\mathbf{u} \times \mathbf{v})$ (c) $\mathbf{u} \times (\mathbf{v} + \mathbf{w})$

   (d) $\mathbf{w} \cdot (\mathbf{w} \times \mathbf{v})$ (e) $\mathbf{w} \times \mathbf{w}$ (f) $(7\mathbf{v} - 3\mathbf{u}) \times (7\mathbf{v} - 3\mathbf{u})$

Exercises

Compute the scalar triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$

1. $\mathbf{u} = (-2, 0, 6),\ \mathbf{v} = (1, -3, 1),\ \mathbf{w} = (-5, -1, 1)$

2. $\mathbf{u} = (-1, 2, 4),\ \mathbf{v} = (3, 4, -2),\ \mathbf{w} = (-1, 2, 5)$

3. $\mathbf{u} = (a, 0, 0),\ \mathbf{v} = (0, b, 0),\ \mathbf{w} = (0, 0, c)$

4. $\mathbf{u} = \mathbf{i},\ \mathbf{v} = \mathbf{j},\ \mathbf{w} = \mathbf{k}$